

Towards a characterization of fields leading to black hole hair

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Abstract. In the present work, it is shown that an asymptotically flat spherical black hole can have a nontrivial signature of any field, for an exterior observer, if the energy–momentum tensor of the corresponding field is either trace-free or if the trace falls off at least as rapidly as the inverse cube of the radial distance. In the absence of a general ‘No Hair Theorem’, this result can provide a characterization of the fields leading to a black hole hair.

Keywords. Black hole; No Hair Theorem.

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1. Introduction

Black holes are easily amongst the most fascinating offshoots of General Theory of Relativity. One important question asked about a black hole is regarding the information one can extract from the exterior gravitational field of such an object. The answer is normally given in terms of a ‘No Hair Theorem’. The so-called theorem says that no information regarding a black hole can be obtained by an exterior observer except for that of the mass (M), electric charge (Q) and the angular momentum (h). This was summarized by Ruffini and Wheeler [1]. This idea was inspired by the uniqueness of Schwarzschild and Reissner–Nordstrom solutions as shown by Israel [2] and the uniqueness of Kerr black hole by Wald [3] and Carter [4]. Although the statement proved to be extremely powerful, it should perhaps be called a ‘No Hair Conjecture’, rather than a theorem as there is hardly any rigorous proof for the same. Attempts are normally made to find examples to either support or to oppose this conjecture.

There have been excellent attempts towards finding a proof to the ‘No Hair Theorem’ but they normally include only one particular field. For example, recently Bhattacharya and Lahiri [5] proved the theorem for an axially symmetric black hole in case of a scalar

field or a massive vector field. A proof for a No Hair Theorem for a spherical black hole regarding Higgs model is also available in [6].

The search for the possibility of information regarding a particular field, i.e., a possible ‘hair’, started way back in the early seventies [7]. The quest gave rise to a classification of black hole hair into two categories, namely a primary hair and a secondary hair [8]. A primary hair is one which is independent of the existence of any other hair. A secondary hair, on the other hand, depends on the existence of other fields and grows on them. The electric field, for example, is a primary hair. The recently discovered dilaton hair [9] in fact grows on the electric charge and ceases to exist if the electric field is switched off. This is an example of a secondary hair.

Indeed there are examples of black hole solutions with a hair other than M , Q or h , which contradict the No Hair Conjecture. But all these counterexamples have some pathological or unwanted features, particularly if the example is that of a primary hair. Most of the black hole solutions with such a hair are unstable [10]. The most talked about counterexample is the existence of a scalar hair for a conformally invariant scalar field, nonminimally coupled to gravity, given by Bekenstein [11]. In this example, the effective Newtonian constant of gravity (G) may become negative! However, for a nonminimally coupled scalar–tensor theory, the possibility of a negative G may not be ruled out. Some axisymmetric black hole solutions endowed with a scalar hair can be seen in [12]. These black holes are, however, not asymptotically flat. There are also examples of the possibility of some hair for tiny black holes (i.e., not of the size of a stellar black hole) as discussed by Weinberg [13].

Anyway, only these kinds of examples are there in the literature regarding the existence of a black hole hair. Except for the attempts with specific fields [5,6], there is hardly any proof of the so-called theorem or even a definite way of characterizing the matter fields which may lead to a hair.

In the present work, an attempt is made to characterize matter fields which might be detected by an exterior observer. It is shown that for a particular hair to exist for an asymptotically flat spherical black hole, the energy–momentum tensor for the corresponding field must either be trace-free or the trace should fall off with the proper radius r at least as fast as $1/r^3$. The scope of this result is a bit limited, as it is achieved only for a spherical black hole, but it definitely gives some clear indications in the absence of a more rigorous theorem.

2. The theorem

A static spherically symmetric line element has the form

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where ν and λ are functions of r alone. This so-called curvature form of the metric has the advantage that the radial coordinate r has the significance of the proper radial distance.

