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PT-symmetric dimer of coupled nonlinear oscillators

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Abstract. We provide a systematic analysis of a prototypical nonlinear oscillator system respecting PT-symmetry, i.e., one of them has gain and the other an equal and opposite amount of loss. We first discuss various symmetries of the model. We show that both the linear system as well as a special case of the nonlinear system can be derived from a Hamiltonian, whose structure is similar to the Pais–Uhlenbeck Hamiltonian. Exact solutions are obtained in a few special cases. We show that the system is a superintegrable system within the rotating wave approximation (RWA). We also obtain several exact solutions of these RWA equations. Further, we point out a novel superposition in the context of periodic solutions in terms of Jacobi elliptic functions that we obtain in this problem. Finally, we briefly mention numerical results about the stability of some of the solutions.

Keywords. PT-symmetry; dimer; rotating wave approximation; novel superposition.

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1. Introduction

The topic of parity–time (PT) symmetry and its relevance to physical applications on the one hand, and its mathematical structure on the other hand have drawn considerable attention both from the physics and the mathematics communities. Originally, this theme was proposed by Bender and co-workers as an additional possibility for operators associated with real measurable quantities within linear quantum mechanics [1]. However, one of the major milestones (and a principal thrust of recent activity) regarding the physical/experimental realizability of the corresponding Hamiltonians stemmed from progress in optics both at the theoretical [2,3] and experimental [4,5] levels. In particular, the realization that in optics, the ubiquitous loss can be counteracted by an overwhelming gain, to create a PT-symmetric set-up e.g. in a waveguide dimer [5] paved the way for numerous developments especially so at the level of nonlinear systems, as several researchers

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studied nonlinear stationary states, stability and dynamics of a few site configurations [6]. Interestingly, most of this nonlinear activity has been centred around Schrödinger-type systems and for good reason, because the original proposal by Bender involved quantum mechanical settings, where this is natural and in addition the optics proposal was placed chiefly on a similar footing (i.e., the Schrödinger model as paraxial approximation to the Maxwell equations).

However, a number of recent studies, especially on the experimental side, have created interest in oscillator systems with PT-symmetry. For example, a mechanical system realizing PT-symmetry has been proposed [7] and there has also been a major thrust in the context of electronic circuits [8]. Other applications are in the area of whispering-gallery microcavities [9]. While most of these applications are in the context of linear systems, recently, a number of nonlinear variants have been explored, like split-ring resonator chain in the context of magnetic metamaterials [10]. Some of these studies have raised some interesting questions. For example, it has been realized that linear oscillator pairs may be Hamiltonian, although one of them has a gain and the other an equal and opposite loss [11]. Recently, it has been shown [12] that this feature of Hamiltonian nature of PT-symmetric system can also be there in a nonlinear system (see below).

Our aim herein [13] is to provide a simple, prototypical nonlinear model, whose linear analogue is effectively the one used in the experimental investigations. The nonlinear structure is such that it allows to obtain a detailed numerical and even considerable analytical insight on the phenomenology of such a nonlinear PT-symmetric oscillator dimers. We shall see that the nonlinear models discussed in the literature are special cases of our rather general nonlinear model.

The plan of the paper is as follows: In §2, we set up the model and discuss the symmetries of the model and then obtain constants of motion of the system in special cases which include linear case as well as when there is no gain or loss in the system. In the linear as well as a special case, we show that the equations of motion can be derived from a Hamiltonian which has a structure similar to that of Pais–Uhlenbeck Hamiltonian. In §3, we show that there are a few special cases in which one can obtain several exact solutions of the system. Remarkably, we also exhibit a novel superposition [14] by showing that if Jacobi elliptic functions cn(x, m) and dn(x, m) are solutions of a nonlinear equation, then their linear superposition is also a solution of the same nonlinear equation. In §4, we treat the problem as a dynamical system and examine the fixed points of the system and their stability. In §5, we consider the RWA approximation and then analyse the symmetries as well as the exact solutions of the RWA equations and show that it is a superintegrable system. In §6, we briefly mention the results of detailed simulations and their comparison with the RWA results. Finally, in §7 we mention some of the possible open problems.

2. The model, its symmetries and constants of motion

We consider the following coupled Klein–Gordon dimer system:

$$\ddot{u} = -u + kv + \gamma\dot{u} + \epsilon u^3 + \delta u v^2, \tag{1}$$

$$\ddot{v} = -v + ku - \gamma\dot{v} + \epsilon v^3 + \delta v u^2. \tag{2}$$

Here $u(t)$ and $v(t)$ denote the amplitude of the two oscillators, the terms proportional to k and δ respectively reflect the linear and nonlinear coupling between the two elements in the dimer, while γ is proportional to the amplification/resistance within the system.

