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Painlevé analysis and some solutions of variable coefficient Benny equation

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Abstract. In this paper, variable coefficient Benny equation (also called the KdV Burgers–Kuramoto equation) has been considered. By using the Painlevé analysis and Lie group analysis methods, the Painlevé properties and symmetries have been studied. Some solutions of the reduced ODEs are obtained.

Keywords. Exact travelling wave solutions; nonlinear physical models.

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1. Introduction

Nonlinear equations arise in different fields of science and finding solutions of nonlinear equations has always been one of the important tasks in mathematics. Now, several researchers have shown considerable interest in studying variable coefficient nonlinear equations. Various methods like inverse scattering transformation (IST) [1,2], Darboux and Bäcklund transformation [3,4], Hirota bilinear method [5] are no longer helpful in finding the solutions of these variable coefficient nonlinear equations. Sophus Lie developed a highly algorithmic method known as the Lie point symmetry method (group method) [6–10]. Using the Lie symmetry method [11], we can study the invariance, symmetry properties and similarity reductions of nonlinear equations. Also, Weiss, Tabor and Carnevale (WTC) [12–14] presented the Painlevé test for nonlinear evolution equations (NLEEs). According to WTC method, a NLEE has Painlevé property if its solutions are single-valued about a movable singularity manifold. Till date, the developments on this method include the Kruskal's simplified method [15] and Conte's invariant method [16]. To a NLEE, regardless of whether it possesses the Painlevé property or not, certain physically interesting solutions can be derived by using the truncated Painlevé expansion. The Benny equation (also called the KdV Burger–Kuramoto equation) [17,18]

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \delta u_{xxxx} = 0, \tag{1}$$

describes the physical processes in the motion of turbulence and other unstable systems. Depending on α, β and δ different forms of Benny equations are known to exist in literature.

- $\beta = \delta = 0$ [19]
- $\alpha = \delta = 0$ [20]
- $\delta = 0$ [21,22]
- $\beta = 0$ [23].

In this paper, we study the variable coefficient version of the Benny equation

$$u_t + uu_x + \alpha(t)u_{xx} + \beta(t)u_{xxx} + \delta(t)u_{xxxx} = 0, \tag{2}$$

for the integrability and solutions using Lie's classical method. This paper is organized as follows. Section 2 is devoted to the Painlevé analysis. Section 3 contains the outline of Lie's classical method to generate various symmetries of the Benny equation and basic vector fields are identified. Section 4 contains solutions of the reduced ordinary differential equation (ODE). Some conclusions are given in §5.

2. Painlevé analysis

A partial differential equation (PDE) which possesses the Painlevé property must be single valued about the movable singularity manifold and the singularity manifold is noncharacteristic. Applying the Painlevé PDE test, one can assume that the solutions of eq. (2) can be expressed in the form of Laurent expansion given as

$$u(x,t) = g(x,t)^{\gamma} \sum_{i=0}^{\infty} u_i g^i(x,t), \tag{3}$$

where $u_i = u_i(x,t)$ and $g = g(x,t)$ are analytic functions in the neighbourhood of $g = 0$. In this case the leading order is $\gamma = -3$ and

$$u_0 = 120\delta(t)g_x^3, \tag{4}$$

where g_x denotes the partial differentiation of $g(x,t)$ with respect to x. For determining the resonances we substitute the Laurent series (3) into eq. (2) and by equating the coefficients of like terms, the polynomial equation is derived as

$$(i-6)\,(i+1)\left(i^2 - 13i + 60\right)\delta(t)g^{(i-3)}u_i = F(u_{i-1}, ..., u_0, g_x, g_{xx}...). \tag{5}$$

Using eq. (5) the resonances are found to be $i = -1, 6$. The arbitrariness of the function $g(x,t) = 0$ is given at $i = -1$ and compatibility conditions occur at $i = 6$ and also in relations between the arbitrary variable coefficient of eq. (2).

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)\right)\delta(t) + (\alpha(t))^3 = 0,$$

$$\beta(t) = 4\sqrt{\alpha(t)\,\delta(t)}. \tag{6}$$

Hence, the equation passes the Painlevé test by using the variable coefficient relation.

Remark 1. Sometimes we obtain the integrable properties like the Bäcklund transformation (BT), Lax pair (LT) [24] etc., by using the Painlevé test. But Bäcklund transformation (BT) is not discussed in this paper.

