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# Variable-coefficient nonisospectral Toda lattice hierarchy and its exact solutions

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**Abstract.** In this paper, a hierarchy of nonisospectral equations with variable coefficients is derived from the compatibility condition of Toda spectral problem and its time evolution. In order to solve the derived Toda lattice hierarchy, the inverse scattering transformation is utilized. As a result, new and more general exact solutions are obtained. It is shown that the inverse scattering transformation can be generalized to solve some other nonisospectral lattice hierarchies with variable coefficients.

**Keywords.** Variable-coefficient nonisospectral Toda lattice hierarchy; exact solution; compatibility condition; inverse scattering transformation.

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## 1. Introduction

It is well known that nonlinear physical phenomena are often related to some nonlinear evolution equations (NLEEs). When the inhomogeneities of the media and nonuniformities of the boundaries are taken into account, the variable-coefficient NLEEs could describe more realistic physical phenomena than their constant-coefficient counterparts [1]. For some important earlier works on integrable inhomogeneous models, one may refer to [2–7]. Constructing such types of NLEEs and obtaining their exact solutions often play important roles in helping us to understand these phenomena. Since Ablowitz successfully solved the isospectral Toda equation, there has been various works for the Toda hierarchy, such as those in [8–10], but almost all these works focus on constant-coefficient and/or isospectral hierarchy. Relating to the same spectral problem, isospectral equations often describe the solitary waves in the lossless and uniform media, while the nonisospectral ones resulting from a spectral problem with a time-dependent spectral parameter $\lambda$ describe solitary waves in a certain type of nonuniform media [11]. In the past several decades, many effective methods [12–22] have been proposed for solving NLEEs. Among them, the inverse scattering transformation (IST) is a systematic method for finding exact solutions of NLEEs. Since its derivation in 1967 [12], IST has achieved considerable improvement and its application slope increased gradually [23–32]. One advantage of IST is that it can be used to solve a whole hierarchy of NLEEs associated with a certain spectral problem.

In this paper, we shall derive a nonisospectral hierarchy of nonlinear lattice equations with coefficients depending on $t$:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L^k \begin{pmatrix} \alpha(t)u_n(v_n - v_{n-1}) + 2\beta(t)u_n \\ \alpha(t)(u_{n+1} - u_n) + \beta(t)v_n \end{pmatrix}, \quad k = 0, 1, 2, \ldots \tag{1}$$

from the compatibility condition of Toda spectral problem and its time evolution, and then solve it through the IST. In eq. (1), $\alpha(t)$ and $\beta(t)$ are nonzero differentiable functions of $t$, and the operator $L$ is employed as

$$L = \begin{pmatrix} u_n(E-1)v_{n-1}(E-1)^{-1}\dfrac{1}{u_n} & u_n(1+E^{-1}) \\ (Eu_nE - u_n)(E-1)^{-1}\dfrac{1}{u_n} & v_n \end{pmatrix}, \tag{2}$$

where $E$ is the shift operator defined as $Ef_n = f_{n+1}$, $E^{-1}f_n = f_{n-1}$, the equality relation $(E-1)^{-1} = (E-E^{-1})^{-1}(1+E^{-1})$ holds provided $f_n$ approaches zero rapidly as $|n| \to \infty$ and the inverse operator of $E - E^{-1}$ is defined as

$$(E - E^{-1})^{-1} f_n = -\sum_{m=n}^{\infty} f_{2m-n+1} \tag{3}$$

or

$$(E - E^{-1})^{-1} f_n = \sum_{-\infty}^{n} f_{2m-n-1}. \tag{4}$$

As supplementary for the operator (2), it is stipulated that $(E-1)^{-1}\alpha(t) = \theta_n(t)$, $(E-1)^{-1}\beta(t) = (\beta(t)/\alpha(t))\theta_n(t)$ and $(E-1)^{-1}0 = 0$. From here onwards $\theta_n(t)$ is a nonzero function of $n$ and $t$, which satisfies the constraint $(E-1)\theta_n(t) = \alpha(t)$.

Particularly, when $\alpha(t) = 1$ and $\beta(t) = 1$, eq. (1) becomes the constant-coefficient nonisospectral Toda lattice hierarchy

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L^k \begin{pmatrix} u_n(v_n - v_{n-1}) + 2u_n \\ (u_{n+1} - u_n) + v_n \end{pmatrix}, \quad k = 0, 1, 2, \ldots. \tag{5}$$

If $k = 0, 1$, then from eq. (1) two nonisospectral lattice equations with variable coefficients can be obtained

$$u_{n,t} = \alpha(t)u_n(v_n - v_{n-1}) + 2\beta(t)u_n, \tag{6}$$

$$v_{n,t} = \alpha(t)(u_{n+1} - u_n) + \beta(t)v_n \tag{7}$$

and

$$u_{n,t} = \alpha(t)u_n(v_n^2 - v_{n-1}^2) + \frac{2\beta(t)}{\alpha(t)}u_n[v_n\theta_{n+1}(t) - v_{n-1}\theta_n(t)]$$
$$+ \alpha(t)u_n(u_{n+2} - u_n) + \beta(t)u_n(v_n - v_{n-1}), \quad (8)$$
$$v_{n,t} = \alpha(t)(u_{n+1}v_{n+1} - u_nv_{n-1}) + \frac{2\beta(t)}{\alpha(t)}[u_{n+1}\theta_{n+2}(t) - u_n\theta_n(t)]$$
$$+ \alpha(t)u_n(u_{n+2} - u_n) + \beta(t)v_n^2. \quad (9)$$

As a reduced case of eqs (6) and (7), the following variable-coefficient nonisospectral Toda lattice equation

$$x_{n,tt} = \alpha(t)(\mathrm{e}^{x_{n-1}-x_n} - \mathrm{e}^{x_n-x_{n+1}}) + \beta(t)x_{n,t} \quad (10)$$

can be obtained if $u_n = \mathrm{e}^{x_{n-1}-x_n}$ and $v_n = -x_{n,t}$.