On the other hand, the term proportional to ϵ reflects the intrinsic nonlinearity in each element of the dimer. It is worth pointing out that the special cases of $\delta = 0$ and $\delta = 3\epsilon$ have been respectively discussed earlier by us [13] and in [12].

2.1 *Linear limit*

On using the ansatz

$$u = A\mathrm{e}^{i\lambda(t+t_0)}, \quad v = B\mathrm{e}^{i\lambda(t+t_0)}, \tag{3}$$

one finds that in the linear limit, i.e. $\delta = \epsilon = 0$, there are two branches of solutions given by

$$\lambda_{\pm} = \sqrt{1 - \gamma^2/2 \pm \sqrt{k^2 - \gamma^2 + \gamma^4/4}}\,, \tag{4}$$

with $\lambda_+(\lambda_-)$ corresponding to symmetric(antisymmetric) linear modes at $\gamma = 0$. Thus, $\gamma = \gamma_{PT}$ is the point of the so called PT-phase transition and bifurcation into the complex plane, where γ_{PT} satisfies the condition

$$\gamma_{PT}^4 - 4\gamma_{PT}^2 + 4k^2 = 0. \tag{5}$$

2.2 *Symmetries*

The coupled eqs (1) and (2) possess a number of symmetries. In particular, these equations remain unchanged under the following transformations:

$$\begin{aligned}
&(1)\ (u, v) \longrightarrow (-u, -v), \quad (2)\ (u, k) \longrightarrow (-u, -k)\\
&(3)\ (v, k) \longrightarrow (-v, -k), \quad (4)\ (\gamma, t) \longrightarrow (-\gamma, -t)\\
&(5)\ (u, v, \gamma) \longrightarrow (\pm v, \pm u, -\gamma),\\
&(6)\ (u, v, \epsilon, \delta) \longrightarrow \left(\alpha u, \alpha v, \frac{\epsilon}{\alpha^2}, \frac{\delta}{\alpha^2}\right).
\end{aligned} \tag{6}$$

Further, while the coupled eqs (1) and (2) are neither invariant under parity P (which corresponds to $(u, v) \longrightarrow (\pm v, \pm u)$), nor under time reversal T (which corresponds to $t \longrightarrow -t$), but they are invariant under the combined PT-symmetry.

2.3 *Constants of motion*

There are three special cases for which we have been able to find a constant of motion.

2.3.1 $\delta = 3\epsilon$.

In the special case when $\delta = 3\epsilon$, one can show that (see also [12]) the equations of motion (1) and (2) can be derived from the Hamiltonian

$$H_2 = p_u p_v + \frac{\gamma}{2}(up_u - vp_v) + \left(1 - \frac{\gamma^2}{4}\right)uv - \frac{k}{2}(u^2 + v^2) - \epsilon(u^3 v + v^3 u). \tag{7}$$

In this case,

$$p_u = \dot{v} + \gamma v/2, \quad p_v = \dot{u} - \gamma u/2. \tag{8}$$

The structure of this Hamiltonian is quite similar to the Pais–Uhlenbeck Hamiltonian. For the special case of the linear coupled system (i.e. $\delta = \epsilon = 0$) this was pointed out a long time ago by Bateman [15].

2.3.2 $\gamma = 0$. When $\gamma = 0$, eqs (1) and (2) can be derived from the Hamiltonian

$$H_1 = \frac{p_u^2 + p_v^2 + u^2 + v^2}{2} - kuv - \frac{\epsilon}{4}(u^4 + v^4) - \frac{\delta}{2}u^2v^2. \tag{9}$$

Note that unlike the previous case, in this case

$$p_u = \dot{u}, \quad p_v = \dot{v}. \tag{10}$$

2.3.3 $\delta = \epsilon, k = 0$. Finally, it is easy to check that when $\delta = \epsilon$ and $k = 0$, there is a constant of motion given by

$$C = u\dot{v} - v\dot{u} + \gamma uv. \tag{11}$$

We thus see that we have three integrable cases, i.e. we have three special cases in which we have two constants of motion in involution. These are
Case I: $\gamma = 0,\ \delta = 3\epsilon$. In this case, the two constants of motion are H_1 and H_2.
Case II: $\gamma = k = 0,\ \ \delta = \epsilon$. In this case, the two constants of motion are H_1 and C.
Case III: $\delta = \epsilon = k = 0$. In this case, the two constants of motion are H_1, H_2.

3. Exact solutions

There are three special cases, when we are able to obtain exact solutions of these coupled equations, which we now discuss one by one.

3.1 $\delta = \epsilon$

It is interesting to note that when $\delta = \epsilon$, there is an exact periodic solution to the coupled eqs (1) and (2) given by

$$u(t) = A\sin[\omega(t + t_0)], \quad v = B\cos[\omega(t + t_0)], \tag{12}$$

with t_0 being an arbitrary constant, provided

$$B = \pm A, \quad \delta = \epsilon, \quad \epsilon A^2 = 1 - \omega^2, \quad k = \mp\gamma\omega. \tag{13}$$

Note that if $\epsilon > 0$ then $\omega^2 < 1$ and hence $\gamma^2 > k^2$, while if $\epsilon < 0$, then $\omega^2 > 1$ and hence $\gamma^2 < k^2$.

