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Finite-time analysis of global projective synchronization on coloured networks

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Abstract. A novel finite-time analysis is given to investigate the global projective synchronization on coloured networks. Some less conservative conditions are derived by utilizing finite-time control techniques and Lyapunov stability theorem. In addition, two illustrative numerical simulations are provided to verify the effectiveness of the proposed theoretical results.

Keywords. Coloured networks; finite-time analysis; global projective synchronization.

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1. Introduction

Complex networks are important and significant to human life, society and governance [1–3]. The earliest research of modern network theory could be traced back to the early 1960s, when Erdös and Rényi proposed a random-graph model [4]. Each pair of nodes, in a random different network, was connected with a certain probability. Then, Watts and Strogatz [5] proposed a novel small-world network model in 1998 which described a transition from a regular network to a random network. After that, Barabsi and Albert [6] presented a scale-free network model. Since then, complex networks have aroused considerable interest due to their real-world application, and many network models such as weighted networks and directed networks have been proposed [7].

A recent research on network model has been given by Wu [8]. A coloured network model was put forward for describing the complexity of some interconnected physical systems. In this model, nodes with different colours indicate that they have different local dynamic behaviour, and a pair of nodes connected by different coloured edges indicate that they have different mutual interaction. The obtained results take an important step forward exploring the model of realistic complex networks. To enrich this new model in synchronization category, Sun *et al* [9] exploited the synchronization problem of two coloured networks via discrete control. However, studies on the synchronization of coloured networks only focus on general synchronization.

In reality, the drive and the response system can be synchronized up to a scaling factor – a constant transformation between drive and response variables, which is called projective synchronization [10]. This proportional feature, in application to secure communications, can be used to extend binary digital to M-ary digital communication for faster communication. Therefore, the projective synchronization should be essentially considered to simulate secure communications.

With the aforementioned background, in this paper, we concentrate on the problems of global projective synchronization of coloured networks. To study more realistic situations, uncertainties are taken into account based on the facts that the parameter fluctuation, external disturbance and parameter uncertainties are unavoidable. In addition, an effective control – finite-time control technique is considered for achieving the synchronization in a faster rate. Furthermore, a detailed theoretical analysis is given to explore sufficient conditions of global projective synchronization for coloured networks.

2. Problem statement

In this paper, we consider a general coloured network consisting of N linearly and diffusively coupled identical nodes described as

$$\dot{x}_i(t) = f_i(t, x_i(t)) + g_i(t, x_i(t)) \cdot \alpha_i + \varepsilon \sum_{j=1, j\neq i}^{N} a_{ij} \Gamma H_{ij}(x_j(t) - x_i(t)), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), ..., x_{in}(t))^{\mathrm{T}} \in R^n$ is the state vector of the ith node; $F_i(t, x, \alpha_i) = f_i(t, x_i(t)) + g_i(t, x_i(t)) \cdot \alpha_i$ represents the local dynamics of node i, which is continuously differentiable; $f_i(\cdot)$ and $g_i(t, x_i(t))$: $R^n \to R^n$ are nonlinear vector-valued functions; Γ is the inner coupling matrix; the matrix $A = (a_{ij})_{N\times N}$ is outer coupling matrix, where $a_{ij} > 0$ if there is a connection between nodes i and $j (i \neq j)$, and $a_{ij} = 0$ otherwise; $H_{ij} = \mathrm{diag}(h_{ij}^1, h_{ij}^2, ..., h_{ij}^n)$ is the inner coupling matrix, which represents the mutual interactions between nodes i and j, and is defined as follows: if the ζth component of node i is affected by that of node j, then $h_{ij}^{\xi} \neq 0$, otherwise $h_{ij}^{\xi} = 0$.