The rest of this paper is organized as follows. In §2, the variable-coefficient nonisospectral Toda lattice hierarchy (1) is derived. In §3, the IST is used to solve the whole hierarchy of nonisospectral lattice eq. (1). In §4, discussions and conclusions are given.

## 2. Variable-coefficient nonisospectral Toda lattice hierarchy

Let us consider the following spectral problem:

$$E\Psi_n = M\Psi_n, \quad M = \begin{pmatrix} 0 & 1 \\ -u_n & \lambda - v_n \end{pmatrix}, \quad \Psi_n = \begin{pmatrix} \varphi_{1,n} \\ \varphi_{2,n} \end{pmatrix} \quad (11)$$

and the time evolution

$$\Psi_{n,t} = N\Psi_n, \quad N = \begin{pmatrix} A_n & \alpha(t)B_n \\ C_n & \beta(t)D_n \end{pmatrix}, \quad (12)$$

where $u_n = u_n(t)$ and $v_n = v_n(t)$ are potentials and $\lambda$ is a spectral parameter. We assume that $u_n$ and $v_n$ are smooth functions of $t$ and $(u_n, v_n)$ approaches (1, 0) rapidly as $|n| \to \infty$. The compatibility condition of eqs (11) and (12) reads as

$$M_t = (EN)M - MN, \quad (13)$$

which gives

$$\alpha(t)u_nB_{n+1} + C_n = 0, \quad (14)$$

$$A_{n+1} + \alpha(t)(\lambda - v_n)B_{n+1} - \beta(t)D_n = 0, \quad (15)$$

$$u_{n,t} = \beta(t)u_nD_{n+1} - u_nA_n + (\lambda - v_n)C_n, \quad (16)$$

$$v_{n,t} = -C_{n+1} - (\lambda - v_n)\beta(t)D_{n+1} - \alpha(t)u_nB_n + (\lambda - v_n)\beta(t)D_n + \lambda_t. \quad (17)$$

Then we obtain

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L_1 \begin{pmatrix} D_n \\ B_n \end{pmatrix} - \lambda L_2 \begin{pmatrix} D_n \\ B_n \end{pmatrix} + \lambda_t \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{18}$$

where

$$L_1 = \begin{pmatrix} \beta(t)u_n(E - E^{-1}) & \alpha(t)u_n(E-1)v_{n-1} \\ \beta(t)v_n(E-1) & \alpha(t)(Eu_nE - u_n) \end{pmatrix}, \tag{19}$$

$$L_2 = \begin{pmatrix} 0 & \alpha(t)u_n(E-1) \\ \beta(t)(E-1) & 0 \end{pmatrix}. \tag{20}$$

Taking $\lambda_t = \alpha(t)\beta(t)\lambda^{k-1}(\lambda^2 - 4)$ and expanding $(D_n, B_n)^T$ as

$$\begin{pmatrix} D_n \\ B_n \end{pmatrix} = \sum_{j=0}^{k} \begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix} \lambda^{k-j}, \tag{21}$$

which satisfies the asymptotic conditions when $u_n = 1$ and $v_n = 0$

$$D_n = \alpha(t)n\lambda^k, \quad B_n = 2\beta(t)n\lambda^{k-1}, \tag{22}$$

and then comparing the coefficients of the same power of $\lambda$, we can obtain

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_t = L_1 \begin{pmatrix} d_{n,k} \\ b_{n,k} \end{pmatrix}, \tag{23}$$

$$L_2 \begin{pmatrix} d_{n,j+1} \\ b_{n,j+1} \end{pmatrix} = L_1 \begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix}, \quad j = 0, 1, \ldots, k-1, \tag{24}$$

$$L_2 \begin{pmatrix} d_{n,0} \\ b_{n,0} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha(t)\beta(t) \end{pmatrix}. \tag{25}$$

If $(E-1)^{-1}0 = 0$, then from eq. (21) we have

$$\begin{pmatrix} d_{n,j+1} \\ b_{n,j+1} \end{pmatrix} = L_2^{-1} L_1 \begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix}, \tag{26}$$

where

$$L_2^{-1} = \begin{pmatrix} 0 & (E-1)^{-1}\dfrac{1}{\beta(t)} \\ (E-1)^{-1}\dfrac{1}{\alpha(t)u_n} & 0 \end{pmatrix}. \tag{27}$$

A direct computation shows that

$$d_{n,0} = (E-1)^{-1}\alpha(t), \quad b_{n,0} = 0 \tag{28}$$

satisfies eq. (25), and therefore we have $d_{n,0} = \theta_n(t)$ as a prior assumption.