3.2 $\epsilon = 0$

When $\epsilon = 0$, the coupled eqs (1) and (2) admit two exact solutions. The first one is

$$u = A\mathrm{e}^{\lambda(t+t_0)}, \quad v = B\mathrm{e}^{-\lambda(t+t_0)}, \tag{14}$$

provided

$$AB = -\frac{k}{\delta}, \quad \lambda = \frac{1}{2}[\gamma \pm \sqrt{\gamma^2 - 4}]. \tag{15}$$

The other solution is

$$u = A\mathrm{e}^{-\lambda(t+t_0)}, \quad v = B\mathrm{e}^{\lambda(t+t_0)}, \tag{16}$$

provided

$$AB = -\frac{k}{\delta}, \quad \lambda = \frac{1}{2}[-\gamma \pm \sqrt{\gamma^2 - 4}]. \tag{17}$$

3.3 $\gamma = 0$

When $\gamma = 0$, the coupled dimer eqs (1) and (2) admit several exact periodic solutions, which we present one by one.

3.3.1 $\gamma = 0,\ \delta + \epsilon > 0$.

It is easy to check that

$$u(t) = \pm v(t) = A\sqrt{m}\,\mathrm{sn}[\beta(t + t_0), m], \tag{18}$$

is an exact solution provided

$$(\epsilon + \delta)A^2 = 2\beta^2, \quad (1 + m)\beta^2 = 1 \mp k. \tag{19}$$

Here m is the modulus of the Jacobi elliptic function [16]. In the limit $m = 1$, we get the corresponding symmetric and antisymmetric hyperbolic soliton solutions

$$u(t) = \pm v(t) = A\tanh[\beta(t + t_0)], \tag{20}$$

provided the constraints as given by eqs (19) with $m = 1$ are satisfied.

It should be emphasized that these solutions are valid even when either δ or ϵ is negative provided their sum is positive.

3.3.2 $\gamma = 0,\ \delta + \epsilon = 0$.

In this case again we have two solutions, symmetric and antisymmetric

$$u(t) = \pm v(t) = A\sin[\sqrt{1 \mp k}(t + t_0)]. \tag{21}$$

Note that these solutions are valid for any value of ϵ (and hence δ) including $\epsilon = \delta = 0$.

3.3.3 $\gamma = 0,\ \delta + \epsilon < 0$.

In this case there are three solutions of the coupled dimer eqs (1) and (2) out of which we present two solutions now and the third one (a novel superposed solution) in the next subsection.

Solution I: It is easy to check that

$$u(t) = \pm v(t) = A\mathrm{dn}[\beta(t + t_0), m] \tag{22}$$

is an exact solution provided

$$(\epsilon + \delta)\beta^2 = -2\beta^2, \quad (2 - m)\beta^2 = -1 \pm k. \tag{23}$$

Solution II: It is easy to check that

$$u(t) = \pm v(t) = A\sqrt{m}\,\mathrm{cn}[\beta(t + t_0), m] \tag{24}$$

is an exact solution provided

$$(\epsilon + \delta)\beta^2 = -2\beta^2, \quad (2m - 1)\beta^2 = -1 \pm k. \tag{25}$$

In the limit $m = 1$, both solutions I and II reduce to the same hyperbolic solution

$$u(t) = \pm v(t) = A\,\mathrm{sech}[\beta(t + t_0)], \tag{26}$$

provided the constraints (25) with $m = 1$ are satisfied.

It must be again emphasized that all these solutions are valid even when ϵ or δ are positive, what is simply required is that their sum be negative.

In addition to the solutions presented so far, there are extra exact solutions in the special cases of (i) $\gamma = 0$ and $\delta = \epsilon$, (ii) $\gamma = 0$ and $\delta = 3\epsilon$ which are presented below.

3.3.4 $\gamma = 0,\ \delta = \epsilon$. In this case remarkably, apart from the solutions discussed above, novel superposition of Jacobi elliptic functions are also exact solutions of the coupled eqs (1) and (2). In particular, it is easy to show that

$$u(t) = A\sqrt{m}\,\mathrm{sn}[\beta(t+t_0), m] + B\sqrt{m}\,\mathrm{cn}[\beta(t+t_0), m] \tag{27}$$

and

$$v(t) = A\sqrt{m}\,\mathrm{sn}[\beta(t+t_0), m] - B\sqrt{m}\,\mathrm{cn}[\beta(t+t_0), m] \tag{28}$$

are exact solutions provided

$$\epsilon(A^2 - B^2) = \beta^2, \quad 4m\epsilon A^2 = 2 + (3m-2)\beta^2, \quad k = \frac{-m\beta^2}{2}. \tag{29}$$