Figure 1 indicates that $F_1 = F_3 = F_4$, $F_2 = F_5 = F_6$, $H_{16} = H_{23}$, $H_{12} = H_{35}$, $H_{24} = H_{36}$. When $n = 3$, $H_{16} = \mathrm{diag}\{1, 1, 0\}$ and $H_{56} = \mathrm{diag}\{1, 0, 1\}$, the first and second components of node 1 are affected by that of node 6, and the first and third components of node 6 are affected by that of node 5, as shown in figure 2. Let $c_{ij} = \mathrm{diag}(c_{ij}^1, c_{ij}^2, ..., c_{ij}^n)$, where $c_{ij}^k = a_{ij} h_{ij}^k$ for $i \neq j$ and $c_{ii}^k = -\sum_{j=1, j\neq i}^{N} c_{ij}^k$. Then the coloured network (1) can be rewritten in Kronecker product form:

$$\dot{x}(t) = f(t, x(t)) \otimes I + g(t, x(t)) \cdot \alpha \otimes I + \varepsilon C \otimes \Gamma x(t), \quad i = 1, 2, ..., N. \tag{2}$$

Let $C_{\xi} = (c_{ij}^{\xi}) \in R^{N\times N}$, $\xi = 1, 2, ..., N$, we regard the coloured network (2) as a combination of n-component subnetworks with a topology determined by C_{ξ}, $\xi = 1, 2, ..., n$. Define the coloured network (2) as the drive network; then the response edge-coloured networks can be described as follows:

$$\dot{y}(t) = f(t, y(t)) \otimes I + g(t, y(t)) \cdot \alpha \otimes I + \varepsilon C \otimes \Gamma y(t) + U(t), \quad i = 1, 2, ..., N, \tag{3}$$

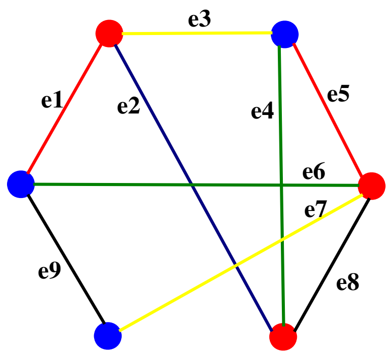

Figure 1. A coloured network consisting of six coloured nodes and nine coloured edges.

where $y(t) = (y_1(t), y_2(t), ..., y_n(t))^{\mathrm{T}} \in R^n$ stands for the state vector of the ith node, $f(t, y(t))$ and $g(t, y(t))$: $R^n \to R^n$ are nonlinear vector-valued functions, $U(t)$ is an adaptive controller.

DEFINITION 1

The drive–response coloured networks are said to achieve adaptive global projective synchronization (GPS) if there exists a scaling constant matrix $P = (p_{ij})_{\mathrm{n}\times\mathrm{n}}$ such that

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t) - Px(t)\| = 0. \tag{4}$$

According to the concept of Definition 1, the synchronization error is defined as $e_i(t) = y(t) - Px(t)$. Then we have the derivative of error dynamic system:

$$\begin{aligned}\dot{e}(t) = \dot{y}(t) - P\dot{x}(t) = f(t, y(t)) - Pf(t, x(t)) + (g(t, y(t)) - Pg(t, x(t)))\cdot\alpha \\ +\varepsilon C \otimes \Gamma(y(t) - Px(t)) + U.\end{aligned}$$

Assumption 1. Assume that $F(\cdot)$ satisfies globally Lipschitz continuous conditions, i.e., there exists a constant matrix $L = (l_{ij})_{\mathrm{n}\times\mathrm{n}} > 0$ such that

$$\|F(t, x, \alpha) - F(t, y, \alpha)\| \le L \|x - y\| .$$

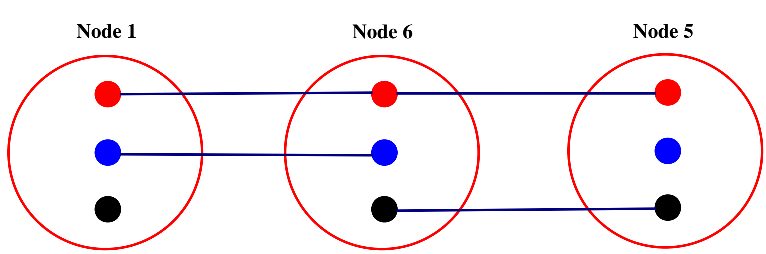

Figure 2. The red, blue and black stand for the first, second and third components of each individual, respectively.

