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Excitations and management of the nonlinear localized gap modes

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Abstract. We discuss about the theory of nonlinear localized excitations, such as soliton and compactons in the gap of the linear spectrum of the nonlinear systems. We show how the gap originates in the linear spectrum using examples of a few systems, such as nonlinear lattices, Bose–Einstein condensates in optical lattice and systems represented by coupled nonlinear evolution equations. We then analytically show the excitation of solitons and compacton-like solutions in the gap of the linear spectrum of a system of coupled Korteweg–de Vries (KdV) equations with linear and nonlinear dispersions. Finally, we discuss about the theory of Feshbach resonance management and dispersion management of the soliton solutions.

Keywords. Gap soliton; gap compacton; soliton management.

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1. Introduction

Nonlinear evolution equations such as sine-Gordon equation, KdV equation etc. have solitary wave solutions which preserve their shape through nonlinear interaction. These stable solutions are also termed as 'solitons'. It has been shown that the equations possessing these solutions have infinite sequence of conservation laws and are integrable. They occur in many branches of science [1].

It is well known that collective nonlinear excitation called 'gap solitons' can exist in spectrum gaps forbidden for linear waves. The existence of gap soliton was shown by Chen and Mills [2] in 1987. Theoretical and numerical studies of the gap solitons have been carried out in many physical fields like nonlinear optical systems [2–5], semiconducting systems [6,7], superlattices [8,9] etc. The interplay of nonlinearity and lattice periodicity also results in the existence of solitary waves such as discrete breathers in the gap frequency range. Grimshaw and Malomed showed that a new type of a two-parameter family of solitons may exist in the narrow gap of the spectrum of two linearly coupled KdV equations with opposite sign of dispersion coefficient [10]. The gap soliton-like structures were demonstrated in Bose–Einstein condensates in the optical dipole trap [11]. Other theoretical studies of the possibility of nonlinear localization in Bose–Einstein condensates have been carried out in [12–14]. The existence of gap solitons in BEC was confirmed experimentally in 2004 for one-dimensional optical lattices [15].

The soliton solutions usually are exponentially localized in space. Rosenau and Hyman introduced a new type of localized solutions of KdV equations with nonlinear dispersion which are termed as 'compactons' [16,17]. These compactons vanish outside a finite core region. Compacton's amplitude depends on its velocity (unlike soliton's which narrows as amplitude (speed) increases) but its width is independent of its amplitude [16,18]. The compacton, like soliton, has a remarkable property that after colliding with another compacton, it re-emerges with the same coherent shape [16]. Unlike soliton collision in an integrable system, the point at which two compactons collide is marked by certain low-amplitude compacton–anticompacton pairs [16,18]. From their linear and nonlinear stability analyses, Dey and Khare have shown that the compactons are stable structures [19]. The stable compacton solutions also exist for KdV equations with higher-order nonlinear dispersion terms [20,21]. It has also been shown that such solution with compact structure can exist even in nonlinear lattice systems [22–27]. The existence of compact discrete breather solutions in two-dimensional Fermi–Pasta–Ulam lattice system was shown by Sarkar and Dey [28,29].

Recently, there have been several studies about the stability of solitons in inhomogeneous media. An example is the propagation of solitons in fibre-opics telecommunications. Solitons usually suffer decay in such media. It has been shown that solitons can be made robust by soliton management techniques [30]. One management technique involves periodic modulation of the nonlinearity parameter(s) of the system using Feshbach resonance mechanism. This method is termed as Feshbach resonance management or nonlinearity management. The other method is called dispersion management as it involves periodic modulation of the dispersion term of the nonlinear evolution equation which can stabilize soliton solutions. An experimental investigation of nonlinearity management in optics using femtosecond pulses and layered Kerr media is reported in [31].

In this paper, we report on the studies of excitations and management of the nonlinear localized solutions which occur in the gap region of the linear spectrum of the system concerned. The plan of the paper is as follows: in §2, we show with examples of three different nonlinear systems, how gap can originate in the linear spectrum of the system concerned. In §3, we demonstrate the existence of gap compacton-like excitation in the gap region of the two linearly coupled KdV equations with mixed dispersion. In §4, we briefly discuss how to stabilize the nonlinear localized solitons using Feshbach resonance and dispersion management mechanisms. Finally we conclude in §5.

2. Origin of gap in the spectrum: Continuous and discrete (lattice) systems

Here we consider examples of different systems to show how gap originates in the linear spectrum of the system.

