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Anomalous hydrodynamics in two dimensions

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Abstract. A new approach is presented to discuss two-dimensional hydrodynamics with gauge and gravitational anomalies. Exact constitutive relations for the stress tensor and charge current are obtained. Also, a connection between response parameters and anomaly coefficients is discussed. These are new results which, in the absence of the gauge sector, reproduce the results found by the gradient expansion approach.

Keywords. Anomalous hydrodynamics; gauge anomaly; gravitational anomaly.

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The chiral anomaly has played a ubiquitous role in modern physics. It has found applications in several diverse fields like quantum wires, quantum Hall effect, chiral magnetic effect and anomalous hydrodynamics, to name a few. In this work, we shall focus on anomalous hydrodynamics.

Fluid dynamics, as a non-relativistic field theory, is well known. Its relativistic extension is necessary if the macroscopic flow has a large velocity (comparable to light) or the microscopic motion of fluid particles is large.

The equations of motion are the conservation of energy–momentum tensor

$$\partial_\mu T^\mu{}_\nu = 0 \tag{1}$$

which, for a charged fluid, has to be supplemented with the conservation of the charge current,

$$\partial_\mu J^\mu = 0. \tag{2}$$

The above conservation laws are augmented by additional relations, known as constitutive relations, which express the energy–momentum tensor/charge in terms of the basic fluid variables like velocity, temperature and chemical potential. For an ideal fluid, these are given by

$$\begin{aligned} T_{\mu\nu} &= (\varepsilon + \mathcal{P})u_\mu u_\nu + \mathcal{P}\eta_{\mu\nu}, \\ J_\mu &= n u_\mu, \end{aligned} \tag{3}$$

where ε is the energy density, $\mathcal{P}$ is the pressure, n is the charge density, $\eta_{\mu\nu}$ is the metric and u_μ is the fluid velocity normalized as $u^2 = u_\mu u^\mu = -1$. Extra terms are necessary in the presence of dissipation. Further complications arise if anomalies are present.

There are usually two approaches to derive the constitutive relations in a general framework. One is based on a Landau-type approach, where positivity of entropy is ensured. The other is based on a derivative expansion approach, where the effective action is expressed as a series of powers of derivatives acting on the fluid variables. For two dimensions, however, conformal flatness of the metric ensures exact solvability of the effective action from which exact constitutive relations may be derived. This will be the approach followed here where exact constitutive relations in the presence of gauge and gravitational anomalies will be obtained.

Before presenting the details, let us make a quick review of anomalies. An anomaly is the breakdown of formal manipulations (ignoring infinities) leading to a modified conservation law. Effectively, this implies the breakdown of a classical symmetry at the quantum level. For the violation of a gauge symmetry, we have gauge anomaly, while for the violation of general coordinate symmetry, we have gravitational anomaly.

For a chiral theory, moreover, two types of anomalies have been widely studied. These are the covariant and consistent anomalies. Covariant anomalies transform covariantly under a gauge (or general coordinate) transformation but do not satisfy the Weiss–Zumino consistency condition. For consistent anomalies, it is just the other way round.

Let us now enumerate the various anomalous conservation laws in (1+1) dimensions. These involve the covariant derivative of the stress tensor,

$$\nabla_\nu T^{\nu\mu} = F^\mu{}_\nu J^\nu + C_g \bar{\epsilon}^{\mu\nu} \nabla_\nu R, \tag{4}$$

the trace of the stress tensor,

$$T^\mu_\mu = C_w R \tag{5}$$

and the covariant derivative of the gauge current,

$$\nabla_\mu J^\mu = C_s \bar{\epsilon}^{\mu\nu} F_{\mu\nu}. \tag{6}$$

Equations (4)–(6) [1–4] characterize the diffeomorphism anomaly, the conformal anomaly and the gauge anomaly, respectively. The coefficients of these anomalies are given by C_g, C_w and C_s. All expressions stand for covariant anomalies. Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge field strength and R is the Ricci scalar. Incidentally, the above functional forms for the covariant anomalies may be deduced from purely algebraic arguments.