Lemma 1 [11]. *Assume that a continuous and positive-definite function V(t) satisfies the following differential inequality*:

$$\dot{V}(t) \le -pV^{\xi}(t) \quad \forall t \ge t_0, \quad V(t_0) \ge 0,$$

where $p > 0$, $0 < \xi < 1$ *are two constants. Then, for any given* t_0, *V(t) satisfies the following inequality*:

$$V^{1-\xi}(t) \le V^{1-\xi}(t_0) - p(1-\xi)(t-t_0), \quad t_0 \le t \le t_1$$

and $V(t) = 0, \forall t \ge t_1$ *with* t_1 *given by*

$$t_1 = t_0 + \frac{V^{1-\xi}(t_0)}{p(1-\xi)}. \tag{5}$$

3. Finite-time analysis of global projective synchronization

In this section, we present our main results that the coloured network (2) achieves global projective synchronization with general coloured network (3) in finite time. Our objective is to find control laws U for stabilizing the error variables of the networks at the origin. To this end, we put forward the following control law for coloured networks:

$$\begin{aligned} U \; = \; & Pf(t,x(t)) - f(t,Px(t)) - (g(t,y(t))) - Pg(t,x(t)))\hat{\alpha} \\ & - De(t) - \omega\,\mathrm{sign}(e(t))\,|e(t)|^{\theta}, \end{aligned} \tag{6}$$

where $D > 0$ is an adaptive control gain matrix that can be suitably chosen by the global function projective error system.

Before giving the main results, we state that Assumption 1 holds in the following theorem and proof.

Theorem 1. *The coloured network* (1) *is globally synchronized in finite time*

$$t_1 = t_0 + \frac{V(t_0)^{(1-\theta)/2}}{2\omega(1-\theta)}$$

for any given t_0 *if there exists a positive-definite matrix D, a constant* ω *and the following condition holds*:

$$[C \otimes \Gamma + (L - D) \otimes I] \le 0,$$

when using the parameter identification:

$$\dot{\hat{\alpha}} = \beta_i e_i^{\mathrm{T}}(t)[g_i(y_i(t)) - Pg_i(x_i(t))]. \tag{7}$$

Proof. Consider the following non-negative function:

$$V(t) = (1/2)\,e^{\mathrm{T}}(t)\,e(t) + (1/2)\,(\alpha - \hat{\alpha})^2.$$

By some detailed calculation, the Dini derivative of *V*(*t*) along the trajectories of error system (4) can be obtained as

$$\begin{aligned} \dot{V}(t) \; = \; & e^{\mathrm{T}}(t)\,\dot{e}(t) - \dot{\hat{\alpha}}(\alpha - \hat{\alpha}) \\ = \; & e^{\mathrm{T}}(t)\,(f(t,y(t)) - Pf(t,x(t))) \otimes I + e^{\mathrm{T}}(t)\,(g(t,y(t)) \end{aligned}$$

$$-Pg(t, x(t))) \cdot (\alpha - \hat{\alpha}) \otimes I$$
$$+\varepsilon e^{\mathrm{T}}(t)\,(C \otimes \Gamma - D)e(t) - e^{\mathrm{T}}(t)\omega \,\mathrm{sign}\,(e(t))\,|e(t)|^{\theta} - \dot{\hat{\alpha}}(\alpha - \hat{\alpha}). \quad (8)$$

Note that

$$e^{\mathrm{T}}\omega \,\mathrm{sign}\,(e(t))\,|e(t)|^{\theta} = \omega|e^{\mathrm{T}}(t)|\,|e(t)|^{\theta} = \omega\,|e(t)|^{\theta+1} \quad (9)$$

and

$$|e(t)|^{\theta+1} \geq (|e(t)|^2)^{(\theta+1)/2} = (e^{\mathrm{T}}(t)\,e(t))^{(\theta+1)/2}. \quad (10)$$

From eqs (8)–(10), we obtain

$$\dot{V}(t) \leq e^{\mathrm{T}}(t)\,[C \otimes \Gamma + (L - D) \otimes I]\,e(t) - \omega\,(e^{\mathrm{T}}(t)\,e(t))^{(\theta+1)/2}.$$

By virtue of Theorem 1, as $[C \otimes \Gamma + (L - D) \otimes I] \leq 0$, we can derive that

$$\dot{V}(t) \leq -\omega\,(e^{\mathrm{T}}(t)\,e(t))^{(\theta+1)/2}.$$

Therefore, on the basis of the Lyapunov stability theorem and Lemma 1, the error dynamical system (4) is asymptotically stable at the origin with the controller (6). Thus, the states of the drive networks and that of the response networks are ultimately asymptotically global projective synchronized in finite time

$$t_1 = t_0 + \frac{V(t_0)^{(1-\theta)/2}}{2\omega(1-\theta)},$$

for any given t_0. That completes the proof. □

As is well-known, synchronization is not only an ubiquitous phenomenon in nature, but also an important research branch in nonlinear science of physics. The highlight of this paper is the employment of finite-time control techniques on the synchronization of coloured networks. Compared to impulsive control or intermittent control, finite-time control technique is an effective way to achieve synchronization at a faster rate, and is easy to apply in the areas of physics and engineering.