2.1 *The lattice problem: Nonlinear lattice*

Here we consider a one-dimensional nonlinear lattice. The Hamiltonian of the system is given by

$$H = \sum_n \left[\frac{1}{2} p_n^2 + V(x_n) + W(x_n - x_{n-1}) \right]. \tag{1}$$

For the lattice to be nonlinear, either the onsite potential $V(x_n)$ is nonlinear or the intersite interaction $W(x_n - x_{n-1})$ is nonlinear (anharmonic) or both V and W are nonlinear. The equation of motion of the system is given by

$$\dot{x}_n = p_n$$

$$\dot{p}_n = -V'(x_n) - W'(x_n - X_{n-1}) + W'(x_{n+1} - X_n).$$

Linearizing the equation of motion around the classical ground state $x_n(t) \sim \mathrm{e}^{i(\omega_q t - qn)}$, we obtain the dispersion relation for small-amplitude plane wave as

$$\omega_q^2 = V''(0) + 4W''(0)\sin^2\left(\frac{q}{2}\right).$$

The plot of the dispersion relation is shown in figure 1 which shows gap region in the linear spectrum. It is well known that discrete breathers, which are time-periodic and space-localized solutions, can be excited in the gap region of the spectrum [32].

2.2 *Presence of periodic potential*

An example of such a system is Bose–Einstein condensates (BEC) in optical lattice. Optical lattices are periodic potential lattices whose presence leads to the modifications of the

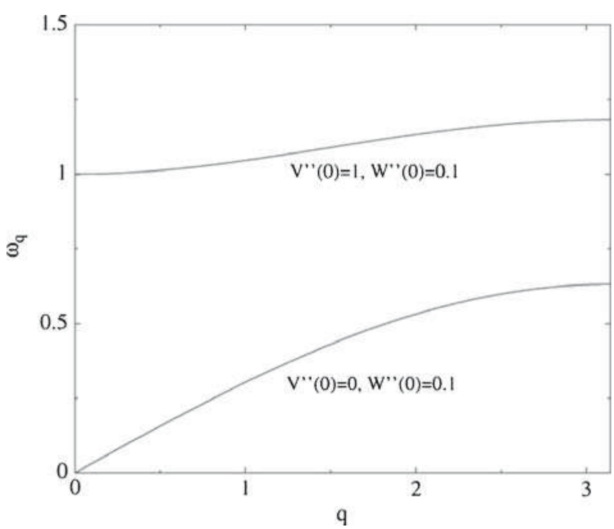

Figure 1. Dispersion relation for small-amplitude plane waves (reproduced from [32]).

linear propagation and dispersion relation. Spectrum of atomic Bloch waves in optical lattice is analogous to single electron states in crystalline solids. To obtain the band-gap spectrum of matter waves we consider a cigar-shaped BEC in a strongly elongated trap with tight confinement of the condensate in the transverse direction and weak confinement along x-direction. The condensate is loaded onto an quasi-one-dimensional optical lattice in the direction of weak confinement. The condensate dynamics is described by the one-dimensional Gross–Pitaevskii (GPE) equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi + g|\psi|^2\psi, \tag{2}$$

where $g = 2a_s/a_0$, $a_0 = \sqrt{\hbar/m\omega_\perp}$, $V(x) = \frac{1}{2}(\Omega^2 x^2 + V_0\sin^2(Kx))$ and $K = \pi a_0/d$. Due to weak confinement along x-direction, it is possible to drop the trapping potential term in the effective potential. For the stationary state $\psi(x,t) = \phi(x,t)\mathrm{e}^{-i\mu t}$, the steady-state wave function $\phi(x)$ satisfies the equation [33]

$$\frac{1}{2}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - [V_0\sin^2(Kx)\phi - \mu\phi] - \sigma|\phi|^2\phi = 0.$$

To obtain the linear band-gap spectra, we linearize the equation by dropping the nonlinear interaction term and the resultant equation can be written as Mathieu's equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} - [2q\cos(2y) - p]\phi = 0,$$

where $y = Kx$, $q = -V_0/(2K^2)$ and $p = 2(q + \mu/K^2)$. By employing the Floquet theorem, the condensate wavefunction can be written as superposition of Bloch waves as

$$\phi(y) = b_1 P_1(y)\mathrm{e}^{i\lambda y} + b_2 P_2(y)\mathrm{e}^{-i\lambda y},$$