For this metric, Einstein field equations,

$$G^\alpha_\beta = R^\alpha_\beta - \frac{1}{2}R\delta^\alpha_\beta = -8\pi GT^\alpha_\beta \quad (2)$$

are written as

$$e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = -8\pi GT_0^0, \quad (3)$$

$$e^{-\lambda} \left(\frac{v'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = -8\pi GT_1^1 \quad (4)$$

and

$$\frac{1}{2} e^{-\lambda} \left(v'' + \frac{1}{2} v'^2 + \frac{v' - \lambda'}{r} - \frac{1}{2} v' \mu' \right) = -8\pi GT_2^2 = -8\pi GT_3^3, \quad (5)$$

where the prime indicates differentiation with respect to r .

The existence of an event horizon characterizes a black hole. The event horizon is a null surface where $g^{11} = 0$ (for a diagonal metric, this would also mean g_{11} is infinity). However, the event horizon should be a regular surface which does not have any singularity on the surface. So this apparent singularity of the metric should be an artifact of choice of coordinates and the physical quantities should be well behaved. For example, the curvature should be finite, the proper volume (given by $\sqrt{-g}$) should also be finite and nonzero.

With a generalized coordinate condition, given by Duan *et al* [14], the metric can in fact be written in spherical polar coordinates in such a way that

$$g_{00}g_{11} = - \left(\frac{dF}{dr} \right)^2, \quad (6)$$

where F is the proper radius. As r is the proper radius in the present form of the metric, this condition yields $e^{\nu+\lambda} = 1$. This also ensures that $\sqrt{-g}$ is nonzero on the horizon. Schwarzschild solution indeed has this property. This condition is actually satisfied in many a situation. For instance, (i) if one assumes a null energy distribution (i.e. $\rho + p = 0$), the field eqs (3) and (4) would lead to the condition $g_{00}g_{11} = -1$; (ii) if one assumes that a radially moving photon moves with a constant speed ($dr/d\lambda = \text{constant}$, where λ is a scalar parameter of the geodesic), the geodesic equation for the coordinate x^0 and the condition for a null particle, $ds^2 = 0$, will combine to give the result $g_{00}g_{11} = -1$. In what follows, the coordinate condition $g_{00}g_{11} = -1$ will be assumed.

A contraction of eq. (2) yields

$$R = R_\alpha^\alpha = 8\pi GT_\alpha^\alpha = 8\pi GT. \quad (7)$$

From the expressions for G_β^α given in the left-hand side of field eqs (3)–(5), one can write the Ricci scalar R as

$$R = \frac{(e^{-\lambda} r^2)''}{r^2} - \frac{2}{r^2}, \quad (8)$$

where the condition $e^{\nu+\lambda} = 1$ has been used. This equation, on integration, yields

$$g_{00} = e^{-\lambda} = 1 + \frac{C_1}{r} + \frac{C_2}{r^2} + \frac{1}{r^2} \int \left(\int R r^2 dr \right) dr, \quad (9)$$

C_1, C_2 being constants of integration. An event horizon is given by a null surface which requires

$$-g^{11} = e^{-\lambda} = 1 + \frac{C_1}{r} + \frac{C_2}{r^2} + \frac{1}{r^2} \int \left(\int Rr^2 dr \right) dr = 0. \quad (10)$$

The real solutions (of r) for this equation will locate the site of event horizons. The number of possible horizons will depend on the degree of the algebraic eq. (10) in r .

If the space-time is asymptotically flat, one has $g_{00} \sim 1$ when r goes to ∞ . So the last term on the right-hand side of eq. (9) should either be zero or be such that it goes to zero as r approaches ∞ . This feature is achieved if R falls off as $1/r^3$ or faster, for large values of r .

