PRAMANA © Indian Academy of Sciences Vol. 86, No. 3
— journal of March 2016
physics pp. 495–502

The entanglement evolution between two entangled atoms

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MS received 25 September 2014; revised 28 December 2014; accepted 12 January 2015
DOI: 10.1007/s12043-015-1030-8; ***e*Publication:** 26 August 2015

Abstract. The entanglement properties of two entangled atoms interacting with the field under intensity-dependent coupling are studied in detail. It is found that the degree of entanglement between the two atoms changes periodically and undergoes the entanglement sudden death (ESD) and sudden birth at some time. The entanglement properties between the field and the atom inside the cavity are dependent on the photon number. Most interestingly, the entanglement between the field and the atom in the field is influenced significantly by manipulating the atom outside the field.

Keywords. Entanglement; entangled atoms; concurrence; manipulation.

PACS Nos 03.65; 03.67

1. Introduction

The concept of entanglement is probably one of the most striking features of quantum mechanics. Entanglement is an important resource for quantum information processing [1–3] and also one of the most important nonclassical properties in quantum theory. Quantum entanglement is seen not only as a puzzle, but also as a resource to be manipulated for communication, information processing and quantum computing [4,5], such as in the investigation of quantum teleportation, dense coding, decoherence in quantum computers and the evaluation of quantum cryptographic schemes [6–10].

In recent years, there has been great interest in a two-level atom interacting with the field [11]. It is shown that the effect of Stark shift changes the quasiperiod of the field entropy. The entanglement between a three-level atom and the field is also studied in [12]. Its results show that, there is a steady-state entanglement between the three-level atom and the field. The entanglement between the atom and the field decreases due to the coherent superposition of the atomic levels. There has also been great interest in atoms interacting with a cavity field [13–16]. It is shown that the intrinsic decoherence does not completely destroy the entanglement, but drives the atoms into a stationary entangled state.

Yu and Eberly [17,18] have shown that the entanglement of two qubits can deteriorate rapidly, to the point of an abrupt total destruction, when coupled to an environment that results in irreversible loss. Quite recently, it has been shown that entanglement in two qubit systems can experience sudden death and sudden birth [19–23]. In earlier studies about the entanglement, two entangled atoms were exposed to the field simultaneously. And also two remote atoms interacting independently with a cavity field are studied in [24]. However, there is the case where the two entangled atoms are far apart, which makes it possible to operate one atom at a time. It is interesting and necessary to see how the entanglement evolves in such a system. It is also intriguing to see how entanglement of atom–atom and atom–field evolves for quantum information processing.

2. Theoretical model

Figure 1 schematically depicts the physical configuration of the composite system under consideration, which consists of two entangled two-level atoms A and B with one atom exposed to the field. Under the resonant condition, which means that the atomic transition frequency is equal to the cavity oscillatory frequency, the Hamiltonian is

$$H = H_0 + V, \tag{1}$$

where

$$H_0 = \omega a^+ a + \omega_0 \sum_{l=1}^{2} S_z^{(l)}, \tag{2}$$

$$V = g(af(a^+a)S^+ + S^- f(a^+a)a^+), \tag{3}$$

where ω is the field frequency, ω_0 is the atomic transition frequency, a and a^+ denote the annihilation and creation operators for the single-mode cavity field and g is the atomic-field coupling constant. The function f, representing the intensity-dependent coupling, is a real function of the photon number operator, for instance $f(a^+a) = \sqrt{a^+a}$. For mathematical simplicity, we consider the case where the field is resonant with the atomic transition, i.e., $\omega = \omega_0$.