Remark 1. It should be emphasized that finite-time control techniques are adopted to guarantee global projective synchronization of coloured networks, while little previous papers have involved this work. The proposed methods are also appropriate for other networks, e.g. stochastic network, community network.

4. Numerical simulation

In this section, two illustrative examples are given to demonstrate the correctness and validity of the theoretical results.

Example 1. Consider an edge-coloured network with ten coupled Lorenz system nodes. Lorenz system is described as

$$F(t, x(t), \alpha) = \begin{pmatrix} a(x_2 - x_1) \\ cx_1 - x_1x_3 - x_2 \\ x_1x_2 - bx_3 \end{pmatrix},$$

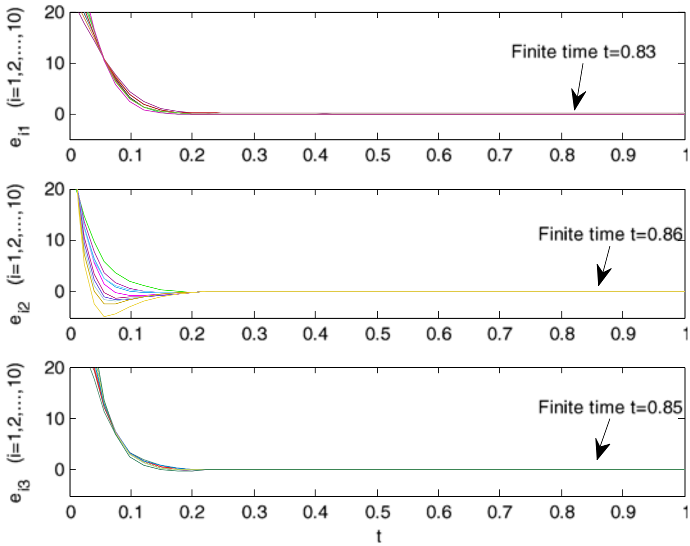

Figure 3. Synchronization errors of the edge-coloured network coupled with ten Lorenz systems.

where the unknown parameters $\alpha = (a, b, c)^{\mathrm{T}}$, $a = 10$, $b = 8/3$, $c = 28$. In this numerical simulation, we assume the control gain $D = \mathrm{diag}[10, 10, \ldots, 10]$, the scaling constant matrix $P = \mathrm{diag}[2, 2, \ldots, 2]$ and the initial values of the drive–response system are chosen as $x_i(0) = (0.3 + 0.1i, 0.3 + 0.1i, 0.3 + 0.1i)^{\mathrm{T}}$, $y_i(0) = (2.0 + 0.7\mathrm{i}, 2.0 + 0.7\mathrm{i}, 2.0 + 0.7\mathrm{i})^{\mathrm{T}}$. For brevity, but without loss of generality, one always assumes $\Gamma = \mathrm{diag}(1, 1, 1)$,

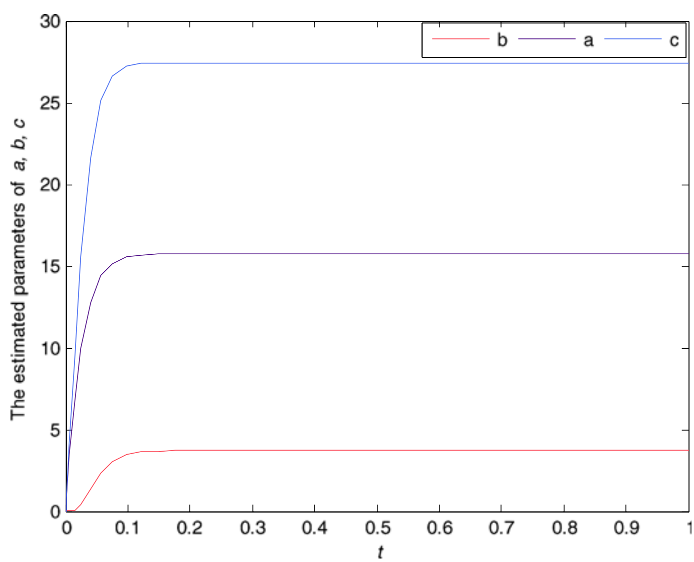

Figure 4. The estimated parameters in node dynamic state.