3. Classical symmetry analysis

In this section, we explain the similarity reduction and exact solutions of the variable coefficient Benny equation. According to the Lie's classical method [25], consider the one-parameter Lie group of transformation in (x, t, u) as

$$\begin{aligned} x^* &= x + \epsilon\xi(x, t, u), \\ t^* &= t + \epsilon\tau(x, t, u), \\ u^* &= u + \epsilon\eta(x, t, u), \end{aligned} \tag{7}$$

where ϵ is the group parameter, which leaves the system invariant. This yields the overdetermined, linear system of equations for the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\eta(x, t, u)$. Assuming that eq. (2) is invariant under the transformation (7), we obtain the following relations from the coefficient of the first order of ϵ:

$$\begin{aligned} &\eta^t + \eta^x u + \eta u_x + \alpha(t)\eta^{xx} + \tau\alpha' u_{xx} + \beta(t)\eta^{xxx} + \tau\beta(t)' u_{xxx} + \delta(t)\eta^{xxxx} \\ &\quad + \tau\delta(t)' u_{xxxx} = 0, \end{aligned} \tag{8}$$

where $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ and η^{xxxx} are extended (prolonged) infinitesimals acting on an enlarged space (jet space) that includes all derivatives of the dependent variables u_t, u_x, u_{xx} and u_{xxt} (for more details refer to [7]). The infinitesimals are determined from the invariance condition by setting the coefficients of different differentials equal to zero. We obtain a large number of PDEs in ξ, τ and η that need to be satisfied. The general solution of this large system provides the following forms for the infinitesimal elements ξ, τ, η and admissible forms of various coefficients in eq. (2).

$$\begin{aligned} \xi &= k_1 xt + k_2 t + k_3 x + k_4, \\ \tau &= k_1 t^2 + (k_3 - k_5)t + k_6, \\ \eta &= k_1(x - ut) + uk_5 + k_2, \end{aligned} \tag{9}$$

where $k_i, i = 1, 2, \ldots, 6$ are arbitrary constants. The functions $\alpha(t)$, $\beta(t)$ and $\delta(t)$ are governed by the conditions

$$\begin{aligned} &\tau\delta(t)' - 4\delta(t)\xi_x + \tau_t\delta(t) = 0, \\ &\tau\beta(t)' + \tau_t\beta(t) - 3\beta(t)\xi_x + 4\delta(t)\eta_{xu} - 6\delta(t)\xi_{xx} = 0, \\ &\tau\alpha(t)' - 2\alpha(t)\xi_x + \tau_t\alpha(t) + 6\delta t\eta_{xxu} - 3\beta(t)\xi_{xx} + 3\beta(t)\eta_{xu} - 4\delta t\xi_{xxx} = 0. \end{aligned} \tag{10}$$

The infinitesimal generators of the corresponding Lie algebra are given by

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ V_4 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\ V_5 &= -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \\ V_6 &= t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + (x - ut)\frac{\partial}{\partial u}. \end{aligned} \tag{11}$$

In general, infinite number of subalgebras of this Lie algebra are formed from any linear combination of generators V_j; $j = 1, \ldots, 6$ and to each subalgebra one can derive the reduction using the characteristic equations.

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}u}{\eta}. \tag{12}$$

The similarity variables and forms can be obtained by solving the characteristic eq. (12). The general solution of these equations involves two constants, one becomes a new independent variable ξ and the other plays the role of a new dependent variable. On substituting the solution of (12) in eq. (2), we get the reduced ODE.

(i) For the generator $V_1 + \lambda_1 V_2$, we have the following similarity variables: $\xi = x - \lambda t$, $u = f(\xi)$ and the group invariant solution is

$$u = f\,(x - \lambda t)\,. \tag{13}$$

By substituting (13) in eq. (2), the equation to the following ODE can be reduced to

$$c_1 f'''' + c_2 f''' + c_3 f'' + f' f - \lambda f' = 0. \tag{14}$$

On integrating, we get

$$c_1 f''' + c_2 f'' + c_3 f' + \frac{f^2}{2} - \lambda f + k_1 = 0, \tag{15}$$

where $f' = \mathrm{d}f/\mathrm{d}\xi$ and $k_i, c_i, i = 1, 2, 3...$ are constants.
(ii) For the generator V_6, we get $\xi = x/t$ and $u = (x/t) + (f\,(\xi)/t)$, the group invariant solution is