Assuming $L = L_1 L_2^{-1}$, from eq. (26) we obtain

$$\begin{pmatrix} d_{n,k} \\ b_{n,k} \end{pmatrix} = L_2^{-1} L^{k-1} \begin{pmatrix} \alpha(t)u_n(v_n - v_{n-1}) + 2\beta(t)u_n \\ \alpha(t)(u_{n+1} - u_n) + \beta(t)v_n \end{pmatrix}, \tag{29}$$

and hence reach the variable-coefficient nonisospectral Toda lattice hierarchy (1) under the necessary constraint that $\lambda \neq \pm 2$. An explanation [33] for this constraint is that, at $\lambda = \pm 2$ one gets $z = \pm 1$ when $z$ falls on the unit circle, where, as known for the standard Toda chain, one gets only continuous spectrum without solitonic solutions. On the other hand, $\lambda = \pm 2$ leading to $\lambda_t = \alpha(t)\beta(t)\lambda^{k-1}(\lambda^2 - 4) = 0$, which will give the corresponding isospectral hierarchy. However, this is not the starting point of this paper.

Finally, we point out that the following theorem holds.

**Theorem 1.** *The variable-coefficient nonisospectral Toda lattice hierarchy* (1) *is Lax integrable*, *the Lax pairs of which can be written as follows*:

$$\varphi_{n+1}(z) + u_n\varphi_{n-1}(z) + v_n\varphi_n(z) = \left(z + \frac{1}{z}\right)\varphi_n(z), \tag{30}$$

$$\varphi_{n,t}(z) = C_n\varphi_{n-1}(z) + \beta(t)D_n\varphi_n(z), \tag{31}$$

*where $C_n$ and $D_n$ are determined by eqs* (14) *and* (15), *$B_n$ and $D_n$ satisfy the asymptotic condition* (22) *and the spectral parameter $\lambda = z + \frac{1}{z}$.*

*Proof.* It is easy to see that when $C_n$ and $D_n$ satisfy eqs (14) and (15), and $B_n$ and $D_n$ possess the asymptotic condition (22), eq. (1) can be expressed by the compatibility condition of the Toda spectral problem (11) and the time evolution (12). It then follows from the definitions of Lax integrability and Lax pairs [26] that the variable-coefficient nonisospectral Toda lattice hierarchy (1) is Lax integrable. Rewriting $\varphi_{2,n} = \varphi_n$ and setting $\lambda = z + \frac{1}{z}$, we can reduce eqs (11) and (12) to the Lax pairs (30) and (31). Thus, the proof concludes. □

In the following section, we shall use the Lax Pairs (30) and (31) to solve the variable-coefficient nonisospectral Toda lattice hierarchy (1) by using the IST.

## 3. Exact solutions

In order to solve the variable-coefficient nonisospectral Toda lattice hierarchy (1), let us first recall the direct and inverse scattering problems [32].

### 3.1 *The direct scattering problem*

If the potentials $u_n$ and $v_n$ satisfy

$$\sum_{n=-\infty}^{\infty} \left|n^j (u_n - 1)\right| < \infty, \quad \sum_{n=-\infty}^{\infty} \left|n^j v_n\right| < \infty, \quad j = 0, 1, 2, \tag{32}$$

then the spectral problem (11) has a set of Jost solutions $\varphi_n(x)$, $\bar{\varphi}_n(x)$, $\phi_n(x)$, $\bar{\phi}_n(x)$, which are bounded for all values of $n$ and satisfy

$$\varphi_n(z) \sim z^n, \quad \bar{\varphi}(z) \sim z^{-n}, \quad n \to +\infty, \tag{33}$$

$$\phi_n(z) \sim z^{-n}, \quad \bar{\phi}(z) \sim z^n, \quad n \to -\infty, \tag{34}$$

where $\varphi_n(z)$ and $\phi_n(z)$ are analytic inside the unit circle, i.e., $|z| \leq 1$ on the complex plane of $z$, while $\bar{\varphi}_n(z)$ and $\bar{\phi}_n(z)$ are analytic outside the unit circle, i.e., $|z| > 1$. In addition, on the circum, i.e., $|z| = 1$, $\varphi_n(z)$ and $\phi_n(z)$ satisfy

$$\bar{\varphi}_n(z) = \varphi_n^*(z), \quad \bar{\phi}_n(z) = \phi_n^*(z). \tag{35}$$

Further, suppose that

$$\phi_n(z) = a(z)\bar{\varphi}_n(z) + b(z)\varphi_n(z), \tag{36}$$

$$\bar{\phi}_n(z) = \bar{a}(z)\varphi_n(z) + \bar{b}(z)\bar{\varphi}_n(z), \tag{37}$$

then from eqs (11), (33) and (34) we have

$$\left(z - \frac{1}{z}\right) a(z) = W(\varphi_n(z), \phi_n(z)), \tag{38}$$

$$\left(z - \frac{1}{z}\right) b(z) = W(\phi_n(z), \bar{\varphi}_n(z)), \tag{39}$$

$$\left(z - \frac{1}{z}\right) \bar{a}(z) = -W(\bar{\varphi}_n(z), \bar{\phi}_n(z)), \tag{40}$$

$$\left(z - \frac{1}{z}\right) \bar{b}(z) = W(\varphi_n(z), \bar{\phi}_n(z)), \tag{41}$$

where $S_n = \prod_{j=n+1}^{\infty} u_j$, the discrete Wronski determinant of $\varphi_n(z)$ and $\phi_n(z)$ are defined as

$$W(\varphi_n(z), \phi_n(z)) = S_{n-1}(\varphi_n(z)\phi_{n-1}(z) - \phi_{n-1}(z)\varphi_n(z)). \tag{42}$$