To facilitate this idea, we now assume that R can be written as a series expansion of r as

$$R = \sum_i a_i r^i + \sum_j b_j r^{-j}, \quad (11)$$

where a_i, b_j are constants, i runs from 0 to m and j runs from 0 to n . It is easy to see that if asymptotic flatness is invoked, i.e., $g_{00} \sim 1$ for r approaching ∞ , one would require to have only inverse powers of r in the expression for g_{00} , which in turn would require that all a_i s are zero. So the Taylor series part does not contribute. From the Laurent series part, one can now evaluate the integral in eq. (9) as

$$\int \left(\int Rr^2 dr \right) dr = \sum_j \alpha_j r^{4-j}, \quad (12)$$

where α_j s are constants. It is now evident that the last term of eq. (10) will satisfy the requirement if $j > 3$. So, $\alpha_j = 0$ for $j < 3$. However, if R is identically zero, asymptotic flatness is already ensured. So now a theorem can be stated as:

If an asymptotically flat spherical black hole solution has to be endowed with a hair (i.e. information) for a particular field, the space-time is either Ricci-flat or the Ricci curvature falls off at least as rapidly as $1/r^3$.

As the Ricci curvature and the trace of the energy-momentum tensor are related as $R = 8\pi GT$, the theorem can be stated in terms of the matter distribution as:

If an asymptotically flat spherical black hole has a hair corresponding to a particular field, then the trace of the energy-momentum tensor is either identically zero or falls off at least as fast as $1/r^3$.

The theorem tells us about the necessary condition for the existence of a hair, but does not ensure anything about the sufficiency condition. It also deserves mention that the standard hair (allowed by the No Hair Conjecture) of mass and the electric charge of a spherical black hole can easily be related to the constants C_1 and C_2 respectively.

Very recently, Faraoni and Sotiriou [15,16] showed the nonexistence of a scalar hair in a nonminimally coupled scalar tensor theory for an axially symmetric, asymptotically flat charged black hole. It is interesting to note that their proof depends on the trace-free property of the energy-momentum tensor of the associated electromagnetic field. The present work, on the other hand, deals with the trace of the field for which the hair is sought rather than that of the other associated fields.

3. Examples

There are two most talked about counterexamples of the ‘No Hair Conjecture’. The first one is the scalar hair discovered way back in 1974 by Bekenstein [11] for a nonminimally coupled, conformally invariant scalar field. In the action, the Ricci scalar R is coupled to the scalar field ϕ as $(1 - (\phi^2/6))R$. Einstein field equations are given by

$$\left(1 - \frac{\phi^2}{6}\right) R_{\nu}^{\mu} = u_{;\alpha}^{\alpha} \delta_{\nu}^{\mu} - 4u_{;\nu}^{;\mu} + 2uu_{;\nu}^{;\mu}, \quad (13)$$

where $u = (1 - (\phi^2/6))$. The wave equation for the scalar field is

$$u_{;\alpha}^{\alpha} = 0. \quad (14)$$

These equations evidently show that the Ricci scalar R and hence the trace of the energy-momentum tensor T are zero (see [17]), directly verifying the result obtained in the present work.

The second serious counterexample is that of the dilaton hair [9]. The relevant metric looks like

$$ds^2 = \frac{1 - (2Me^{\phi_0})/r}{1 - (Q^2e^{3\phi_0})/(Mr)} dt^2 - [(1 - 2Me^{\phi_0}/r)(1 - Q^2e^{3\phi_0})/(Mr)]^{-1} dr^2 - r^2 d\Omega^2. \quad (15)$$

Here M , Q and ϕ_0 are the mass, the electric charge and the scalar charge respectively. This scalar charge is visible for an exterior observer. If one now calculates the Ricci scalar, it is seen that the asymptotic behaviour is dominated by $1/r^3$. This is again consistent with the theorem discussed in the present work. The counterexamples of the ‘No Hair Conjecture’ are therefore found to be consistent with the theorem developed here.

It also deserves mention that the conclusions regarding a black hole hair survive a conformal transformation for the metric [18] provided there is no singularity in the transformation. So, the conclusions derived here in the string frame will be valid for a conformally transformed frame as well, provided the field does not diverge anywhere (see also [16]).