$$u = \xi + \frac{f\,(\xi)}{t}. \tag{16}$$

By substituting (16) in eq. (2), the equation to the following ODE is reduced to

$$c_4 f'''' + c_5 f''' + c_6 f'' + f' f = 0, \tag{17}$$

on integrating with respect to ξ gives

$$c_4 f''' + c_5 f'' + c_6 f' + \frac{f^2}{2} + k_2 = 0. \tag{18}$$

(iii) Also, for the generator V_3, it can be seen that this equation has a trivial solution $u = c$, where c is an arbitrary constant.

4. Analysis of ODE

First, the modified (G'/G)-expansion method is applied [26–29] for eq. (15) by taking integration constant as zero. Assume that the solution of eq. (15) can be expressed by a polynomial in (G'/G) as follows:

$$f(\xi) = a_0 + \sum_{i}^{n} \left\{ a_i \left(\frac{G'}{G}\right)^{i} + b_i \left(\frac{G'}{G}\right)^{-i} \right\}, \tag{19}$$

where a_0, a_i and b_i are constants and the positive integer n can be determined by considering the homogeneous balance of the highest-order derivatives and highest-order nonlinear appearing in ODEs (15). The function $G(\xi)$ is the solution of the auxiliary linear ODE given as

$$G''(\xi) + \mu G(\xi) = 0, \tag{20}$$

where μ is a constant to be determined. Depending upon μ different types of solutions are obtained as follows:

(1) If $\mu \prec 0$, we find the hyperbolic-type solutions, and we have

$$\frac{G'}{G} = \sqrt{-\mu}\left(\frac{A\sinh\sqrt{-\mu}\xi + B\cosh\sqrt{-\mu}\xi}{A\cosh\sqrt{-\mu}\xi + B\sinh\sqrt{-\mu}\xi}\right). \tag{21}$$

(2) If $\mu \succ 0$, we find the trigonometric-type solutions

$$\frac{G'}{G} = \sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}\xi - B\cos\sqrt{\mu}\xi}{A\sin\sqrt{\mu}\xi + B\cos\sqrt{\mu}\xi}\right), \tag{22}$$

where A, B are arbitrary constants. We get $n = 3$ by balancing the highest-order derivatives and nonlinear terms. Using eq. (19) it follows that

$$\begin{aligned} f(\xi) = {} & a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2 + a_3\left(\frac{G'}{G}\right)^3 + b_1\left(\frac{G'}{G}\right)^{-1} \\ & + b_2\left(\frac{G'}{G}\right)^{-2} + b_3\left(\frac{G'}{G}\right)^{-3}. \end{aligned} \tag{23}$$

On substituting eq. (23) in the ODE eq. (15), collecting all the terms with the same powers of (G'/G) together and equating their coefficients to zero yields a system of algebraic equations for $a_0, a_1, a_2, a_3, b_1, b_2, b_3$ and μ which are solved to get the following results:

Case 1

$$a_0 = 0, \quad b_1 = 0, \quad b_3 = 0, \quad a_2 = -15c_2, \quad a_3 = 120c_1,$$
$$b_2 = -12c_2\mu^2, \quad a_1 = \frac{15}{76}\,\frac{608\,c_1^2\mu - c_2^2 + 16\,c_3c_1}{c_1}, \quad \mu = \frac{\lambda}{8c_2}. \tag{24}$$

Case 2

$$a_0 = 0, \quad b_2 = 0, \quad a_2 = -15c_2, \quad a_3 = 120c_1, \quad b_3 = -120c_1\mu^3,$$
$$b_1 = -\frac{60}{19}\,\mu(c_3 + 38\,c_1\mu), \quad a_1 = \frac{15}{76}\,\frac{608\,c_1^2\mu - c_2^2 + 16\,c_3c_1}{c_1},$$
$$\mu = \frac{1}{6080}\,\frac{56\,c_2c_3c_1 - 13c_2^3 + 608\,\lambda c_1^2}{c_2{c_1}^2}. \tag{25}$$

Case 3

$$a_0 = 0, \quad b_1 = 0, \quad a_1 = 0, \quad a_2 = -12c_2, \quad a_3 = 0,$$
$$b_2 = -15c_2\mu^2, \quad b_3 = -120c_1\mu^3, \quad \mu = \frac{\lambda}{8c_2}. \tag{26}$$

From eq. (19) and the general solution of eq. (20), we deduce the travelling wave solutions of eq. (2) as follows.