As $a(z)(\bar{a}(z))$ has only a finite number of simple zeros (simple poles) at $z_1, z_2, \ldots, z_l$ in the unit circle, $\varphi_n(z_j)$ and $\phi_n(z_j)$ are linearly dependent. Therefore, there exists a constant $b_j$ such that

$$\phi_n(z_j) = b_j\varphi_n(z_j), \quad j = 1, 2, \ldots, l. \tag{43}$$

Then from eq. (30) we have

$$\sum_{n=-\infty}^{\infty} S_n\varphi_n^2(z_j) = -\frac{z_j a_z(z_j)}{b_j}, \quad j = 1, 2, \ldots, l. \tag{44}$$

If the following equality

$$\sum_{n=-\infty}^{\infty} c_j^2 S_n\varphi_n^2(z_j) = 1, \tag{45}$$

holds, where $z_j$ is the simple zero of $a(z)$, then we call $c_j$ and $c_j\varphi_n(z_j)$ the normalization constant and the normalization eigenfunction of $\varphi_n(z_j)$, respectively.

Obviously, eqs (44) and (45) suggest that

$$c_j^2 = -\frac{b_j}{z_j a_z(z_j)}, \quad j = 1, 2, \ldots, l. \tag{46}$$

The set

$$\left\{|z| = 1,\ R(z) = \frac{b(z)}{a(z)},\ z_j,\ c_j, \quad j = 1, 2, \cdots, l\right\} \tag{47}$$

is named the scattering data of the spectral problem (30).

### 3.2 *The inverse scattering problem*

Let $\varphi_n(z)$, $\bar{\varphi}_n(z)$ and $\phi_n(z)$, $\bar{\phi}_n(z)$ be expanded as

$$\varphi_n(z) = \sum_{j=n}^{\infty} K_{n,j} z^j, \quad \bar{\varphi}_n(z) = \sum_{j=n}^{\infty} K_{n,j} z^{-j} \tag{48}$$

$$\phi_n(z) = \sum_{j=-\infty}^{n} J_{n,j} z^{-j}, \quad \bar{\phi}_n(z) = \sum_{j=-\infty}^{n} J_{n,j} z^j, \tag{49}$$

from eqs (30), (48) and (49), we have

$$u_n K_{n-1,n-1} = K_{n,n}, \tag{50}$$

$$u_n K_{n-1,n} + v_n K_{n,n} = K_{n,n+1}, \tag{51}$$

$$K_{n+1,n+1} + u_n K_{n-1,n+1} + v_n K_{n,n+1} = K_{n,n} + K_{n,n+2}, \tag{52}$$

and so forth, then we can determine $u_n$ and $v_n$ as follows:

$$u_n = \frac{K_{n,n}}{K_{n-1,n-1}}, \quad v_n = \frac{K_{n,n+1}}{K_{n,n}} - \frac{K_{n-1,n}}{K_{n-1,n-1}}. \tag{53}$$

For the given scattering data (47) for the spectral problem (30), $\tilde{K}_{n,m} = (K_{n,m}/K_{n,n})$ satisfy the discrete Gel'fand–Levitan–Marchenko (GLM) equation

$$\tilde{K}_{n,m} + F_{n,m} + \sum_{s=n+1}^{\infty} \tilde{K}_{n,s} F_{s+m} = 0, \quad m > n \tag{54}$$

and

$$\tilde{K}_{n,n}^{-2} = 1 + F_{2n} + \sum_{j=n+1}^{\infty} \tilde{K}_{n,j} F_{j+n}, \quad m = n, \tag{55}$$

where

$$F_m = \sum_{j=1}^{l} c_j^2 z_j^m + \frac{1}{2\pi i} \oint_{|z|=1} R(z) z^{m-1} \mathrm{d}z. \tag{56}$$

For the sake of convenience, we still consider $\tilde{K}_{n,m}$ as $K_{n,m}$ in the following discussion.

### 3.3 *Time evolution of the scattering data*

**Theorem 2.** *The scattering data* (47) *for the spectral problem* (30) *possess the following time evolutions*:

$$R[t, z(t)] = R[0, z(0)]\exp\left\{2\int_0^t \left[\alpha(s)\beta(s)\lambda^{k-1}(s)\left(z(s) - \frac{1}{z(s)}\right)\right]\mathrm{d}s\right\}, \tag{57}$$

$$c_j^2(t) = c_j^2(0)\exp\left\{\int_0^t \left[\alpha(s)\beta(s)\lambda^k(s)\left(k + 2 - \frac{4}{1 + z_j^2(s)}\right)\right]\mathrm{d}s\right\}, \tag{58}$$

*where*

$$\lambda(t) = z(t) + \frac{1}{z(t)}, \quad \lambda_j(t) = z_j(t) + \frac{1}{z_j(t)},$$