Remarkably, even

$$u(t) = A\sqrt{m}\,\mathrm{sn}[\beta(t+t_0), m] + B\,\mathrm{dn}[\beta(t+t_0), m] \tag{30}$$

and

$$v(t) = A\sqrt{m}\,\mathrm{sn}[\beta(t+t_0), m] - B\,\mathrm{dn}[\beta(t+t_0), m], \tag{31}$$

are exact solutions of the coupled eqs (1) and (2) provided

$$\epsilon(A^2 - B^2) = \beta^2, \quad 4\epsilon A^2 = 2 + (3-2m)\beta^2, \quad k = \frac{-\beta^2}{2}. \tag{32}$$

Note that for all these solutions if $\epsilon > 0$, then $A^2 > B^2$, while if $\epsilon < 0$, then $B^2 > A^2$. In the limit $m = 1$, both these solutions go over to the corresponding hyperbolic soliton solutions provided the above constraints with $m = 1$ are satisfied.

3.3.5 $\gamma = 0,\ \delta = 3\epsilon$. We introduce new variables x, y

$$x = u + v, \quad y = u - v. \tag{33}$$

Then it is easy to show that the equations of motion for x and y decouple, i.e.

$$\ddot{x} = -(1-k)x + \epsilon x^3, \tag{34}$$

$$\ddot{y} = -(1+k)y + \epsilon y^3. \tag{35}$$

Now depending on whether ϵ (and hence δ) is positive or negative, one has different solutions. When $\epsilon > 0$, then the solutions are

$$x(t) = A\sqrt{m}\,\mathrm{sn}[\beta(t+t_0), m], \tag{36}$$

provided

$$\epsilon A^2 = 2\beta^2, \quad (1+m)\beta^2 = 1 - k > 0. \tag{37}$$

Similarly,

$$y(t) = B\sqrt{m_1}\,\mathrm{sn}[\beta_1(t+t_1), m_1], \tag{38}$$

is an exact solution provided

$$\epsilon B^2 = 2\beta_1^2, \quad (1 + m_1)\beta_1^2 = 1 + k > 0. \tag{39}$$

In the limit $m = m_1 = 1$, these solutions go over to the corresponding hyperbolic soliton solutions.

On the other hand, if $\epsilon < 0$, then there are three solutions in the cases of both x and y, out of which we present two of them below and the third (a novel superposed solution) in the next subsection.

Solution I: One pair of solutions are

$$x(t) = A\sqrt{m}\mathrm{cn}[\beta(t + t_0), m], \tag{40}$$

provided

$$\epsilon A^2 = -2\beta^2, \quad (2m - 1)\beta^2 = -(1 - k), \tag{41}$$

while

$$y(t) = B\sqrt{m_1}\mathrm{cn}[\beta_1(t + t_1), m_1], \tag{42}$$

is an exact solution provided

$$\epsilon B^2 = -2\beta_1^2, \quad (2m_1 - 1)\beta_1^2 = -(1 + k). \tag{43}$$

Solution II: The other solutions are

$$x(t) = A\mathrm{dn}[\beta(t + t_0), m], \tag{44}$$

provided

$$\epsilon A^2 = -2\beta^2, \quad (2 - m)\beta^2 = -(1 - k) > 0 \tag{45}$$

while

$$y(t) = B\mathrm{dn}[\beta_1(t + t_1), m_1], \tag{46}$$

is an exact solution provided

$$\epsilon B^2 = -2\beta_1^2, \quad (2 - m_1)\beta_1^2 = -(1 + k) > 0. \tag{47}$$

In the limit $m = m_1 = 1$, both the solutions I and II go over to the corresponding hyperbolic soliton solutions.

Once we know the solutions for x and y, one can immediately obtain solutions in the case of u and v. Here t_0, t_1 are arbitrary constants.

3.4 *Novel superposition for nonlinear equations*

Recently, Khare and Saxena [14] have pointed out a novel superposition in the case of periodic solutions of nonlinear equations. In this study, it is shown that whenever a nonlinear equation has periodic solutions in terms of Jacobi elliptic functions dn and cn, then the same equation also admits solutions in terms of a linear superposition of the two. We now discuss two such examples in the context of the present dimer problem.