We assume that initially the two atoms are at the following entangled states:

$$|\psi_{\mathrm{a}}\rangle = \sqrt{\gamma}\,|eg\rangle\,\sqrt{1-\gamma}\,|ge\rangle, \tag{4}$$

Figure 1. Schematic diagram of the physical model.

where $|e\rangle$ and $|g\rangle$ are the excited state and the ground state respectively. The initial state of the field is the Fock state

$$|\psi_{\rm f}\rangle = |n\rangle. \tag{5}$$

Therefore, the initial-state vector of the atom–field system is

$$|\psi_{\rm af}(0)\rangle = \sqrt{\gamma}\,|egn\rangle + \sqrt{1-\gamma}\,|gen\rangle. \tag{6}$$

At any time, the state vector of the atom–field system has the general form

$$\begin{aligned}|\psi_{\rm af}(t)\rangle \;=\; & A(n,t)|e,e,n-1\rangle + B(n,t)\,|e,g,n\rangle \\ & +C(n,t)|g,e,n\rangle + D(n,t)|g,g,n+1\rangle.\end{aligned} \tag{7}$$

Substituting eq. (7) into the Schrödinger equation

$$i\frac{\rm d}{{\rm d}t}\,|\psi_{\rm af}(t)\rangle = V|\psi_{\rm af}(t)\rangle, \tag{8}$$

we obtain

$$i\frac{{\rm d}A(n,t)}{{\rm d}t} = ngB(n,t), \tag{9}$$

$$i\frac{{\rm d}B(n,t)}{{\rm d}t} = ngA(n,t), \tag{10}$$

$$i\frac{{\rm d}C(n,t)}{{\rm d}t} = g(n+1)D(n,t), \tag{11}$$

$$i\frac{{\rm d}D(n,t)}{{\rm d}t} = g(n+1)C(n,t). \tag{12}$$

From the above four equations, the four coefficients can be derived as

$$\begin{aligned}A(n,t) \;=\; & -i\sqrt{\gamma}\sin ngt, \quad B(n,t) = \sqrt{\gamma}\cos ngt, \\ C(n,t) \;=\; & \sqrt{1-\gamma}\cos(n+1)gt, \quad D(n,t) = -i\sqrt{1-\gamma}\sin(n+1)gt.\end{aligned} \tag{13}$$

3. Entanglement evolution of the entangled atoms

To quantify and trace the amount of entanglement appropriately, Wootters concurrence [25] is employed, which is defined mathematically as

$$C = 2\max\left\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right\}, \tag{14}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of the density matrix operator $\rho = \rho_{\rm AB}(\sigma_y^{\rm A} \otimes \sigma_y^{\rm B})\rho_{\rm AB}^*(\sigma_y^{\rm A} \otimes \sigma_y^{\rm B})$. Here $\rho_{\rm AB}^*$ denotes the complex conjugation of $\rho_{\rm AB}$, and $\sigma_y^{\rm A(B)}$ is the standard Pauli matrix acting in the space of qubit A (or B). The concurrence varies from $C(\rho) = 0$ for an unentangled state to $C(\rho) = 1$ for a maximally entangled state. The reduced density matrix of the two atoms $\rho_{\rm a}$, in the standard product basis $|ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle$, can be written as

$$\rho_{\rm a}(t) = {\rm Tr}_{\rm f}\rho_{\rm af}(t) = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{15}$$

where

$$\rho_{11}=\gamma\sin^2(ngt),\quad \rho_{22}=\gamma\cos^2(ngt),\quad \rho_{33}=(1-\gamma)\cos^2[(n+1)gt],$$
$$\rho_{44}=(1-\gamma)\sin^2\big[(n+1)gt\big],\quad \rho_{23}=\rho_{32}=\sqrt{\gamma}\sqrt{1-\gamma}\cos(ngt)\cos[(n+1)gt].$$

The concurrence of ρ_a could be obtained as

$$C=2\max\{0,|\rho_{23}|-\sqrt{\rho_{11}\rho_{44}}\}. \tag{16}$$

First, we focus on the disentanglement of two atoms. It can be seen that the entanglement evolves continuously with certain periods. There is no entanglement sudden death as shown in figure 2a. In figure 2a, at time $t=(2n-1)\pi/2$, $n=1,2,3,\ldots$, the concurrence achieves its minimum. At time $t=n\pi$, the concurrence achieves its maximum. For $n=1$, the entanglement evolution of two atoms is shown in figure 2b. It can be seen that the concurrence decays exponentially to zero when the initial entanglement is large. The entanglement undergoes the entanglement sudden death (ESD) and sudden birth. As the photon number increases, the concurrence decays to zero in a finite time. For $n=3$, figure 2c reveals the time evolvement of concurrence. We see from figure 2b that the