It should be good to check how the examples favouring the ‘No Hair Conjecture’ behave *vis-à-vis* the present theorem. We pick up one example, namely, Penney’s well-known solution [19] for a scalar field distribution along with an electromagnetic field. The solution is given by

$$ds^2 = e^{\gamma} dt^2 - e^{\alpha} dr^2 - e^{\beta} d\Omega^2, \quad (16)$$

where $\alpha + \gamma = 0$. The metric functions are given as

$$e^{\alpha} = \left(r^2 - 2mr + \frac{K\epsilon^2}{2A^2}\right)^{-A} \left(\frac{b(r-a)^A - a(r-b)^A}{b-a}\right)^2$$

and

$$e^{\beta} = \left(r^2 - 2mr + \frac{K\epsilon^2}{2A^2}\right) e^{\alpha}.$$

The constants are related by $2A^2ab = K\epsilon^2$, $a + b = m$ and $A^2Kc^2 = (1 - A^2)(2A^2m^2 - K\epsilon^2)$. Here m and ϵ are the mass and charge of the distribution and c is the scalar charge, which is zero if $A = 1$. If one demands a nontrivial scalar hair for an exterior observer out of this solution, the so-called horizon becomes singular and one does not have a black hole. Thus, the solution does not yield a nontrivial scalar hair. It can be shown that the trace of the energy–momentum tensor in fact falls off as $1/\rho^2$ where ρ is the proper radial distance. So T falls off slower than $1/\rho^3$, the minimum rate required by the theorem, and thus does not give rise to a nontrivial hair. It should be noted that the radial coordinate in Penney’s work is not the proper radius, and the asymptotic behaviour of T or R is carefully examined against the proper radius ρ , given by $\rho = e^\beta$, and not against the radial coordinate r .

Recently, Martinez and Troncoso [20] reported the existence of a scalar hair for a minimally coupled scalar field endowed with a potential $V(\phi) = \cosh^4 \alpha\phi$, where α is a constant and ϕ is the scalar field. But the solution is not asymptotically flat and hence does not come in the purview of the present theorem. Incidentally, the solution is asymptotically anti-de Sitter.

4. Discussion

In this paper, a theorem for characterizing the fields leading to a hair for an asymptotically flat black hole has been proved. The term ‘hair’ indicates a nontrivial information regarding the corresponding field for an exterior observer. The characterization is restricted as the theorem is proved only for a spherical black hole. An axially symmetric black hole, however, would add only the information regarding the angular momentum of the black hole, which does not add to the trace of the energy–momentum tensor. So the theorem is at least intuitively correct for axially symmetric black holes as well. The two most talked about counterexamples of the ‘No Hair Conjecture’ are absolutely compatible with the present theorem. The Bekenstein black hole [11] provides an example of a primary hair, while the dilaton black hole gives an example of a secondary hair which grows on mass and charge of the black hole. So both classes of black hole hair are included in the purview of the theorem. The theorem in fact proves a necessary condition and not a sufficient condition on the matter distribution for the existence of a hair. It is interesting to note that common examples (e.g. Penney’s solution [19]) in favour of the ‘No Hair Conjecture’ as well as the counterexamples are both completely consistent with the theorem discussed in the present work.

Another point that deserves mention is that the theorem that has been proved, does not discuss any particular field like a scalar field or a vector field and is fairly general. Although the proof is based on fields which are minimally coupled to gravity, this in principle takes care of the nonminimally coupled theories as well. This is because by virtue of a conformal transformation, one can reduce such theories into a minimally coupled one at least formally, and it has been shown that the conclusion regarding the existence of a black hole hair is independent of this choice of frame [18].

The proof of the theorem indeed depends on the choice of coordinate condition leading to $g_{00}g_{11} = -1$. But this is perhaps not too severe a restriction as discussed in §2. What is more important is that all sorts of examples appear to be consistent with the result proved in this work.

The present work depends crucially on the asymptotic flatness. Asymptotically nonflat solutions may also be looked at. One such example is already there [20]. The other direction of investigation will certainly be to include nonspherical black holes. As already mentioned, some work has started in that direction too [12,15].

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