Example I: We have shown above that when $\gamma = 0, \delta + \epsilon < 0$, then the coupled dimer eqs (1) and (2) have periodic solutions in terms of the Jacobi elliptic functions dn and cn (see eqs (22)–(25)). We now show that, remarkably, even a linear superposition of the above two are exact solutions of the coupled dimer eqs (1) and (2). In particular, it is easy to check that

$$u(t) = v(t) = \frac{A}{2}(\mathrm{dn}[\beta(t+t_0), m] \pm \sqrt{m}\mathrm{cn}[\beta(t+t_0), m]), \tag{48}$$

is an exact solution provided

$$(\epsilon + \delta)A^2 = -2\beta^2, \quad (1+m)\beta^2 = 2(k-1) > 0, \tag{49}$$

while

$$u(t) = -v(t) = \frac{A}{2}(\mathrm{dn}[\beta(t+t_0), m] \pm \sqrt{m}\mathrm{cn}[\beta(t+t_0), m]), \tag{50}$$

is an exact solution provided

$$(\epsilon + \delta)A^2 = -2\beta^2, \quad (1+m)\beta^2 = -2(k+1) > 0. \tag{51}$$

In the limit $m = 1$, while one of these superposed solutions goes over to the same hyperbolic solution (26), the other one goes to the vacuum solution $u = v = 0$.

Example II: We have shown in the last subsection that when $\gamma = 0, \ \delta = 3\epsilon < 0$, then the coupled dimer eqs (1) and (2) have periodic solutions in terms of the Jacobi elliptic functions dn and cn (see eqs (40)–(43) and (44)–(47)). We now show that, remarkably, even a linear superposition of the above two are also exact solutions of the coupled dimer eqs (1) and (2). In particular, it is easy to check that

$$x(t) = (A/2)\mathrm{dn}[\beta(t+t_0), m] + (D/2)\sqrt{m}\mathrm{cn}[\beta(t+t_0), m], \tag{52}$$

is an exact solution provided

$$D = \pm A\,, \ \ \epsilon A^2 = -2\beta^2\,, \ \ (1+m)\beta^2 = -2(1-k) > 0 \tag{53}$$

while

$$y(t) = (B/2)\mathrm{dn}[\beta_1(t+t_1), m_1] + (E/2)\sqrt{m_1}\mathrm{cn}[\beta_1(t+t_1), m_1]\,, \tag{54}$$

is an exact solution provided

$$E = \pm B, \quad \epsilon B^2 = -2\beta_1^2, \quad (1+m_1)\beta_1^2 = -2(1+k) > 0. \tag{55}$$

In the limit $m = m_1 = 1$, while one of the superposed solution goes over to the same hyperbolic soliton solution (sech), the other one goes to the vacuum solution $x(t) = y(t) = 0$.

Another kind of novel superposition has also been pointed out [14]. In particular, it is shown that whenever a nonlinear equation admits periodic solution in terms of dn^2, then the same equation also admits solutions in terms of the superposition of dn^2 and cndn, irrespective of whether cndn is an exact periodic solution of that nonlinear equation or not.

4. Fixed point analysis

Let us now discuss the coupled dimer problem as a dynamical system and look at the fixed points and their stability. Let us define

$$\dot{u} = w, \quad \dot{v} = x, \tag{56}$$

in terms of which we have four equations

$$\dot{u} = f_1(w, x, u, v) = w, \tag{57}$$

$$\dot{v} = f_2(w, x, u, v) = x, \tag{58}$$

$$\dot{w} = f_3(w, x, u, v) = -u + kv + \gamma w + \epsilon u^3 + \delta u v^2, \tag{59}$$

$$\dot{x} = f_4(w, x, u, v) = -v + ku - \gamma x + \epsilon v^3 + \delta v u^2. \tag{60}$$

Hence the fixed points are

$$u = v = 0, \quad u = v = \pm\sqrt{\frac{1-k}{\epsilon+\delta}}, \quad u = -v = \pm\sqrt{\frac{1+k}{\epsilon+\delta}},$$
$$u^2 + v^2 = \frac{1}{\epsilon}, \quad uv = \frac{k}{\epsilon - \delta}. \tag{61}$$

Note however that in the special case when $\delta = \epsilon$, the asymmetric fixed points with $u^2 + v^2 = 1/\epsilon, uv = k/(\epsilon - \delta)$ are not admitted and we only have three fixed points in that case. Thus so long as $\delta \neq \epsilon$, the exact coupled eqs (1) and (2) admit symmetric (S), antisymmetric (AS) as well as asymmetric fixed points. We shall see in the next section that the RWA for the same model only admits S and AS but not asymmetric solutions.