If $\mu \prec 0$, then Case 1 gives the following values of $u(x,t)$:

$$\begin{aligned}
u_1(x,t) &= \frac{15}{76}\,\frac{608c_1^2\frac{\lambda}{8c_2} - c_2^2 + 16c_3c_1}{c_1} \\
&\times \left(\sqrt{-\frac{\lambda}{8c_2}}\left(\frac{A\sinh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t) + B\cosh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t)}{A\cosh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t) + B\sinh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t)}\right)\right) \\
&- 15c_2\left(\sqrt{-\frac{\lambda}{8c_2}}\left(\frac{A\sinh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t) + B\cosh\sqrt{-\frac{\lambda}{8c_2}}\xi}{A\cosh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t) + B\sinh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t)}\right)\right)^2 \\
&+ 120c_1\left(\frac{\lambda}{8c_2}\right)^2\left(\sqrt{-\frac{\lambda}{8c_2}}\left(\frac{A\sinh\sqrt{-\frac{\lambda}{8c_2}}\xi + B\cosh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t)}{A\cosh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t) + B\sinh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t)}\right)\right)^3 \\
&- 12c_2\left(\frac{\lambda}{8c_2}\right)^2\left(\sqrt{-\frac{\lambda}{8c_2}}\left(\frac{A\sinh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t) + B\cosh\sqrt{-\frac{\lambda}{8c_2}}\xi}{A\cosh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t) + B\sinh\sqrt{-\frac{\lambda}{8c_2}}(x-\lambda t)}\right)\right)^{-2}.
\end{aligned} \tag{27}$$

Case 2 gives the solution

$$u_2(x,t) = \frac{15}{76}\,\frac{608\,c_1^2\mu - c_2^2 + 16\,c_3c_1}{c_1} \times \left(\sqrt{-\mu}\left(\frac{A\sinh\sqrt{-\mu}(x-\lambda t) + B\cosh\sqrt{-\mu}(x-\lambda t)}{A\cosh\sqrt{-\mu}(x-\lambda t) + B\sinh\sqrt{-\mu}(x-\lambda t)}\right)\right)$$
$$- 15c_2\sqrt{-\mu}\left(\frac{A\sinh\sqrt{-\mu}(x-\lambda t) + B\cosh\sqrt{-\mu}(x-\lambda t)}{A\cosh\sqrt{-\mu}(x-\lambda t) + B\sinh\sqrt{-\mu}(x-\lambda t)}\right)^2$$
$$+ 120c_1\sqrt{-\mu}\left(\frac{A\sinh\sqrt{-\mu}(x-\lambda t) + B\cosh\sqrt{-\mu}(x-\lambda t)}{A\cosh\sqrt{-\mu}(x-\lambda t) + B\sinh\sqrt{-\mu}(x-\lambda t)}\right)^3$$
$$- \frac{60}{19}\mu\;(c_3+38\,c_1\mu)\sqrt{-\mu}\left(\frac{A\sinh\sqrt{-\mu}(x-\lambda t) + B\cosh\sqrt{-\mu}(x-\lambda t)}{A\cosh\sqrt{-\mu}(x-\lambda t) + B\sinh\sqrt{-\mu}(x-\lambda t)}\right)^{-1}$$
$$- 120(c_1\mu^3)\sqrt{-\mu}\left(\frac{A\sinh\sqrt{-\mu}(x-\lambda t) + B\cosh\sqrt{-\mu}(x-\lambda t)}{A\cosh\sqrt{-\mu}(x-\lambda t) + B\sinh\sqrt{-\mu}(x-\lambda t)}\right)^{-3}, \quad (28)$$

where

$$\mu = \frac{1}{6080}\,\frac{56\,c_2c_3c_1 - 13\,c_2^3 + 608\,\lambda\,c_1^2}{c_2c_1^2}.$$