where λ is the Floquet exponent. According to Floquet theory, for $\lambda > 0$ the spectrum consists of bands in which only amplitude-bounded solutions exist (not in general periodic). The bands are separated by regions or gaps where the solutions decay or increase exponentially for λ nonvanishing imaginary [33]. This is shown in figure 2. The band edges correspond to exact periodic solutions. Nonlinear localized solutions or gap solitons in BEC are possible with repulsive interactions in the gap region of the linear spectrum. The chemical potential μ for the gap solitons lies in the band gaps as shown in figure 3 [33]. To study the linear stability of the gap solitons, we consider small perturbation to the solution as

$$\psi(x,t) = \phi(x)\mathrm{e}^{-i\mu t} + \epsilon[u(x)\mathrm{e}^{i\beta t} + w^*(x)\mathrm{e}^{i\beta^* t}]\mathrm{e}^{-i\mu t}.$$

Substituting this in the GPE (eq. (2)) and linearizing the equation around the gap soliton we get the stability equation to first order in ϵ as [33]

$$\begin{bmatrix} \hat{\mathcal{L}}_0 & -\sigma\phi^2 \\ \sigma\phi^{*2} & -\hat{\mathcal{L}}_0 \end{bmatrix}\begin{pmatrix} u \\ w \end{pmatrix} = \beta\begin{pmatrix} u \\ w \end{pmatrix},$$

where $\hat{\mathcal{L}}_0 = (1/2)(\mathrm{d}^2/\mathrm{d}x^2) - V(x) - 2\sigma|\phi|^2 + \mu$. Solution of this equation for purely imaginary or complex eigenvalue β denotes oscillatory instability.

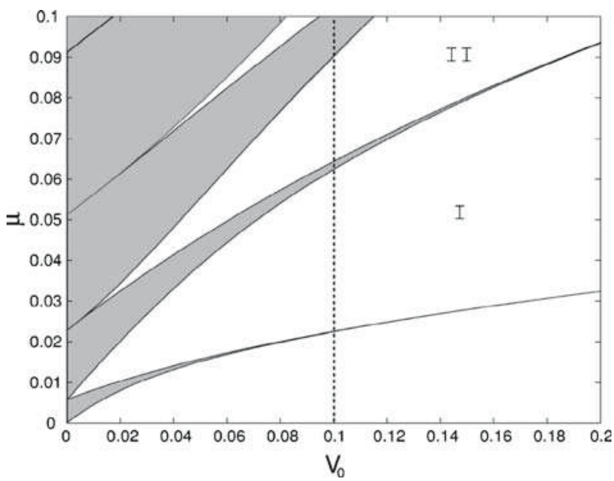

Figure 2. Band-gap diagram for Bloch waves in the linear regime (reproduced from [33]).

2.3 *Coupled nonlinear dynamical equations*

Gap can also appear in the linear spectrum of the coupled nonlinear dynamical equations. For example, dynamics of systems such as (a) spinor Bose–Einstein condensates, (b) multispecies Bose–Einstein condensates, (c) nonlinear optics etc. can be represented by the coupled nonlinear dynamical equations. Let us consider the coupled equations

$$iu_t + iu_x + \gamma\left(\frac{1}{2}|u|^2 + |v|^2\right)u + \kappa v = 0, \tag{3a}$$

$$iv_t - iv_x + \gamma\left(\frac{1}{2}|v|^2 + |u|^2\right)v + \kappa u = 0, \tag{3b}$$

which is a standard model of the nonlinear optical fibre equipped with a Bragg grating. κ is the Bragg-reflectivity coefficient and γ is the nonlinearity coefficient. To get the

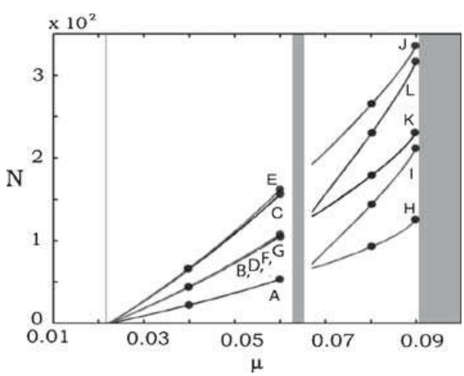

Figure 3. Families of different gap solitons (reproduced from [33]).