It is straightforward to set up the stability matrix by calculating derivatives of f_i $(i = 1, 2, 3, 4)$ with respect to u, v, w, x. We then find that the eigenvalues satisfy the equation

$$\lambda^4 - \lambda^2[\gamma^2 - 2 + (3\epsilon + \delta)(u^2 + v^2)] + \gamma\lambda[(3\epsilon - \delta)(v^2 - u^2)]$$
$$+[(3\epsilon u^2 + \delta v^2 - 1)(3\epsilon v^2 + \delta u^2 - 1) - (k + 2\delta uv)^2] = 0. \tag{62}$$

Hence the characteristic values at the various fixed points can be worked out and one can understand the nature of the fixed point as a function of parameters. For example,

$u = v = 0$:

$$\lambda^2 = \frac{\gamma^2}{2} - 1 \pm \sqrt{\frac{\gamma^4}{4} - \gamma^2 + k^2}. \tag{63}$$

$u = v = \pm\sqrt{\dfrac{1-k}{\epsilon+\delta}}$:

$$\lambda^4 - \lambda^2\left[\gamma^2 + \frac{4\epsilon - 2k(3\epsilon+\delta)}{\epsilon+\delta}\right] + \frac{4(1-k)(\epsilon - \delta - 2\epsilon k)}{\epsilon+\delta} = 0. \tag{64}$$

$u = -v = \pm\sqrt{\dfrac{1+k}{\epsilon+\delta}}$:

$$\lambda^4 - \lambda^2\left[\gamma^2 + \frac{4\epsilon + 2k(3\epsilon+\delta)}{\epsilon+\delta}\right] + \frac{4(1+k)(\epsilon - \delta + 2\epsilon k)}{\epsilon+\delta} = 0. \tag{65}$$

At the asymmetric point, in general we get a quartic equation. However, in the special case when $\delta = 3\epsilon$ one obtains quadratic equation with simple solution, i.e.

$$\delta = 3\epsilon, \quad u^2 + v^2 = \frac{1}{\epsilon}, \quad uv = \frac{-k}{2\epsilon}:$$
$$\lambda^2 = \frac{\gamma^2}{2} + 2 \pm \sqrt{\frac{\gamma^4}{4} + 2\gamma^2 + 4k^2}. \tag{66}$$

Thus while two of the roots are real, the other two roots are complex.

5. Rotating wave approximation

We now consider the model within the rotating wave approximation (RWA). The advantage is that it approximates the system by a nonlinear Schrödinger type PT-symmetric dimer for which everything can be solved analytically, including the stationary states, the symmetry breaking bifurcations and even the full dynamics.

We start with the ansatz

$$u(t) = \phi_1(t)\mathrm{e}^{i\omega_b t} + \phi_1^*(t)\mathrm{e}^{-i\omega_b t}, \tag{67}$$

$$v(t) = \phi_2(t)\mathrm{e}^{i\omega_b t} + \phi_2^*(t)\mathrm{e}^{-i\omega_b t}. \tag{68}$$

The RWA approximation assumes that $\dot{\phi}_{1,2} \ll \omega_b \phi_{1,2}$ and $\ddot{\phi}_{1,2} \ll \omega_b \dot{\phi}_{1,2}$ (i.e. $\phi_{1,2}$ varies slowly on the scale of the oscillations of the actual exact time periodic scale), and further discards the term multiplying $\exp(\pm 3i\omega_b t)$. In this approximation, the coupled dimer eqs (1) and (2) transform into a set of coupled nonlinear Schrödinger-type PT-symmetric dimer

$$2i\omega_b\dot{\phi}_1 = [\omega_b^2 - 1 + 3\epsilon|\phi_1|^2 + 2\delta|\phi_2|^2 + i\omega_b\gamma]\phi_1 + (k + \delta\phi_1^*\phi_2)\phi_2, \tag{69}$$

$$2i\omega_b\dot{\phi}_2 = [\omega_b^2 - 1 + 3\epsilon|\phi_2|^2 + 2\delta|\phi_1|^2 - i\omega_b\gamma]\phi_2 + (k + \delta\phi_2^*\phi_1)\phi_1. \tag{70}$$

5.1 *Symmetries of RWA equations*

It is interesting to note that the RWA eqs (67) and (69) remain invariant under the following transformations:

$$\begin{aligned}
&(1)\ (\phi_1, \phi_2) \longrightarrow (-\phi_1, -\phi_2), \quad (2)\ (\phi_1, k) \longrightarrow (-\phi_1, -k),\\
&(3)\ (\phi_2, k) \longrightarrow (-\phi_2, -k), \quad (4)\ (\phi_1, \phi_2, \gamma) \longrightarrow (-\phi_2, -\phi_1, -\gamma),\\
&(5)\ (t, \gamma, \omega_b) \longrightarrow (-t, -\gamma, -\omega_b),\\
&(6)\ (\phi_1, \phi_2, \epsilon, \delta) \longrightarrow \left(\alpha\phi_1, \alpha\phi_2, \frac{\epsilon}{\alpha^2}, \frac{\delta}{\alpha^2}\right).
\end{aligned} \tag{71}$$