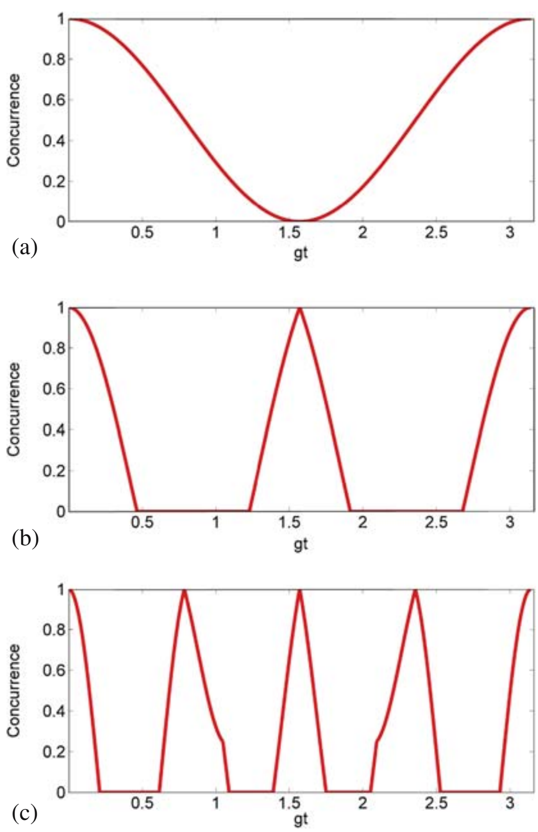

Figure 2. Atom–field entanglement vs. the scaled time gt when $\gamma=1/2$ for (**a**) $n=0$, (**b**) $n=1$, (**c**) $n=3$.

concurrence has an opposite effect with figure 2a at time $t = (2n-1)\pi/2$, $n = 1, 2, 3, \ldots$. For $n = 3$, the tendency is similar to that of the above case. It reveals that a period revival of full entanglement between atoms is possible.

4. Manipulation of the atom outside the field

Following the above section, we need to study how the field and the atom inside the cavity behave if the atom outside the field is manipulated. Guo and Yang [26] have found that the atom inside the cavity will emit squeezed light. In this section, we shall study how the manipulation of the atom outside the cavity affects the entanglement properties between the field and the atom inside the cavity. We shall perform a rotation operation on the atom outside the cavity and then measure it. The rotation operation leads to

$$\hat{R}(\theta)\,|g\rangle_{\mathrm{A}} = -\cos\theta\,|g\rangle_{\mathrm{A}} + \sin\theta\,|e\rangle_{\mathrm{A}}, \tag{17}$$

$$\hat{R}(\theta)\,|e\rangle_{\mathrm{A}} = -\sin\theta\,|g\rangle_{\mathrm{A}} + \cos\theta\,|e\rangle_{\mathrm{A}}. \tag{18}$$

Then, we study the atom outside the cavity. If the atom outside the cavity is detected in the ground state $|g\rangle$, the state of the system composed of the field and the atom inside the cavity will be given as

$$|\psi_{\mathrm{bf}}(t)\rangle = c(t)\,|e\rangle_{\mathrm{B}} + s(t)\,|g\rangle_{\mathrm{B}}, \tag{19}$$

where

$$c(t) = \frac{1}{\sqrt{2}}\cos\sqrt{n+1}gt\cos\theta\,|n\rangle + \frac{i}{\sqrt{2}}\sin\sqrt{n}gt\sin\theta\,|n-1\rangle, \tag{20}$$

$$s(t) = -\left(\frac{1}{\sqrt{2}}\cos\sqrt{n}gt\sin\theta\,|n\rangle + \frac{i}{\sqrt{2}}\sin\sqrt{n+1}gt\cos\theta\,|n+1\rangle\right). \tag{21}$$