linearized gap spectrum we linearize the equations by dropping the cubic terms. We look for linear-wave solutions as $\{u(x,t), v(x,y) \sim e^{(ipx-i\omega t)}\}$. Substituting in the coupled equations we get the dispersion relation as

$$\omega^2 = p^2 + \kappa^2.$$

This implies that there are no linear waves with frequency in the gap region of the spectrum given by

$$-\kappa \le \omega \le \kappa.$$

It has been shown that there exist gap soliton solutions in the gap region of the spectrum given by [30]

$$u_{\mathrm{GS}}(x,t) = \sqrt{\frac{2\kappa}{3\gamma}}(\sin\theta)e^{-i(\kappa\cos\theta)t}\,\mathrm{sech}\left(\kappa x\sin\theta - \frac{i\theta}{2}\right),$$

$$u_{\mathrm{GS}}(x,t) = -\sqrt{\frac{2\kappa}{3\gamma}}(\sin\theta)e^{-i(\kappa\cos\theta)t}\,\mathrm{sech}\left(\kappa x\sin\theta + \frac{i\theta}{2}\right).$$

Note that the frequencies of the soliton family $\omega_{\mathrm{sol}} = \kappa\cos\theta$, exactly cover the band gap and hence these solitons are called gap solitons.

3. Gap compacton-like solutions

We consider a system of coupled Korteweg–de Vries equations with linear and nonlinear dispersions. We show the existence of gap compacton-like excitations within the gap of the linear spectrum of these coupled equations. We provide the main results here. Details of the calculations are given in [34]. This system of coupled KdV equations can be written as

$$u_t + \alpha_1(u^2)_x + u_{3x} + \beta_1(u^2)_{3x} = -\lambda v_x, \tag{4a}$$

$$v_t - \Delta v_x + \alpha_2(v^2)_x - \alpha v_{3x} + \beta_2(v^2)_{3x} = -\beta\lambda u_x \tag{4b}$$

where $-\Delta$ is the relative group velocity of linear long wave in the two subsystems, $\alpha > 0$ (corresponding to the oppositely signed dispersion in the subsystem), β_1 and β_2 are the nonlinear dispersion coefficients, β is an independent parameter (here we take $\Delta > 0$ and $\beta > 0$), λ is a small coupling constant and α_1 and α_2 are constants. In the absence of nonlinear dispersion terms ($\beta_1 = \beta_2 = 0$) and for particular value $\alpha_1 = \alpha_2 = -1/2$, the system of coupled KdV equations (eq. (4)) supports gap solitons [10]. However, we keep the value of α_1 and α_2 as arbitrary.

It has been shown that the uncoupled equations (λ=0) have exact travelling compacton solutions [34] given by

$$u(\eta) = E_1\cos^{\delta_1}(D_1\eta), \quad \text{for } |D_1\eta| \le \pi/2, \quad u = 0 \text{ otherwise} \tag{5a}$$

and

$$v(\eta) = E_2\cos^{\delta_2}(D_2\eta), \quad \text{for } |D_2\eta| \le \pi/2, \quad v = 0 \text{ otherwise} \tag{5b}$$

for $\delta_1 = \delta_2 = 2$ where

$$D_1^2 = \frac{\alpha_1}{16\beta_1}, \quad E_1 = \frac{4D_1^2 - c}{12\beta_1 D_1^2} \tag{6a}$$

$$D_2^2 = \frac{\alpha_2}{16\beta_2}, \quad E_2 = \frac{\Delta - c - 4\alpha D_2^2}{12\beta_2 D_2^2} \tag{6b}$$

and $\eta = x + ct$. The solutions satisfy the condition for the compacton solution which states that the width is independent of its amplitude [16].

3.1 *Existence of the gap*

To show that a weak coupling opens a gap in the system's linear spectrum and to find its width, we consider the solution of the linear case ($\beta_1 = \beta_2 = 0$) of eq. (4) to have the form

$$u = u_0 e^{i(kx-wt)} \quad \text{and} \quad v = v_0 e^{i(kx-wt)}.$$

By substituting in eq. (4) ($\beta_1 = \beta_2 = 0$) we get

$$\begin{bmatrix} (c+k^2) & -\lambda \\ -\beta\lambda & c+\Delta-\alpha k^2 \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = 0, \tag{7}$$

and from the condition of the existence of nontrivial solutions for the coupled linear equations, i.e. the determinant of the matrix should be zero, we get the new dispersion relation for the coupled equations as

$$c^2 + (\Delta - \alpha k^2 + k^2)c + k^2(\Delta - \alpha k^2) - \lambda^2\beta = 0. \tag{8}$$

The plot of linear spectrum for the uncoupled equations ($\lambda = 0$) is shown in figure 4 [34]. We can see from the figure that there is no gap in the spectrum for the uncoupled case. For $\lambda \neq 0$, a gap opens in the spectrum as shown in figure 5 [34].