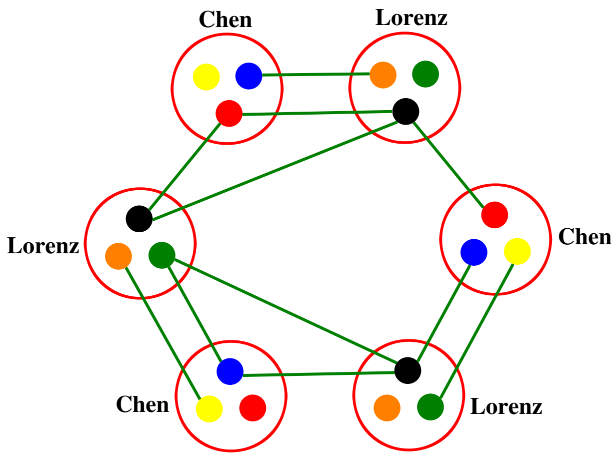

Figure 5. The topology of the coloured network coupled with three Chen systems and three Lorenz systems.

$\|\Gamma\| = 1$, $L = 1$, $\theta = 0.5$, $\beta_1 = \beta_2 = \beta_3 = 1$, $\omega_1 = \omega_2 = \omega_3 = 10$, and the estimated parameters have initial conditions: $\hat{a} = 0$, $\hat{b} = 0$, $\hat{c} = 0$. Figure 3 implies that the error dynamical systems are globally stable and the synchronization is achieved in finite time. The values of estimated parameters a, b, c are given in figure 4.

Example 2. Two general coloured networks, whose topology coupled with three Chen systems and three Lorenz systems shown in figure 5, are considered.

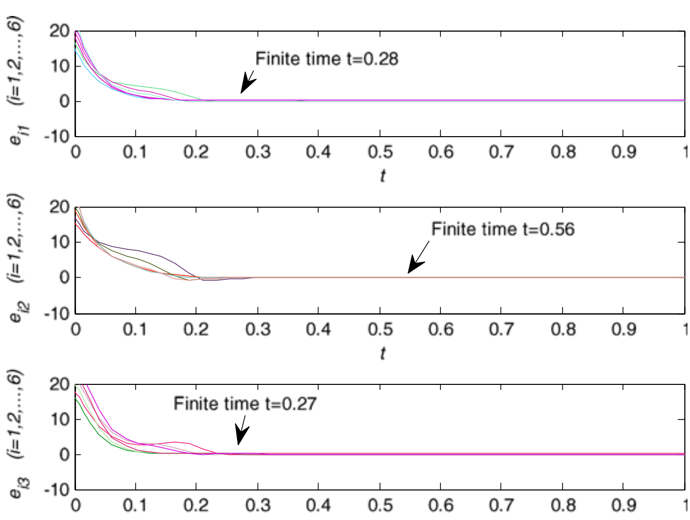

Figure 6. Synchronization errors of the coloured network coupled with three Chen systems and three Lorenz systems.

The Chen system is given as

$$F(t, x(t), \alpha) = \begin{pmatrix} q_1(x_2 - x_1) \\ (28{-}35)x_1 - x_1x_3 + 28x_2 + q_2 \\ x_1x_2 - q_3x_3 \end{pmatrix},$$

where the unknown parameter $\alpha = (q_1, q_2, q_3)^{\mathrm{T}}$. $q_1 = 35$, $q_2 = 0$, $q_3 = 28$. $\theta = 0.6$, $P_1 = \mathrm{diag}(1, 1, 1)$, $D = (20, 20, \ldots, 20)^{\mathrm{T}}$, $\omega = (20, 8, 20)^{\mathrm{T}}$. The initial values of the drive–response system chosen are the same as the ones in Example 1. The synchronization error