We assume $\gamma = 1/2$ in the initial state of the two atoms. Using the above equations, the density matrix of the field and the atom inside the cavity at any time t can be obtained as

$$\rho(t) = \begin{pmatrix} |c(t)\rangle\langle c(t)| & |c(t)\rangle\langle s(t)| \\ |s(t)\rangle\langle c(t)| & |s(t)\rangle\langle s(t)| \end{pmatrix}. \tag{22}$$

We can obtain the expression for the density matrix of the atom

$$\rho_{\mathrm{a}}(t) = \begin{pmatrix} \langle c(t)|\,|c(t)\rangle & \langle c(t)|\,|s(t)\rangle \\ \langle s(t)|\,|c(t)\rangle & \langle s(t)|\,|s(t)\rangle \end{pmatrix}. \tag{23}$$

Following the work by Phoenix and Knight [27], the reduced entropy of the field (atom) is

$$S_{\mathrm{f}}(t) = S_{\mathrm{a}}(t) = -\mathrm{Tr}[\rho_{\mathrm{a}}(t)\ln\rho_{\mathrm{a}}(t)] = -\sum_{j=1}^{4}\lambda_j\ln\lambda_j, \tag{24}$$

where λ_j, $j = 1, 2, 3, 4$, are the eigenvalues of the corresponding reduced density matrix (21), which are found to satisfy the equation

$$\lambda^4 - \lambda^3 + P\lambda^2 + Q\lambda = 0, \tag{25}$$

where

$$P = \rho_{11}\rho_{22} + \rho_{11}\rho_{33} + \rho_{11}\rho_{44} + \rho_{22}\rho_{44} + \rho_{33}\rho_{44}, \tag{26}$$

$$Q = -\rho_{11}\rho_{22}\rho_{44} - \rho_{11}\rho_{33}\rho_{44}. \tag{27}$$

The eigenvalue of the reduced density matrix $\rho_{\mathrm{a}}(t)$ can be obtained. Therefore, we can use the quantum reduced entropy theory [28] to study the entanglement properties between the field and the atom inside the cavity.

Figure 3 shows the entanglement evolution as a function of gt at different rotation operations. We find that after the manipulation, the oscillations between the field and the atom inside the cavity are periodic for the rotation angle $\theta = \pi/2$. On the other hand, the disentanglement of the field–atom will not be found. The entanglement properties of

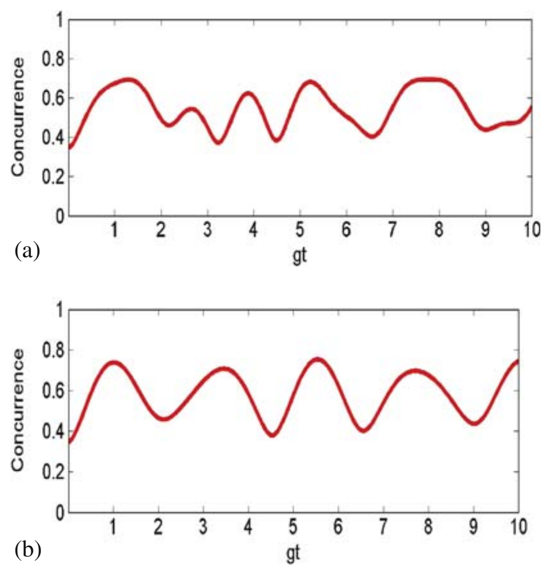

Figure 3. Atom–field entanglement vs. the scaled time gt after measuring the atom outside the cavity when $\gamma = 1/2$, $n = 1$. (**a**) $\theta = \pi/4$, (**b**) $\theta = \pi/2$.

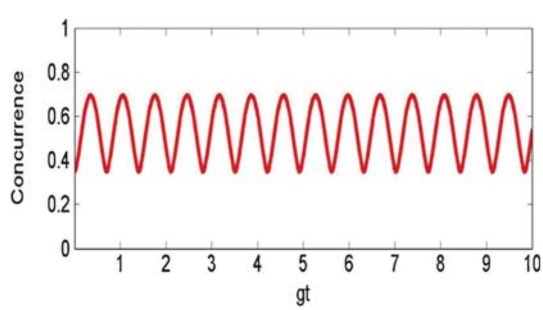

Figure 4. Atom–field entanglement vs. the scaled time gt after measuring the atom outside the cavity when $\gamma = 1/2$, $n = 5$, $\theta = \pi/2$.

the field and the atom inside the cavity are affected by manipulating the atom outside the cavity. If we detect the atom outside the cavity in an excited state in the above process, it is easy to get similar results.