*$z(t)$ and $z_j(t)$ satisfy*

$$z_t(t) = \alpha(t)\beta(t)\lambda^{k-1}(t)[z^2(t) - 1], \tag{59}$$

$$z_{j,t}(t) = \alpha(t)\beta(t)\lambda_j^{k-1}(t)[z_j^2(t) - 1], \tag{60}$$

*while $c_j(0)$ and $R[0, z(0)]$ are the corresponding scattering data of eq.* (47) *when $u_n(t) = u_n(0)$ and $v_n(t) = v_n(0)$.*

*Proof.* First, we consider the case of the spectrum being continuous. As in [32], we suppose that there exist two functions $p(t)$ and $q(t)$ such that

$$\phi_{n,t}(z) - C_n\phi_{n-1}(z) - \beta(t)D_n\phi_n(z) = p(t)\bar{\phi}_n(z) + q(t)\phi_n(z). \tag{61}$$

When $n \to -\infty$, from eqs (14), (22) and (34) we know $\phi_n(z) \sim z^{-n}$, $\bar{\phi}(z) \sim z^n$, $C_n \sim -2\alpha(t)\beta(t)(n+1)\lambda^{k-1}$, $D_n \sim \alpha(t)n\lambda^k$, thus $p(t) = 0$. Then eq. (61) reduces to

$$-nz_t + 2\alpha(t)\beta(t)(n+1)\lambda^{k-1}z^2 - n\alpha(t)\beta(t)\lambda^k z = q(t)z, \tag{62}$$

we further compare the coefficients of $n$ in eq. (62) and hence obtain

$$z_t = \alpha(t)\beta(t)\lambda^{k-1}(z^2 - 1), \quad q(t) = 2\alpha(t)\beta(t)\lambda^{k-1}z. \tag{63}$$

Therefore, eq. (61) becomes

$$\phi_{n,t}(z) - C_n\phi_{n-1}(z) - \beta(t)D_n\phi_n(z) = 2\alpha(t)\beta(t)\lambda^{k-1}z\phi_n(z). \tag{64}$$

Substituting (36) into eq. (64) and letting $n \to +\infty$, we have

$$a_t(t, z)z^{-n} + b_t(t, z)z^n = 2\alpha(t)\beta(t)\lambda^{k-1}b(t, z)z^n\left(z - \frac{1}{z}\right) \tag{65}$$

from which we have

$$a_t(t, z) = 0, \quad b_t(t, z) = 2\alpha(t)\beta(t)\lambda^{k-1}b(t, z)\left(z - \frac{1}{z}\right), \tag{66}$$

which means

$$a(t, z) = a(0, z), \tag{67}$$

$$b(t,z)=b(0,z)\exp\left\{2\int_0^t\left[\alpha(s)\beta(s)\lambda^{k-1}(s)\left(z(s)-\frac{1}{z(s)}\right)\right]\mathrm{d}s\right\}, \tag{68}$$

and then we reach eq. (57) by using $R(t,z)=(b(t,z)/a(t,z))$.

Secondly, we consider the time evolutions of discrete scattering data. Taking $z=z_j$ in (61), we have the linear relationship

$$\varphi_{n,t}(z_j)-C_n\varphi_{n-1}(z_j)-\beta(t)D_n\varphi_n(z_j)=p(t)\bar{\varphi}_n(z_j)+q(t)\varphi_n(z_j), \tag{69}$$

where $\bar{\varphi}_n(z_j)$ and $\varphi_n(z_j)$ satisfy the asymptotic conditions (33) and (34), from which we have $p(t)=0$ because $\varphi_n(z)\sim z^n$ while $\bar{\varphi}_n(z)\sim z^{-n}$. In this case, eq. (69) reads as

$$\varphi_{n,t}(z_j)-C_n\varphi_{n-1}(z_j)-\beta(t)D_n\varphi_n(z_j)=q(t)\varphi_n(z_j). \tag{70}$$

Let $\theta_n(z_j)=\sqrt{S_n}\varphi_n(z_j)$. Then we have

$$\sqrt{u_{n+1}}\theta_{n+1}(z_j)+\sqrt{u_n}\theta_{n-1}(z_j)+v_n\theta_n(z_j)=\lambda_j\theta_n(z_j) \tag{71}$$

and therefore eq. (70) can be rewritten as

$$\begin{aligned}\theta_{n,t}(z_j)\ &+\ \frac{1}{2}\beta(t)(D_{n+1}-D_n)\theta_n(z_j)-\frac{1}{2}\alpha(t)(\lambda_j-v_n)B_{n+1}\theta_n(z_j)\\ &+\ \alpha(t)\sqrt{u_n}B_{n+1}\theta_{n-1}(z_j)=q(t)\theta_n(z_j),\end{aligned} \tag{72}$$

where

$$S_{n,t}=-\left[\beta(t)D_{n+1}+\beta(t)D_n-\alpha(t)(\lambda_j-v_n)B_{n+1}\right]S_n.$$