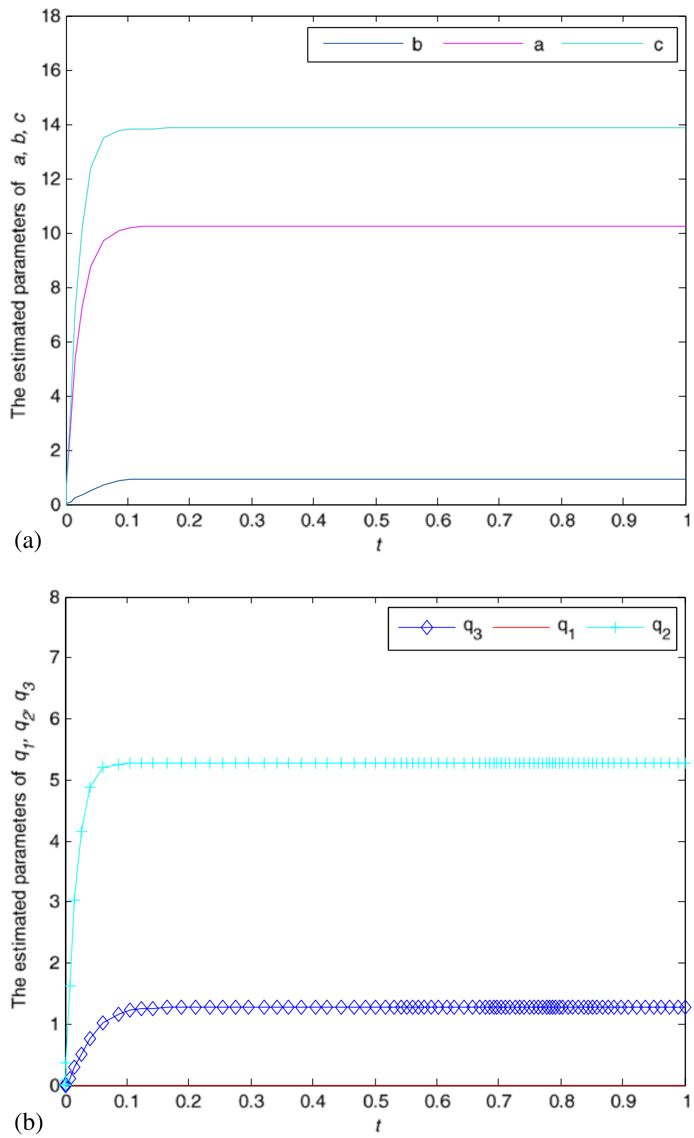

Figure 7. (**a**) The estimated parameters in Lorenz system and (**b**) in Chen system.

orbit of the general coloured networks is shown in figure 6. Figure 7 shows the estimated parameters of a, b, c in Lorenz systems and q_1, q_2, q_3 in Chen systems. Therefore, this example also shows the feasibility and effectiveness of theoretical results.

From these two examples, we can easily obtain the facts that the finite-time control technique is an effective way to achieve synchronization on coloured networks. The difference between the two examples is that while the first example studies general networks – special edge-coloured networks which have identical dynamic nodes, the second one explores general coloured networks with two kinds of dynamic nodes. It also shows the application of finite-time control technique in a wide range.

Remark 2. The topology of coloured network coupled with one Chen system, two Rössler systems and two Lorenz systems in figure 5 (in ref. [9]) are not accurate. In fact, different nodes of coloured networks indicate different dynamical behaviour. The dynamical behaviour of different nodes should be assigned to different colours. However, different systems have the same colour in ref. [9]. That is the reason why the figure in [9] is incorrect. Analogous figure is more precise in figure 5.

Remark 3. General synchronization of coloured networks has been extensively studied, in which all the nodes synchronize with each other in a common manner. However, in real complex networks, different communities usually synchronize with each other in a different manner. So in this paper we consider global projective synchronization. General projective synchronization can be realized if $P = \phi_i$.

Remark 4. In the existing research of synchronization of coloured networks, certain networks are often considered [8,9]. However, information may not be available in many practical cases. The uncertain networks (1) can be seen as the special case of the coloured networks.

5. Conclusion

This paper investigated global projective synchronization of coloured networks in finite time. An uncertain coloured network is considered in many practical cases. An effective method – a finite-time control technique – was applied to achieve synchronization of the coloured networks instead of using impulsive control or intermittent control, to reduce the synchronization time. Rigorous and effective criteria for the coloured networks were established by theoretical analysis. Finally, numerical simulations were provided to verify the feasibility and effectiveness of theoretical results.

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