Let us now give the general set up of (1+1)-dimensional static space–time. The general form of the metric is given by

$$\mathrm{d}s^2 = -\mathrm{e}^{2\sigma(r)}\mathrm{d}t^2 + g_{11}(r)\mathrm{d}r^2. \tag{7}$$

It is sometimes useful to work in null coordinates that are defined as

$$u = t - r^*, \quad v = t + r^*, \tag{8}$$

$$\frac{\mathrm{d}r}{\mathrm{d}r^*} = -\frac{\mathrm{e}^\sigma}{\sqrt{g_{11}}}.$$

In these coordinates, the metric takes the form

$$ds^2 = -\frac{1}{2}e^{2\sigma}(dudv + dvdu). \tag{9}$$

The antisymmetric tensor $\bar{\epsilon}_{\mu\nu}$ introduced earlier in (4) and (6) is expressed in terms of the numerical tensor $\epsilon_{\mu\nu}$ by

$$\bar{\epsilon}_{\mu\nu} = \sqrt{-g}\epsilon_{\mu\nu} = \frac{e^{2\sigma}}{2}\epsilon_{\mu\nu},$$

$$\epsilon_{uv} = -\epsilon_{vu} = 1 \tag{10}$$

where

$$g = \det g_{\mu\nu} = -\frac{e^{4\sigma}}{4} \tag{11}$$

in null coordinates.

The passage to hydrodynamics is effected by introducing the velocity u^μ of the time-independent equilibrium fluid fields satisfying $u^\mu u_\mu = -1$ (co-moving frame),

$$u^\mu = e^{-\sigma(r)}(1,0), \quad u_\mu = -e^{\sigma(r)}(1,0), \quad \mu = t, r. \tag{12}$$

In null coordinates,

$$u_\mu = -\frac{e^{\sigma(r)}}{2}(1,1), \quad u^\mu = e^{-\sigma(r)}(1,1), \quad \mu = u, v. \tag{13}$$

Also, we introduce the dual velocity,

$$\tilde{u}_\mu = \bar{\epsilon}_{\mu\nu}u^\mu = \frac{e^{\sigma(r)}}{2}(1,-1) \tag{14}$$

$$\tilde{u}^\mu = e^{-\sigma(r)}(1,-1), \quad \mu = u, v$$

and the chiral velocity,

$$u^c_\mu = u_\mu - \tilde{u}_\mu = -\bar{\epsilon}_{\mu\nu}u^{\nu c}. \tag{15}$$

The gauge field has the components

$$A_\mu = (A_t(r), 0)$$

in terms of which the chemical potential is

$$\mu = A_t(r)/\sqrt{-g_{00}} = A_t(r)e^{-\sigma}. \tag{16}$$

The fluid temperature T is expressed in terms of the equilibrium temperature T_0 by a Tolman-like relation,

$$T = T_0 e^{-\sigma}. \tag{17}$$

Finally, the Ricci scalar takes the form,

$$R = \frac{1}{g_{11}^2}(g'_{11}\sigma' - 2g_{11}\sigma'^2 - 2g_{11}\sigma'') = -2u^\mu \nabla^\nu \nabla_\mu u_\nu. \tag{18}$$

In terms of these expressions, the constitutive relations for the energy–momentum tensor and gauge current may be written. The final results are [5–7],

$$\begin{aligned}
T_{\mu\nu} = &\left[C_1 T^2 - C_w \left(u^\alpha \nabla^\beta \nabla_\beta u_\alpha\right) + \mu^2 \left(\frac{1}{2\pi} - C_s\right)\right] g_{\mu\nu} \\
&+ \left[2C_w \left(u^\alpha \nabla^\beta - u^\beta \nabla^\alpha\right) \nabla_\alpha u_\beta + 2C_1 T^2 + 2\mu^2 \left(\frac{1}{2\pi} - C_s\right)\right] u_\mu u_\nu \\
&- \left[2C_g \left(u^\alpha \nabla^\beta - u^\beta \nabla^\alpha\right) \nabla_\alpha u_\beta + C_2 T^2 + C_s \mu^2\right] \left(u_\mu \tilde{u}_\nu + \tilde{u}_\mu u_\nu\right) \\
&+ \left\{\left(\frac{C}{\pi} - 2(C+P)C_s\right) \frac{T}{T_0}\mu + \left(\frac{C^2 + P^2}{2\pi} - C_s (C+P)^2\right) \frac{T^2}{T_0{}^2}\right\} \left(2u_\mu u_\nu + g_{\mu\nu}\right) \\
&+ \left\{\left(\frac{P}{\pi} - 2(C+P)C_s\right) \frac{T}{T_0}\mu + \left(\frac{CP}{\pi} - C_s (C+P)^2\right) \frac{T^2}{T_0{}^2}\right\} \left(u_\mu \tilde{u}_\nu + \tilde{u}_\mu u_\nu\right)
\end{aligned}$$

and

$$\begin{aligned}
J_\mu = &-2C_s \mu \left(u_\mu + \tilde{u}_\mu\right) + \frac{\mu}{\pi} u_\mu + \left(\frac{C}{\pi} - 2(C+P)C_s\right) \frac{T}{T_0} u_\mu \\
&+ \left(\frac{P}{\pi} - 2(C+P)C_s\right) \frac{T}{T_0} \tilde{u}_\mu.
\end{aligned} \tag{19}$$