Thus, eq. (70) becomes

$$\begin{aligned}\theta_{n,t}(z_j)\ &+\frac{1}{2}\beta(t)(D_{n+1}-D_n)\theta_n(z_j)+\frac{1}{2}\alpha(t)(B_{n+1}-B_n)\sqrt{u_n}\theta_{n-1}(z_j)\\ &+\frac{1}{2}\alpha(t)[B_n\sqrt{u_n}\theta_{n-1}(z_j)-B_{n+1}\sqrt{u_{n+1}}\theta_{n+1}(z_j)]=q(t)\theta_n(z_j).\end{aligned} \tag{73}$$

Further, multiplying eq. (73) by $2\theta_n(z_j)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=-\infty}^{\infty}\theta_n^2(z_j)\ &+\ \sum_{n=-\infty}^{\infty}\left[\beta(t)(D_{n+1}-D_n)\theta_n^2(z_j)\right.\\ &\left.+\ \alpha(t)(B_{n+1}-B_n)\sqrt{u_n}\theta_{n-1}(z_j)\theta_n(z_j)\right]+\sum_{n=-\infty}^{\infty}\alpha(t)(B_n\sqrt{u_n}\theta_{n-1}\theta_n\\ &-B_{n+1}\sqrt{u_{n+1}}\theta_{n+1}\theta_n)=2\sum_{n=-\infty}^{\infty}q(t)\theta_n^2.\end{aligned} \tag{74}$$

As $\theta_n(z_j)$ is a normalization eigenfunction, we have

$$q(t)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\beta(t)(D_{n+1}-D_n)\theta_n^2+\alpha(t)(B_{n+1}-B_n)\sqrt{u_n}\theta_{n-1}\theta_n\right], \tag{75}$$

the inner product of which can be written as

$$q(t) = \frac{1}{2}\left((E-1)\begin{pmatrix}\beta(t)D_n\\ \alpha(t)B_n\end{pmatrix}, \begin{pmatrix}\theta_n^2(z_j)\\ \sqrt{u_n}\theta_{n-1}(z_j)\theta_n(z_j)\end{pmatrix}\right). \tag{76}$$

To compute the inner product (76), we recall the recursion (26) and then obtain the relation

$$(E-1)\begin{pmatrix}d_{n,j+1}\\ b_{n,j+1}\end{pmatrix} = P(E-1)\begin{pmatrix}d_{n,j}\\ b_{n,j}\end{pmatrix}, \tag{77}$$

where

$$q(t) = \begin{pmatrix} v_n & (Eu_nE-u_n)(E-1)^{-1}\\ 1+E^{-1} & (E-1)v_{n-1}(E-1)^{-1}\end{pmatrix}. \tag{78}$$

It is required for us to introduce two skew-symmetric operators

$$T = \begin{pmatrix} u_nE^{-1}-Eu_n & v_n(1-E)\\ (E^{-1}-1)v_n & E^{-1}-E\end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1-E\\ E^{-1}-1 & 0\end{pmatrix}, \tag{79}$$

then we can easily verify that $P = TJ^{-1}$ and

$$T\begin{pmatrix}\theta_n^2(z_j)\\ \sqrt{u_n}\theta_{n-1}(z_j)\theta_n(z_j)\end{pmatrix} = \lambda_j J\begin{pmatrix}\theta_n^2(z_j)\\ \sqrt{u_n}\theta_{n-1}(z_j)\theta_n(z_j)\end{pmatrix}. \tag{80}$$

From eq. (80), we obtain

$$\begin{aligned}
q(t) &= \frac{1}{2}\left((E-1)\begin{pmatrix}\beta(t)D_n\\ \alpha(t)B_n\end{pmatrix}, \begin{pmatrix}\theta_n^2(z_j)\\ \sqrt{u_n}\theta_{n-1}z_j\theta_n(z_j)\end{pmatrix}\right)\\
&= \frac{1}{2}\left((E-1)\sum_{m=0}^{k}\begin{pmatrix}\beta(t)d_{n,m}\\ \alpha(t)b_{n,m}\end{pmatrix}\lambda_j^{k-m}, \begin{pmatrix}\theta_n^2(z_j)\\ \sqrt{u_n}\theta_{n-1}(z_j)\theta_n(z_j)\end{pmatrix}\right)\\
&= \frac{1}{2}\left((E-1)\sum_{m=0}^{k}\begin{pmatrix}\beta(t)d_{n,0}\\ \alpha(t)b_{n,0}\end{pmatrix}\lambda_j^{k}, \begin{pmatrix}\theta_n^2(z_j)\\ \sqrt{u_n}\theta_{n-1}(z_j)\theta_n(z_j)\end{pmatrix}\right)\\
&= \frac{1}{2}(k+1)\left((E-1)\begin{pmatrix}n\beta(t)\\ \alpha(t)\end{pmatrix}\lambda_j^{k}, \begin{pmatrix}\theta_n^2(z_j)\\ \sqrt{u_n}\theta_{n-1}(z_j)\theta_n(z_j)\end{pmatrix}\right)\\
&= \frac{1}{2}\alpha(t)\beta(t)(k+1)\lambda_j^k.
\end{aligned}$$

We note that $\theta_n(z_j) \to c_j(t)z_j^n$ when $n \to +\infty$, then eq. (72) reduces to

$$c_{j,t}(t) + nc_j(t)z_{j,t}\frac{1}{z_j} + \frac{1}{2}\alpha(t)\beta(t)\lambda_j^k c_j(t) + n\alpha(t)\beta(t)\lambda_j^{k-1}c_j(t)$$

$$-\alpha(t)\beta(t)(n+1)\lambda_j^{k-1}c_j(t)z_j = \frac{1}{2}\alpha(t)\beta(t)(k+1)\lambda_j^k c_j(t), \tag{81}$$

from which we have

$$z_{j,t} = \alpha(t)\beta(t)\lambda_j^{k-1}(z_j^2 - 1), \tag{82}$$

$$c_{j,t}(t) = \frac{1}{2}\alpha(t)\beta(t)\lambda_j^k c_j(t)\left(k + 2 - \frac{4}{z_j^2 + 1}\right). \tag{83}$$

□

Therefore, we complete the proof of Theorem 2.