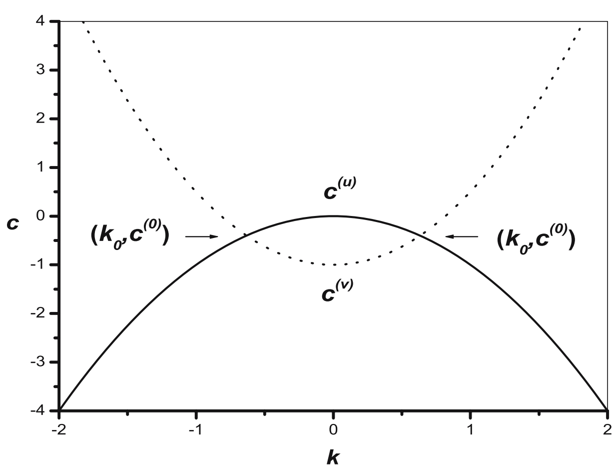

Figure 4. The linear spectrum of the system when coupling is absent (reproduced from [34]).

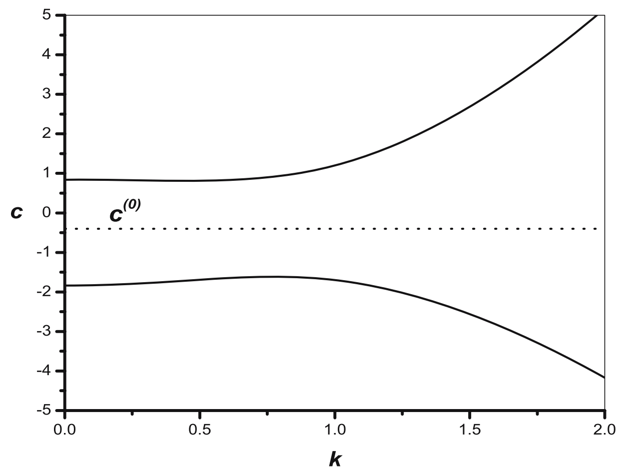

Figure 5. The opening of the gap in the linear spectrum of the system when coupling is present. The dotted line shows the intersection velocity $c^{(0)}$ (reproduced from [34]).

3.2 *Dynamics of the system inside the spectral gap region*

To study the dynamics of the system inside the spectral gap region we expand the wave field as [10]

$$u = U_1(x,t)\mathrm{e}^{ik_0(x-c^{(0)}t)} + U_2(x,t)\mathrm{e}^{2ik_0(x-c^{(0)}t)} + U_0(x,t) + \text{c.c.} \tag{9a}$$

$$v = V_1(x,t)\mathrm{e}^{ik_0(x-c^{(0)}t)} + V_2(x,t)\mathrm{e}^{2ik_0(x-c^{(0)}t)} + V_0(x,t) + \text{c.c.} \tag{9b}$$

To look for localized solutions inside the gap of the spectrum we consider weak nonlinearity effect [35–38] and assume that the amplitude of U and V are small and slowly varying. We also assume the differentiation of the slowly-varying functions $|U_1^2|$ and $|V_1^2|$ to be of the order of coupling constant λ [10]. Substituting eq. (9) in eq. (4), the amplitudes for the second harmonic are completely determined as [34]

$$U_2 = -\frac{1}{3}\left[\frac{4\Delta\beta_1 - \alpha_1(1+\alpha)}{\Delta}\right]U_1^2 \tag{10a}$$

$$V_2 = \frac{1}{3}\left[\frac{4\Delta\beta_2 - \alpha_2(1+\alpha)}{\alpha\Delta}\right]V_1^2 \tag{10b}$$

and the differential equations for the amplitudes for the first harmonics as

$$U_t - U_\eta + i|U|^2U - iMU = -i\epsilon V \tag{11a}$$

$$V_t + V_\eta - i|V|^2V - iNV = -i\epsilon\gamma U \tag{11b}$$

and that for the zeroth harmonics as

$$M_t + \frac{2-\alpha}{1+\alpha}M_\eta = \frac{6\alpha_1}{\alpha_1(1+\alpha) - 4\Delta\beta_1}(|U|^2)_\eta, \tag{12a}$$

$$N_t + \frac{1-2\alpha}{1+\alpha} N_\eta = \frac{6\alpha_2\alpha}{\alpha_2(1+\alpha) - 4\Delta\beta_2} (|V|^2)_\eta, \tag{12b}$$