Case 3 gives the solution

$$u_3(x,t) =$$
$$- 12c_2\sqrt{-\left(\frac{\lambda}{8c^2}\right)}\left(\frac{A\sinh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) + B\cosh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}{A\cosh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) + B\sinh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}\right)^2$$
$$- 15c_2\left(\frac{\lambda}{8c^2}\right)^2\sqrt{-\left(\frac{\lambda}{8c^2}\right)}\left(\frac{A\sinh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) + B\cosh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}{A\cosh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) + B\sinh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}\right)^{-2}$$
$$- 120c_1\left(\frac{\lambda}{8c^2}\right)^3\sqrt{-\left(\frac{\lambda}{8c^2}\right)}\left(\frac{A\sinh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) + B\cosh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}{A\cosh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) + B\sinh\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}\right)^{-3}. \quad (29)$$

Remark 1. The graphical presentation of solution $u_1(x,t)$ is illustrated in figure 1.

If $\mu \succ 0$, we obtain the following results for $u_4(x,t)$.

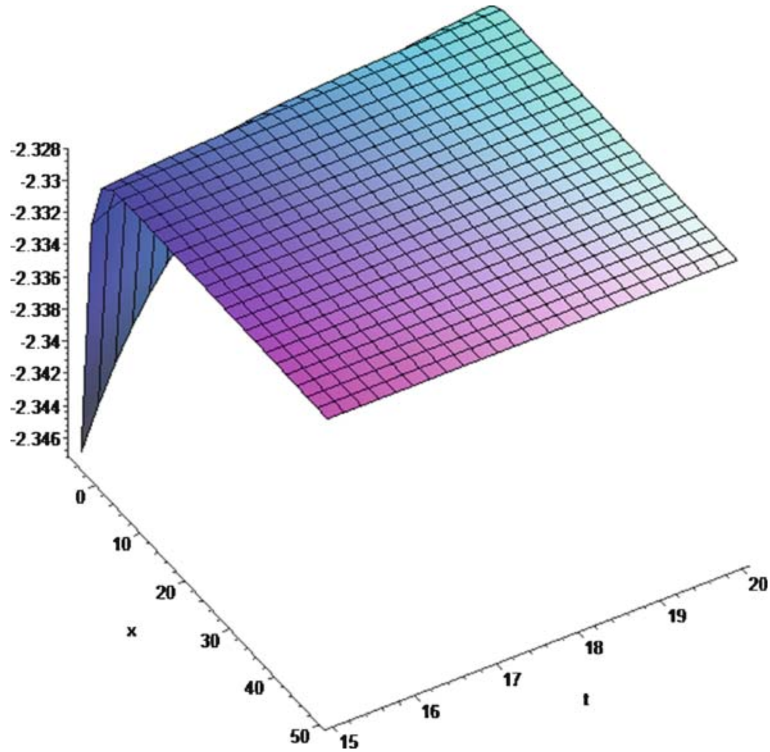

Figure 1. Figure showing the solution $u_1(x,t)$ for $B=0, c_1=c_2=1, \lambda=-1$.

Case 1 gives the value as

$$u_4(x,t)=\frac{15}{76}\,\frac{608\,c_1^2\frac{\lambda}{8c_2}-c_2^2+16\,c_3c_1}{c_1}$$

$$\times\left(\sqrt{\frac{\lambda}{8c_2}}\left(\frac{A\cos\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)-B\sin\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)}{A\sin\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)+B\cos\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)}\right)\right)$$

$$-15c_2\left(\sqrt{\frac{\lambda}{8c_2}}\left(\frac{A\cos\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)-B\sin\sqrt{\frac{\lambda}{8c_2}}\xi}{A\sin\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)+B\cos\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)}\right)\right)^2$$

$$+120c_1\left(\frac{\lambda}{8c_2}\right)^2\left(\sqrt{\frac{\lambda}{8c_2}}\left(\frac{A\cos\sqrt{\frac{\lambda}{8c_2}}\xi-B\sin\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)}{A\sin\sqrt{\frac{\lambda}{8c_2}}\xi+B\cos\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)}\right)\right)^3$$

$$-12c_2\left(\frac{\lambda}{8c_2}\right)^2\left(\sqrt{\frac{\lambda}{8c_2}}\left(\frac{A\cos\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)-B\sin\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)}{A\sin\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)+B\cos\sqrt{\frac{\lambda}{8c_2}}(x-\lambda t)}\right)\right)^{-2}. \tag{30}$$