In what follows, we shall derive the reflectionless potentials $u_n(t)$ and $v_n(t)$. In the case, the reflection coefficient $R(t, z) = 0$.

We take

$$F_m = \sum_{j=1}^{l} c_j^2(t) z_j^m, \tag{84}$$

$$K_{n,m}(t) = \sum_{j=1}^{l} c_j(t) z_j^m g_{n,j}(t). \tag{85}$$

Then the GLM eq. (54) becomes

$$g_{n,j}(t)+c_j(t)z_j^n+\sum_{k=1}^{l} c_j(t)c_k(t)\frac{z_j^{n+1}z_k^{n+1}}{1 - z_j z_k} g_{n,k}(t) = 0, \quad j = 1, 2, \ldots, l. \tag{86}$$

If we take

$$(D_n(t))_{j,k} = c_j(t)c_k(t)\frac{z_j^{n+1}z_k^{n+1}}{1 - z_j z_k}, \tag{87}$$

$$g_n(t) = (g_{n,1}(t), g_{n,1}(t), \ldots, g_{n,l}(t))^T, \tag{88}$$

$$h_n(t) = (c_1(t)z_1^n, c_2(t)z_2^n, \ldots, c_l(t)z_l^n)^T, \tag{89}$$

then eq. (86) can be rewritten as

$$(I + D_n(t))g_n(t) = -h_n(t). \tag{90}$$

Using the known result [32], we have the relationship that

$$K_{n,m}(t) = -\mathrm{tr}\left[(I + D_n(t))^{-1}h_n(t)h_m(t)^T\right], \quad m > n \tag{91}$$

and eq. (55) in the case $m = n$ can be written as

$$K_{n,n}^{-1}(t) = 1 + \sum_{j=1}^{l} c_j^2(t)z_j^{2n} + \sum_{j,k=1}^{l} c_j^2(t)c_k(t)\frac{z_j^{n+1}z_k^{n+1}}{1 - z_j z_k} z_k^n g_{n,k}(t), \tag{92}$$

$$K_{n,n}(t) = \frac{\det(I + D_n(t))}{\det(I + D_{n-1}(t))}, \tag{93}$$

from (53) we finally obtain the exact solutions of the variable-coefficient nonisospectral Toda lattice hierarchy (1):

$$u_n = \frac{\det(I + D_n(t))\det(I + D_{n-2}(t))}{\det(I + D_{n-1}(t))^2}, \tag{94}$$

$$\begin{aligned} v_n = & -\frac{\operatorname{tr}\left[(I + D_n(t))^{-1}h_n(t)h_{n+1}(t)^T\right]\det(I + D_{n-1}(t))}{\det(I + D_n(t))} \\ & +\frac{\operatorname{tr}\left[(I + D_{n-1}(t))^{-1}h_{n-1}(t)h_n(t)^T\right]\det(I + D_{n-2}(t))}{\det(I + D_{n-1}(t))}. \end{aligned} \tag{95}$$

In particular, when $l = 1$, solutions (94) and (95) become

$$u_n = \frac{(1 - z_1^2 + c_1^2(t)z_1^{2n+2})(1 - z_1^2 + c_1^2(t)z_1^{2n-2})}{(1 - z_1^2 + c_1^2(t)z_1^{2n})^2}, \tag{96}$$

$$v_n = \frac{c_1^2(t)z_1^{2n-1}(1 - z_1^2 + c_1^2(t)z_1^{2n-2})}{(1 - z_1^2 + c_1^2(t)z_1^{2n})^2} - \frac{c_1^2(t)z_1^{2n+1}(1 - z_1^2 + c_1^2(t)z_1^{2n})}{(1 - z_1^2 + c_1^2(t)z_1^{2n+2})^2}. \tag{97}$$

For the variable-coefficient nonisospectral Toda lattice eqs (6) and (7), we can determine the single-wave solutions $u_n$ and $v_n$ by replacing $c_1^2(t)$ and $z_1(t)$ in eqs (96) and (97) with

$$c_1^2(t) = c_1^2(0)\exp\left\{\int_0^t \left[\alpha(s)\beta(s)\left(2 - \frac{4}{1 + z_1^2(s)}\right)\right]\mathrm{d}s\right\},$$

$$z_1(t) = \frac{1}{2}\omega\exp\left[\int_0^t \alpha(s)\beta(s)\mathrm{d}s\right] \pm \frac{1}{2}\left\{4 + \omega^2\exp\left[2\int_0^t \alpha(s)\beta(s)\mathrm{d}s\right]\right\}^{1/2},$$

where $\omega = z_1(0) - (1/z_1(0))$, the sign $\pm$ depends on $z_1(0)$, to be more precise, if $z_1(0) > 0$, then it is taken $+$, otherwise $z_1(t)$ takes '$-$'.