where

$$U_1 = \left[\frac{3\Delta}{2k_0(\alpha_1 - \beta_1 {k_0}^2)[\alpha_1(1+\alpha) - 4\Delta\beta_1]} \right]^{1/2} U,$$

$$V_1 = \left[\frac{3\Delta\alpha}{2k_0(\alpha_2 - \beta_2 {k_0}^2)[\alpha_2(1+\alpha) - 4\Delta\beta_2]} \right]^{1/2} V,$$

$$M = 4k_0(\beta_1 {k_0}^2 - \alpha_1)U_0,$$
$$N = 4k_0(\beta_2 {k_0}^2 - \alpha_2)V_0 \quad \text{and} \quad \eta = \Delta^{-1}x + \frac{2-\alpha}{1+\alpha}t, \quad t = t.$$

We look for travelling solitary wave solutions for eqs (11) and (12) as

$$U(\eta - wt) = \mathrm{e}^{-i\sigma t} A(\eta - wt), \tag{13a}$$

$$V(\eta - wt) = \mathrm{e}^{-i\sigma t} B(\eta - wt), \tag{13b}$$

$$M = M(\eta - wt), \quad N = N(\eta - wt), \tag{14}$$

where the frequency σ is assumed to be of order λ. Substituting in eqs (11) and (12) we get three equations, one for the amplitude

$$\frac{\mathrm{d}R^2}{\mathrm{d}(\eta - wt)} = 2\epsilon\sqrt{\frac{\gamma}{1-w^2}} R^2 \sin(\varphi - \psi) \tag{15a}$$

and the phases

$$\frac{\mathrm{d}\varphi}{\mathrm{d}(\eta - wt)} = \epsilon\sqrt{\frac{\gamma}{1-w^2}} \cos(\varphi - \psi) - \frac{(\sigma + C(1-w)R^2)}{(1+w)}, \tag{15b}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}(\eta - wt)} = \frac{(\sigma - D\gamma(1+w)R^2)}{(1-w)} - \epsilon\sqrt{\frac{\gamma}{1-w^2}} \cos(\varphi - \psi), \tag{15c}$$

where $A = \sqrt{(1-w)}R\mathrm{e}^{i\phi}$ and $b = \sqrt{\gamma(1+w)}R\mathrm{e}^{i\psi}$ [34]. We note that the solution must satisfy the condition ($w^2 < 1$) for it to be inside the gap spectrum. Solving these coupled equations we get

$$\sin(\varphi - \psi) = [1 - (\Omega + 2R^2W)^2]^{1/2} \tag{16}$$

and

$$\frac{\mathrm{d}R^2}{\mathrm{d}(\eta - wt)} = 2\epsilon\sqrt{\frac{\gamma}{1-w^2}} R^2 [1 - (\Omega + 2R^2W)^2]^{1/2}, \tag{17}$$

where Ω, W, Q_1 and Q_2 are constants [34].

To obtain the gap compacton solution of eq. (17), we rewrite this equation in the compacton equation form [16] as

$$\left(\frac{\mathrm{d}y}{\mathrm{d}l}\right)^2 + p(y, p_0) = p_1, \tag{18a}$$

where

$$p(y, p_0) = 4a^2W^2y^4 + 4a^2\Omega W y^3 - a^2(1-\Omega^2)y^2 + p_0 \tag{18b}$$

$y = R^2$, $l = \eta - wt$, $a = 2\epsilon\sqrt{\gamma/(1-w^2)}$ and $p_0 = p_1 = 0$. Solution of this equation gives a gap soliton [34]

$$R^2 = (2W)^{-1}\frac{(1-\Omega^2)}{2\cosh^2[Z\sqrt{1-\Omega^2}] - (1-\Omega)} \tag{19a}$$

with phase difference

$$\tan\left(\frac{\varphi-\psi}{2}\right) = \sqrt{\frac{1-\Omega}{1+\Omega}}\tanh[Z\sqrt{1-\Omega^2}] \tag{19b}$$

for $\Omega^2 < 1$. Similarly for $\Omega^2 = 1$, we get weak gap soliton solutions with nonexponential decay as [34]

$$R^2 = (2W)^{-1}\left[\frac{1}{1+4Z^2}\right], \quad \tan\left(\frac{\varphi-\psi}{2}\right) = 2Z$$

$$\text{for } \Omega = -1,\ W > 0, \tag{20a}$$

$$R^2 = (2W)^{-1}\left[\frac{-1}{1+4Z^2}\right], \quad \tan\left(\frac{\varphi-\psi}{2}\right) = (2Z)^{-1}$$

$$\text{for } \Omega = +1,\ W < 0. \tag{20b}$$

On the other hand for $\Omega^2 > 1$ we obtain a gap compacton-like solution given by

$$R^2 = (2W)^{-1}\frac{(1-\Omega^2)}{(1+\Omega) - 2\cos^2[Z\sqrt{\Omega^2-1}]} \tag{21a}$$

with phase difference

$$\tan\left(\frac{\varphi-\psi}{2}\right) = \sqrt{\frac{\Omega-1}{\Omega+1}}\cot[Z\sqrt{\Omega^2-1}]. \tag{21b}$$