The above results are the solutions of the anomalous Ward identities (4) and (6) which may be explicitly checked by direct substitution. Here C_1, C_2, P and C are the integration constants that will be fixed by choosing appropriate boundary conditions.

It is possible to fix the constants by assuming that the metric is a solution of a black hole whose vacuum is determined by the Israel–Hartle–Hawking condition [8]. In fact, this is the only vacuum that is physically meaningful in the present context. It is defined by taking $T_{\mu\nu}/J_\mu$ in Kruskal variables, corresponding to both outgoing and ingoing modes, as regular, near the horizon. This implies T_{uu}, T_{vv}, J_u, $J_v \to 0$ near the horizon. A little algebra shows [6],

$$P = C = 0 \tag{20}$$

from the condition on J_u, J_v. Likewise [6],

$$C_1 = 4\pi^2 C_w, \quad C_2 = 8\pi^2 C_g,$$

from the condition on T_{uu}, T_{vv}.

Finally, we discuss a connection between response parameters and anomaly coefficients. The first thing is to write the constitutive relation (19) after putting the constants $P = C = 0$,

$$J_\mu = -2C_s \mu (u_\mu + \tilde{u}_\mu) + \frac{\mu}{\pi} u_\mu. \tag{21}$$

In the hydrodynamic expansion approach we have

$$J_\mu = -2C_s\mu\tilde{u}_\mu + \left(\frac{\partial P}{\partial\mu} - \frac{a_2'}{T^2}S_2 + \frac{a_2}{T}S_4\right)u_\mu, \tag{22}$$

where S_2, S_4 are certain combinations of gauge field that occur in the second-order expansion, and

$$P = T^2 p_0\left(\frac{\mu}{T}\right). \tag{23}$$

A comparison of the above relations yields

$$\frac{\partial P}{\partial\mu} = T^2\frac{\partial p_0}{\partial\mu} = \left(-2C_s + \frac{1}{\pi}\right)\mu; \quad a_2 = a_2' = 0$$

so that

$$p_0 = \left(\frac{1}{\pi} - C_s\right)\frac{\mu^2}{T^2} + Q(\text{int. const}). \tag{24}$$

The integration constant Q is fixed by using the known expression for p_0 in the absence of the gauge field [9],

$$p_0|_{\mu=0} = Q = 4\pi^2 C_w. \tag{25}$$

Inserting this value of Q in (24) gives the desired general result,

$$p_0 = 4\pi^2 C_w + \left(\frac{1}{\pi} - C_s\right)\frac{\mu^2}{T^2} \tag{26}$$

while the other relation is unchanged,

$$C_2 = 8\pi^2 C_g. \tag{27}$$

Relations (26) and (27) give the connection between the response parameters p_0 and C_2 with the anomaly coefficients. In the absence of the gauge field ($\mu = 0$), results found earlier [9] are reproduced. As a simple consistency check, it is easy to show that the constitutive relation for $T_{\mu\nu}$ found in (19) agrees with the form obtained by the derivative expansion method, provided the above identifications are used.

We have presented a new approach to analyse various features of anomalous hydrodynamics in two (1+1) dimensions. Exploiting the conformal flatness of the metric, it was possible to provide exact expressions for the constitutive relations for the stress tensor and the gauge current. It may be recalled that the standard approach, based on a gradient expansion, fails to provide concrete results in the presence of a gauge field. Likewise, relation (26) is another result that has not been provided within the gradient expansion scheme. In this sense our approach is economical and more general. We would like to extend this approach to study other relevant issues and also include dimensions higher than two.

The power of the present approach would lie in its applicability to dimensions greater than two. If the flow is chiral, i.e., it has some sort of handedness, then, anomalies will always occur. Of course, it is possible to redefine the anomalous stress tensor or the gauge current by the introduction of local counterterms. However, that just changes the structure of the anomaly, a typical example being the shift of the covariant anomaly considered in the paper to the consistent anomaly. The analysis has to be redone to account for this change and it will lead to different constitutive relations.

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