Case 2 gives the solution

$$u_5(x,t) = \frac{15}{76}\,\frac{608\,c_1^2\mu - c_2^2 + 16c_3c_1}{c_1}$$
$$\times\left(\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}(x-\lambda t) - B\sin\sqrt{\mu}(x-\lambda t)}{A\sin\sqrt{\mu}(x-\lambda t) + B\cos\sqrt{\mu}(x-\lambda t)}\right)\right)$$
$$-15c_2\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}(x-\lambda t) - B\sin\sqrt{\mu}(x-\lambda t)}{A\sin\sqrt{\mu}(x-\lambda t) + B\cos\sqrt{\mu}(x-\lambda t)}\right)^2$$
$$+120c_1\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}(x-\lambda t) - B\sin\sqrt{\mu}(x-\lambda t)}{A\sin\sqrt{\mu}(x-\lambda t) + B\cos\sqrt{\mu}(x-\lambda t)}\right)^3$$
$$-\frac{60}{19}\mu\,(c_3+38\,c_1\mu)\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}(x-\lambda t) - B\sin\sqrt{\mu}(x-\lambda t)}{A\sin\sqrt{\mu}(x-\lambda t) + B\cos\sqrt{\mu}(x-\lambda t)}\right)^{-1}$$
$$-120(c_1\mu^3)\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}(x-\lambda t) - B\sin\sqrt{\mu}(x-\lambda t)}{A\sin\sqrt{\mu}(x-\lambda t) + B\cos\sqrt{\mu}(x-\lambda t)}\right)^{-3}, \tag{31}$$

where

$$\mu = \frac{1}{6080}\,\frac{56\,c_2c_3c_1 - 13\,c_2^3 + 608\,\lambda\,c_1^2}{c_2c_1^2}.$$

Case 3 gives the solution

$$u_6(x,t) =$$
$$-12c_2\sqrt{\left(\frac{\lambda}{8c^2}\right)}\left(\frac{A\cos\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) - B\sin\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}{A\sin\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) + B\cos\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}\right)^2$$
$$-15c_2\left(\frac{\lambda}{8c^2}\right)^2\sqrt{\left(\frac{\lambda}{8c^2}\right)}\left(\frac{A\cos\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) - B\sin\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}{A\sin\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) + B\cos\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}\right)^{-2}$$
$$-120c_1\left(\frac{\lambda}{8c^2}\right)^3\sqrt{\left(\frac{\lambda}{8c^2}\right)}\left(\frac{A\cos\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) - B\sin\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}{A\sin\sqrt{-\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t) + B\cos\sqrt{\left(\frac{\lambda}{8c^2}\right)}(x-\lambda t)}\right)^{-3}. \tag{32}$$

Remark 2. The graphical presentation of solution $u_4(x,t)$ is illustrated in figure 2.

The tanh method [30,31] is applied for the third-order eq. (18) by considering the constant of integration to be zero. Assume that the solution for eq. (18) is given as

$$f(\xi) = \sum_{i=0}^{n} a_i(\tanh(\mu\xi))^i,$$

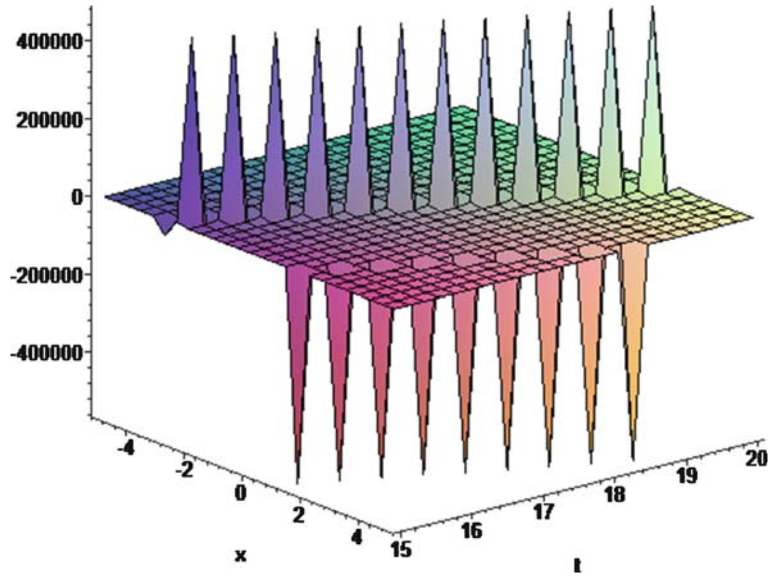

Figure 2. Figure showing the solution $u_4(x,t)$ for $B=0, c_1=c_2=c_3=1, \lambda=1$.