Similarly, on substituting

$$c_1^2(t) = c_1^2(0)\exp\left\{\int_0^t \left[\alpha(s)\beta(s)\left(z_1(s) + \frac{1}{z_1(s)}\right)\left(3 - \frac{4}{1 + z_1^2(s)}\right)\right]\mathrm{d}s\right\},$$

$$z_1(t) = \frac{z_1(0) + 1 + [z_1(0) - 1]\exp\left[2\int_0^t \alpha(s)\beta(s)\mathrm{d}s\right]}{z_1(0) + 1 - [z_1(0) - 1]\exp\left[2\int_0^t \alpha(s)\beta(s)\mathrm{d}s\right]},$$

into eqs (96) and (97), we can finally determine the single-wave solutions $u_n$ and $v_n$ of eqs (8) and (9).

## 4. Discussions and conclusions

In summary, we have derived a new variable-coefficient nonisospectral Toda hierarchy (1) from the spectral problem (11) by the discrete zero curvature equations. In order to obtain exact solutions of the derived hierarchy, we utilized the IST. As a result, new and more general exact solutions were obtained. To the best of our knowledge, the IST has not been extended to such a whole hierarchy of nonlinear lattice equations with variable coefficients and the obtained solutions have not been reported in literature. This paper shows that the IST can be generalized for solving the whole hierarchy of nonisospectral nonlinear differential-difference equations with variable coefficients. The application of the IST to other variable-coefficient hierarchies of nonlinear lattice equations is worth studying which will be the topic for our future study.

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## References

[1] Y Liu, Y T Gao, Z Y Sun and X Yu, *Nonlinear Dyn.* **66**, 575 (2011)
[2] M Bruschi, D Levi and O Ragnisco, *Nuovo Cimento A* **53**, 21 (1979)
[3] V Konotov, O Chubaikalo and L Vazquez, *Phys. Rev. E* **48**, 563 (1993)
[4] V N Serkin and A Hasegawa, *Phys. Rev. Lett.* **85**, 015601 (2000)
[5] A Kundu, *J. Math. Phys.* **44**, 4589 (2003)
[6] A Kundu, *Phys. Rev. Lett.* **99**, 154101 (2007)
[7] A Kundu, *Phys. Rev. E* **79**, 4502 (2009)
[8] F J Yu, *Acta Math. Sci.* **35**, 974 (2012)
[9] F J Yu and H Q Zhang, *Chaos, Solitons and Fractals* **32**, 1053 (2007)
[10] N C Lee, H F Shen and M H Tu, *J. Math. Phys.* **51**, 122704 (2012)
[11] T K Ning, C F Wu and D J Zhang, *Physica A* **377**, 421 (2007)
[12] C S Gardner, J M Greene, M D Kruskal and R M Miura, *Phys. Rev. Lett.* **19**, 1095 (1967)
[13] R Hirota, *Phys. Rev. Lett.* **27**, 1192 (1971)
[14] M R Miurs, *Bäcklund transformation* (Springer, Berlin, 1978)
[15] J Weiss, M Tabor and G Carnevale, *J. Math. Phys.* **24**, 522 (1983)
[16] M L Wang, *Phys. Lett. A* **213**, 279 (1996)
[17] E G Fan, *Phys. Lett. A* **300**, 243 (2002)
[18] E G Fan, *Chaos, Solitons and Fractals* **16**, 819 (2003)
[19] J H He and X H Wu, *Chaos*, *Solitons and Fractals* **30**, 700 (2006)
[20] S Zhang and T C Xia, *J. Phys. A: Math. Theor.* **40**, 227 (2007)
[21] S Zhang, J Wang, A X Peng and B Cai, *Pramana – J. Phys.* **71**, 763 (2013)

[22] C Q Dai, X G Wang and G Q Zhou, *Phys. Rev. A* **89**, 013834 (2014)
[23] P D Lax, *Commun. Pure Appl. Math.* **21**, 467 (1968)
[24] V E Zakharov and A B Shabat, *Sov. Phys. JETP* **34**, 62 (1972)
[25] M Wadati, *J. Phys. Soc. Jpn.* **32**, 1289 (1973)
[26] M J Ablowitz, D J Kaup, A C Newell and H Segur, *Phys. Rev. Lett.* **30**, 1262 (1973)
[27] C S Garder, J M Greene, M D Kruskal and R M Miura, *Commun. Pure Appl. Math.* **27**, 97 (1974)
[28] H Flaschka, *Prog. Theor. Phys.* **51**, 703 (1974)
[29] W L Chan and K S Li, *J. Math. Phys.* **30**, 2521 (1989)
[30] B Z Xu and S Q Zhao, *Appl. Math. Ser. B* **9**, 331 (1994)
[31] Y B Zeng, W X Ma and R L Lin, *J. Math. Phys.* **41**, 5453 (2000)
[32] D Y Chen, *Introduction to soliton* (Science Press, Beijing, 2006)
[33] L D Faddeev and L A Takhtajan, *Hamiltonian methods in the theory of solitons* (Springer, Berlin, 1987)