5.2 *Exact solutions*

We now obtain several stationary solutions of the RWA eqs (69) and (70). To that purpose, we write

$$\phi_1 = A\mathrm{e}^{i\theta_1}, \quad \phi_2 = B\mathrm{e}^{i\theta_2}, \quad \phi = \theta_2 - \theta_1, \tag{72}$$

and then on separating the real and imaginary parts, we obtain

$$EA = pB\cos\phi + qAB^2\cos 2\phi + A(A^2 + 2qB^2), \tag{73}$$

$$EB = pA\cos\phi + qBA^2\cos 2\phi + B(B^2 + 2qA^2), \tag{74}$$

$$A\sin\phi(p + 2qAB\cos\phi) = -\Gamma B, \tag{75}$$

$$B\sin\phi(p + 2qAB\cos\phi) = -\Gamma A. \tag{76}$$

Here

$$E = \frac{1-\omega_b^2}{3\epsilon}, \quad p = \frac{k}{3\epsilon}, \quad q = \frac{\delta}{3\epsilon}, \quad \Gamma = \frac{\gamma\omega_b}{3\epsilon}. \tag{77}$$

Let us first discuss the Hamiltonian case, i.e. $\gamma = 0$.

5.3 $\gamma = 0$

In this case we have either $\sin\phi = 0$ or $\cos\phi = -p/2qAB$. Let us discuss these two cases separately one by one.

5.3.1 $\gamma = 0, \sin\phi = 0$.

In this case the symmetric, antisymmetric and asymmetric solutions exist. In particular, it is easy to check that the symmetric and antisymmetric solutions are

$$B = \pm A, \quad A^2 = \frac{E \mp p}{1+3q} = \frac{1-\omega_b^2 \mp k}{3(\epsilon+\delta)}, \tag{78}$$

provided $\delta + \epsilon \neq 0$. When $\delta + \epsilon = 0$ then $E = \pm p$, i.e. $1 - \omega_b^2 = \pm k$ and A cannot be determined.

On the other hand, if $\delta \neq \epsilon$ then one has the asymmetric solution

$$B = \frac{k}{3(\epsilon-\delta)A}, \quad A^2 = \frac{(1-\omega_b^2) \pm \sqrt{(1-\omega_b^2)^2 - \frac{4k^2\epsilon^2}{(\epsilon-\delta)^2}}}{6\epsilon}. \tag{79}$$

Note that when $\delta = \epsilon$, then this asymmetric solution is not allowed.

5.3.2 $\gamma = 0, \cos\phi = -p/2qAB$.

In this case too all three, i.e. symmetric, antisymmetric and asymmetric solutions are possible. In particular, the symmetric and the antisymmetric solutions are given by

$$B = \pm A\,, \quad A^2 = \frac{E}{1+q} = \frac{1-\omega_b^2}{(\delta+3\epsilon)}\,,$$
$$\cos\phi = \mp\frac{p(1+q)}{2qE} = \mp\frac{k(\delta+3\epsilon)}{2\delta(1-\omega_b^2)}\,, \tag{80}$$

provided $q \neq -1$, i.e. $\delta \neq -3\epsilon$. When $q = -1$, then the solution takes the simpler form

$$B = \pm A, \quad \delta = -3\epsilon, \quad \omega_b^2 = 1, \quad \cos\phi = \pm\frac{k}{6\epsilon A^2}. \tag{81}$$

Note that in this case A cannot be determined.

In the special case of $q = 1$, i.e. $\delta = 3\epsilon$, one also has the asymmetric solution

$$\delta = 3\epsilon, \quad E = A^2 + B^2, \quad \cos\phi = -\frac{p}{2AB}. \tag{82}$$

Note that in this case A, B are arbitrary but with the proviso that they must satisfy the constraint (82).

5.4 $\gamma \neq 0$

Let us now discuss the general case when $\gamma \neq 0$. In this case, the asymmetric solution is no longer a stationary solution and only symmetric and antisymmetric solutions exist as exact stationary states. On dividing eq. (75) by eq. (76), we immediately find that $B = \pm A$. In both the cases, eqs (73)–(76) can be simplified as a quartic equation for A^2:

$$\begin{aligned}
&(1+3q)^2(1+q)^2A^8 - 4E(1+q)(1+2q)(1+3q)A^6 \\
&\quad +[4(1+2q)^2E^2 + 2(1+3q)(1+q)(\Gamma^2+E^2) - (1+q)^2p^2]A^4 \\
&\quad +2E[(1+q)p^2 - 2(1+2q)(\Gamma^2+E^2)]A^2 \\
&\quad +(\Gamma^2+E^2)(\Gamma^2+E^2-p^2) = 0.
\end{aligned} \tag{83}$$

On the other hand, the phase fulfills the equation

$$\tan(\phi) = -\frac{\Gamma}{E-(1+q)A^2}\,. \tag{84}$$

In fact, it is also possible to eliminate A^2 and show that in both the cases of $B = \pm A$, the phase ϕ satisfies the constraint

$$Eq\sin(2\phi) \pm p(1+q)\sin(\phi) + \Gamma[1+q+2q\cos^2(\phi)] = 0. \tag{85}$$

Note that there are two special cases (i.e. when $q = -1, -1/3$) when A^2 only satisfies quadratic equation.