The gap compacton-like solutions are shown in figure 6. From the figure we can see that the nonzero amplitude of the solution outside the compact support can be reduced to a very small value for suitable choice of the parameters. Due to their presence in the gap region of the spectrum, the gap compacton-like solutions as obtained here are stable, as they do not decay by resonating with linear phonon band.

4. Management of localized solutions

Integrable systems with soliton solutions are exceptional. Any additional term which takes into account the physical effects breaks the exact integrability. Solitary waves or localized solutions of the nonintegrable systems are not generally stable. Such solutions can be stabilized using soliton 'management' techniques [30]. Usual methods of managements are (a) Feshbach resonance management or nonlinearity management and (b) dispersion management.

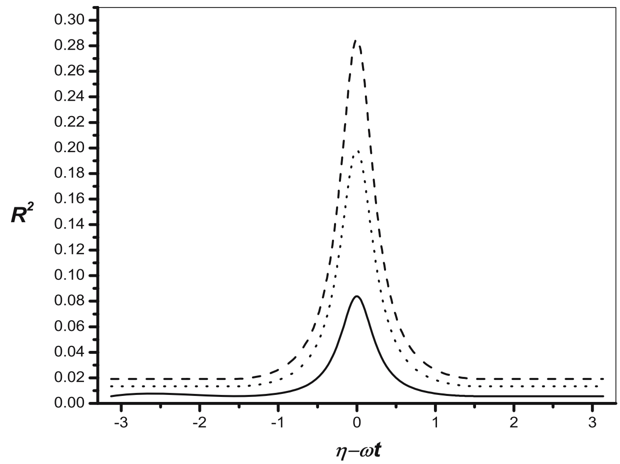

Figure 6. The compacton-like solution for three different values of parameters β_1 and β_2. From top to bottom, $\beta_1 = \beta_2 = 4$, $\beta_1 = \beta_2 = 2$ and $\beta_1 = \beta_2 = 1$, respectively. The parameters are $\alpha_1 = \alpha_2 = 1$, $\lambda = 0.9$, $\Delta = 1$, $\alpha = 1.5$ and $\beta = 1.9$ (reproduced from [34]).

4.1 *Feshbach resonance management*

Feshbach resonance management involves periodic modulations of the nonlinear parameters of the system. Here we consider an example of the nonlinearity management through Feshbach resonance mechanism in Bose–Einstein condensate on an optical lattice. The nonlinear parameter g_0 of the Gross–Pitaevskii equation, describing the dynamics of the condensate, is modulated as

$$i\frac{\partial\psi}{\partial t} = \left[\frac{1}{2}\nabla^2 + \epsilon(1-\cos(2z)) + (g_0 + g_1\sin(\Omega t)|\psi|^2\right]\psi, \qquad (22)$$

where g_0 and g_1 denote the DC and AC parts of the Feshbach-controlled nonlinearity. An analytical approach to solve the above equation is time-dependent variational analysis with a variational ansatz for the wavefunction as [30]

$$\psi(r,t) = A(t)\exp\left(-\frac{r^2}{2a^2(t)} + \frac{1}{2}ib(t)r^2 + i\delta(t)\right),$$

where the parameters $A,\ a,\ b$ and δ are, respectively, the amplitude, width, chirp and phase, which are assumed to be real functions of time. Substituting this in eq. (22) and integrating, we get an effective Lagrangian in terms of these time-dependent variational parameters $a,\ b$ and δ. The variational Euler–Lagrange equations which follow from the effective Lagrangian give the coupled ordinary differential equations in time for these parameters. Solutions of these coupled equations provide the nature of the time-evolution of these parameters and the stability of the wavefunction [30].