where $a_i, i = 1,2,3...$ and μ are constants to be determined and n is obtained by balancing the nonlinear terms and higher-order derivatives of the equation. From eq. (18) we get $n = 3$. Substitution of the form $f(\xi)$ in eq. (18) brings several possibilities for the values of the constants, and all these possibilities correspond to the following solutions of eq. (2).

Case 1

$$a_0 = 2c_2\mu^2, \quad a_1 = (-16c_1\mu^3 + 2c_3\mu), \quad a_2 = 0, \quad a_3 = 0,$$

$$\mu = \frac{1}{8}\left(\frac{\sqrt{2}\sqrt{10\,c_1c_3 - c_2^2 + \sqrt{36\,c_1^2c_3^2 - 20\,c_1c_3c_2^2 + c_2^4}}}{c_1}\right),$$

which corresponds to the following solution of eq. (2):

$$u_7 = 2c_2\mu^2 + (-16c_1\mu^3 + 2c_3\mu)(\tanh(\mu\xi)). \tag{33}$$

Case 2

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = -\frac{1}{20}\frac{(-15c_2^2 + 45c_1c_3)\left(-c_3 + \frac{1}{45}\frac{15c_2^2+45c_1c_3}{c_1}\right)}{c_1c_2}$$

$$a_3 = \frac{1}{225}\frac{(15c_2^2 + 45c_1c_3)}{(c_1^2)} \quad \text{and} \quad \mu = \frac{1}{30}\frac{\sqrt{15c_2^2 + 45c_1c_3}}{c_1}.$$

The corresponding solution is

$$u_8 = -\frac{1}{20}\frac{(-15c_2^2+45c_1c_3)\left(-c_3+\frac{1}{45}\frac{15c_2^2+45c_1c_3}{c_1}\right)}{c_1c_2}(\tanh(\mu\xi))^2 + \frac{1}{225}\frac{(15c_2^2+45c_1c_3)}{(c_1^2)}(\tanh(\mu\xi))^3. \quad (34)$$

Case 3

$$a_0 = 0, \quad a_1 = \frac{45}{361}\frac{\sqrt{-19c_1c_2c_3}}{c_1}, \quad a_2 = 0,$$

$$a_3 = \frac{15}{6859}\frac{(-19c_1c_3)}{c_1^2} \quad \text{and} \quad \mu = \frac{1}{38}\frac{\sqrt{-19c_1c_3}}{c_1}.$$

In this case, the solution of eq. (2) is given as

$$u_9 = \frac{45}{361}\frac{\sqrt{-19c_1c_2c_3}}{c_1}(\tanh(\mu\xi)) + \frac{15}{6859}\frac{(-19c_1c_3)}{c_1^2}(\tanh(\mu\xi))^3. \quad (35)$$

Case 4

$$a_0 = \frac{2}{5}\frac{(c_2c_3)}{c_1}, \quad a_1 = 0, \quad a_2 = \frac{-4}{5}\frac{(c_2c_3)}{c_1},$$

$$a_3 = 0 \quad \text{and} \quad \mu = \frac{1}{10}\frac{(\sqrt{5})(\sqrt{(c_1c_3)})}{c_1}$$

solution of eq. (2) is

$$u_{10} = \frac{2}{5}\frac{c_2c_3}{c_1} + \frac{-4}{5}\frac{c_2c_3}{c_1}(\tanh(\mu\xi))^2. \quad (36)$$

5. Conclusion

In summary, we have carried out the Painlevé analysis and investigated the symmetries of the Benny equation with three time-dependent variable coefficients using the Lie's classical method. The reduced equations were presented, and the exact solutions were investigated simultaneously using the modified (G'/G)-expansion and tanh methods. These solutions were expressed in terms of the hyperbolic functions and trigonometric functions. Thus, we have found some new exact solutions that might prove to be potentially useful for applications in mathematical physics and applied mathematics. The availability of mathematical computer software like *Maple* facilitates the tedious algebraic calculations.

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