5.5 *A novel exact solution of the RWA equations*

In addition to the solutions just discussed, the coupled eqs (69) and (70) also have a novel exact solution

$$\phi_1 = \mathrm{e}^{(\gamma/2)t-(ia/\gamma)\sinh(\gamma t+\phi)-ibt}, \quad \phi_2 = \mathrm{e}^{-(\gamma/2)t-(ia/\gamma)\sinh(\gamma t+\phi)-ibt}, \tag{86}$$

provided

$$b = \frac{\omega_b^2-1}{2\omega_b}, \quad a = \frac{\sqrt{3\epsilon(k+3\delta)}}{\omega_b}, \quad \mathrm{e}^{2\phi} = \frac{3\epsilon}{k+3\delta}\,. \tag{87}$$

5.6 *Superintegrability of RWA*

We now show that our RWA equations, in fact, correspond to a superintegrable system. First, note that the RWA eqs (69) and (70) can be derived from the Hamiltonian

$$H = \frac{3\epsilon}{4\omega_b}(|\phi_1|^4 + |\phi_2|^4) + \frac{\delta}{2\omega_b}|\phi_1|^2|\phi_2|^2 + \frac{(\omega_b^2 - 1)}{2\omega_b}(|\phi_1|^2 + |\phi_2|^2)$$
$$+\frac{i\gamma}{2}(|\phi_1|^2 - |\phi_2|^2) + \frac{k}{2\omega_b}(\phi_1^*\phi_2 + \phi_2^*\phi_1)$$
$$+\frac{\delta}{4\omega_b}(\phi_1^*\phi_2 + \phi_2^*\phi_1)^2. \tag{88}$$

We now show that, in addition, there are other constants of motion associated with the RWA equations. To that end, we introduce the so-called Stokes variables

$$X = \frac{1}{2}(\phi_1^*\phi_2+\phi_2^*\phi_1), \quad Y = \frac{i}{2}(\phi_1^*\phi_2-\phi_2^*\phi_1), \quad Z = \frac{1}{2}(|\phi_1|^2-|\phi_2|^2). \tag{89}$$

Using the RWA eqs (69) and (70), it is easy to compute $\dot{X}, \dot{Y}, \dot{Z}$ and $\dot{r}$ where $r^2 = X^2 + Y^2 + Z^2$ and using them one can show that when $\delta \neq \epsilon$ or $\delta \neq 3\epsilon$, the two constants of motion for the RWA case are C_1, C_2 given by

$$C_1 = (aX - b)^2 + Y^2, \quad a^2 = \frac{3(\epsilon - \delta)}{3\epsilon - \delta}, \quad b^2 = \frac{k^2}{3(\epsilon - \delta)(3\epsilon - \delta)}, \tag{90}$$

$$C_2 = r + f\sin^{-1}\left(\frac{AX - b}{\sqrt{C_1}}\right), \quad f = -\frac{\gamma\omega_b}{a(3\epsilon - \delta)}. \tag{91}$$

Note that when $\delta = \epsilon$, the two constants are $C_1 = Y^2 - (kX/\epsilon)$ and $C_2 = r - (\gamma\omega_b Y/k)$. On the other hand, when $\delta = 3\epsilon$, the two constants are $C_1 = X$ and $C_2 = r - (\gamma\omega_b Y)/(6\epsilon C_1 + k)$.

6. Numerical analysis

We now briefly discuss the numerical analysis of some of the exact solutions (see [13] for details). We have first studied the stability of various solutions obtained within RWA *vis à vis* the exact dimer model. For $\gamma = 0$, we find that for the hard nonlinearity case, i.e. $\epsilon, \delta < 0$, the agreement between the two is even quantitatively accurate. However, in the soft nonlinearity case, i.e. $\epsilon, \delta > 0$, RWA becomes less accurate as the frequency ω decreases away from the linear limit. When $\gamma \neq 0$, we have made detailed comparison between the RWA and exact dimer model when $\delta = \epsilon, \delta = 3\epsilon/2$ and $\delta = 3\epsilon$. When $\delta = \epsilon$, the exact solution at $\delta = \epsilon$ appears to defy the PT-phase transition existing in case γ is nonzero. Finally, we have also numerically examined the dynamical evolution of some of the unstable modes [13]. In some cases we also found that the instabilities identified in the analysis were either due to indefinite growth or bounded quasiperiodic oscillations.

7. Open problems

There are several possible open questions. One question is, can one extend the analysis to the case of three or even four coupled oscillators. Secondly, while here we have only considered the cubic nonlinearity, it would be worthwhile to examine the effect of quadratic nonlinearities and see how their nonlinear states get deformed when $\gamma \neq 0$.

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