Comparing with figure 3b, we plot the entanglement as a function of gt with the photon number $n = 5$ in figure 4. It is evident that the photon number does not change the maximum and minimum of the entanglement between the two atoms. However, the period of the entanglement between the two atoms is changed.

5. Conclusions

In summary, we have studied the entanglement properties of two entangled atoms. Our calculations have shown that entanglement between the two atoms changes periodically and undergoes the entanglement sudden death (ESD) and sudden birth at some time. We have performed a rotation operation on the atom and then measured it. The periodic oscillations between the field and the atom have also occurred for the rotation angle $\theta = \pi/2$.

Appendix. Derivation of the elements in eq. (15)

We rewrite the wave function (7) as

$$|\psi_{\mathrm{af}}(t)\rangle = A(n,t)|ee\rangle|n-1\rangle + [B(n,t)\,|eg\rangle + C(n,t)|ge\rangle]|n\rangle + D(n,t)|gg\rangle|n+1\rangle. \tag{A1}$$

The density matrix of the atom–field system is

$$\rho_{\mathrm{af}}(t) = |\psi_{\mathrm{af}}(t)\rangle\langle\psi_{\mathrm{af}}(t)|. \tag{A2}$$

By tracing over the field, we get the reduced density matrix for the atoms

$$\begin{aligned}\rho_{\mathrm{a}}(t) &= \mathrm{Tr}_{\mathrm{f}}[\rho_{\mathrm{af}}(t)] \\ &= \langle n-1|\,\rho_{\mathrm{af}}(t)\,|n-1\rangle + \langle n|\,\rho_{\mathrm{af}}(t)\,|n\rangle + \langle n+1|\,\rho_{\mathrm{af}}(t)\,|n+1\rangle \\ &= |A(n,t)|^2\,|ee\rangle\,\langle ee| + \left[B(n,t)\,|eg\rangle + C(n,t)\,|ge\rangle\right]\left[B^*(n,t)\,\langle eg| \right.\\ &\quad \left. + C^*(n,t)\,\langle ge|\right] + |D(n,t)|^2\,|gg\rangle\,\langle gg|.\end{aligned} \tag{A3}$$

The corresponding matrix elements of the atomic density matrix are

$$\begin{aligned}\rho_{11}(t) &= \langle ee|\rho_{\mathrm{a}}(t)|ee\rangle = |A(n,t)|^2 = \gamma\,\sin^2(ngt), \\ \rho_{22}(t) &= \langle eg|\rho_{\mathrm{a}}(t)|eg\rangle = |B(n,t)|^2 = \gamma\,\cos^2(ngt), \\ \rho_{33}(t) &= \langle ge|\rho_{\mathrm{a}}(t)|ge\rangle = |C(n,t)|^2 = (1-\gamma)\,\cos^2[(n+1)gt)], \\ \rho_{44}(t) &= \langle gg|\rho_{\mathrm{a}}(t)|gg\rangle = |D(n,t)|^2 = (1-\gamma)\,\sin^2[(n+1)gt)], \\ \rho_{23}(t) &= \langle eg|\rho_{\mathrm{a}}(t)|ge\rangle = B(n,t)C^*(n,t) \\ &= \sqrt{\gamma(1-\gamma)}\cos(ngt)\cos[(n+1)gt)], \\ \rho_{32}(t) &= \langle ge|\rho_{\mathrm{a}}(t)|eg\rangle = B^*(n,t)C(n,t) \\ &= \sqrt{\gamma(1-\gamma)}\cos(ngt)\cos[(n+1)gt)].\end{aligned} \tag{A4}$$

Other matrix elements are zero. As the coefficients $B(n,t)$ and $C(n,t)$ are real, we have

$$\rho_{23}(t) = \rho_{32}(t)$$

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