4.2 *Dispersion management*

In this case, the dispersion term of the nonlinear evolution equation is modified. As an example we may consider the dispersion management in a nonlinear planar waveguide. Here, an optical beam propagates across a layered structure that does not affect the nonlinearity [30]. As a model, we can consider nonlinear Schrödinger equation (NLS) with modulated dispersion

$$iu_z - \frac{1}{2}\beta(z)u_{\tau\tau} + \gamma|u|^2u = 0. \tag{23}$$

Here $\beta(z)$ can be a piece-wise constant function. We can also consider harmonic modulated dispersion

$$iu_z + \frac{1}{2}(1 + \epsilon \sin z)u_{\tau\tau} + \gamma|u|^2u = 0. \tag{24}$$

Variational method is used to solve the above equation. The ansatz for the solution is taken as similar to the exact solution of the nonlinear Schrödinger equation but the variational parameters are considered functions of evolution variables z as

$$u(z,\tau) = A(z)\mathrm{sech}\left(\frac{\tau}{a(z)}\right)\exp[i\phi(z) + ib(z)\tau^2].$$

Substituting this in eq. (23) or eq. (24) and integrating, we obtain an effective Lagrangian in terms of variational parameters. Solving the coupled ODEs involving the variational parameters, which follow from the Euler–Lagrange equation of the effective Lagrangian, we can study the stability of the solutions of NLS equation [30].

4.3 *Dispersion management in KdV system*

Although the Feshbach resonance and dispersion management techniques for soliton management have been quite sucessful for optical solitons and solitons in BEC, use of such techniques for soliton management in KdV or KdV system of equations has received very little attention. There are a few reports of studies involving dispersion management in KdV system. Clarke *et al* [39] studied the variable coefficient KdV equation which has the form of KdV equation with a periodically-varying third-order dispersion coefficient

$$u_t + c(\epsilon x)u_x + \epsilon[r(\epsilon x)uu_x + s(\epsilon x)u_{xxx}] = 0(\epsilon^2), \tag{25}$$

where $\epsilon \ll 1$ is a small parameter. By changing the variables, eq. (25) can be reduced to a simpler form [39]

$$u_\chi + 6uu_\theta + D(\chi)u_{\theta\theta\theta} = 0, \tag{26}$$

where $\chi = -\int(\epsilon r/6c^2)\mathrm{d}x$, $\theta = t - \int(\mathrm{d}x/c)$ and the local variable dispersion coefficient $D(\chi) = 6s/(rc^2)$. The variable dispersion coefficient $D(\chi)$ is assumed to have a general form

$$D = SD_0(\chi/T) + D_1(\chi), \tag{27}$$

where D_0 is a periodic function with zero mean value. The function D_1 then represents the local average dispersion which can be subjected to a long-range modulation. For a weakly nonlinear case, eq. (26) can be reduced to a simple integral equation. This method of reducing the dispersion-modulated equation to an integral equation has been used also for the fibre-optic (optical soliton) dispersion management model [40]. Solutions of the integral equation are then obtained numerically using iterative scheme. Numerical calculations are done by taking dispersion $D_0(\chi)$ either as piece-wise constant (as is also done for the optical soliton case) or as other forms such as smooth sinusoidal modulation. The average dispersion D_1 is taken as constant. Shape of the localized solution so obtained can be plotted as a function of D_1 [39]. It is found that for both piece-wise constant dispersion and smooth sinusoidal modulation of the dispersion, the corresponding solitary wave solutions are very similar to the sech^2 solitons of the constant dispersion KdV equation. The method can be extended to KdV equations with fifth-order dispersion.

The other method of solving the variable-coefficient KdV system of equation is to use the Painleve test and Lax pair formalism widely used for the completely integrable PDEs (partial differential equations) [41]. It involves reducing the variable-coefficient KdV system of equations to a generalized integral form through Painleve analysis. Here one finds the particular form and relations between the variable-coefficients for which the variable-coefficient KdV system of equation passes the Painleve test. The existence of the Lax pair for the generalized integral equation then provides conditions for the existence of the soliton solutions. Knowing the Lax pair one can use the inverse scattering method to find the N-soliton solutions of the given variable-coefficient KdV system of equation [41].

5. Conclusions

In conclusion, we have shown how gaps can occur in the linear spectrum of nonlinear systems using a few examples from the areas of recent research interest. We have shown how gap solitons and gap compacton-like solutions can occur in the gap of the linear spectrum of a system of coupled KdV equations with linear and nonlinear dispersion. We have also discussed how the propagation of these nonlinear localized solutions in inhomogeneous media can be stabilized by periodic modulation of either the nonlinearity or the dispersion of the corresponding nonlinear evolution equations using management techniques.

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