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Construction of classical and quantum integrable field models unravelling hidden possibilities

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Abstract. Reviewing briefly the concept of classical and quantum integrable systems, we propose an alternative Lax operator approach, leading to quasi-higher-dimensional integrable model, unravelling some hidden dimensions in integrable systems. As an example, we construct a novel integrable quasi-two-dimensional NLS equation at the classical and the quantum levels with intriguing application in rogue wave modelling.

Keywords. Classical and quantum integrable models; Yang–Baxter equation; higher space-Lax operator; quasi-higher-dimensional field models; rogue waves; integrable defect models.

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1. Introduction

Nonlinear systems, though more prevalent, are difficult to solve in general. However, as we know now, there are nonlinear models, mostly in (1 + 1)-dimensional space-time, which belong to integrable systems, both at classical and quantum levels. Some of them are only classically integrable, some are integrable only as quantum models, while some others are integrable both at the classical and the quantum levels.

1.1 *Classical and quantum integrable systems*

The aims, scopes and the methods of the integrable classical and quantum systems however, are different, though they share the basic ideas at a fundamental level (see [2]). The aim of the classical integrable systems is to focus mainly on evolution equations, investigate their various properties like the symmetry, separability, transformations, integrability criteria etc. and whenever possible, to find exact solutions in the form of soliton, breather or rational solutions. A representative example is the NLS equation

$$iq_t + q_{xx} + 2|q|^2 q = 0, \tag{1}$$

for complex scalar function $q(x,t)$ in x, t, which will be our main focus in the present investigation. The subscripts x and t indicate partial derivatives in the corresponding variables.

The quantum integrable systems on the other hand aim mainly at solving the eigenvalue problem (EVP) for the quantum Hamiltonian (energy) operator: $\mathcal{H}|\Psi_n\rangle = E_n|\Psi_n\rangle$. For the quantum NLS field model the Hamiltonian may be given by

$$\mathcal{H} = \int \mathrm{d}x : (q_x^\dagger(x,t) q_x(x,t) + \alpha(q^\dagger(x,t) q(x,t))^2) :, \tag{2}$$

where $q, q^\dagger$ are the creation/annihilation operators acting in a Hilbert space, in the second quantized NLS field model. The problems relevant to the quantum models also seek to find the excited states, finite-temperature thermodynamic quantities, various correlation functions etc. in an analytic way.

1.2 *Liouville integrability*

This is a strong and beautiful integrability criterion related to rich Nöther symmetries of such systems, demanding the existence of independent higher conserved quantities: C_j, $j = 1, 2, \ldots, N$ in involution, where the total number N of the conserved charges should coincide with the degrees of freedom of the system. For field models having ∞ degrees of freedom, the integrability naturally demands the existence of $N \to \infty$ number of conserved quantities.

For classical integrable systems one should have $\{C_j, C_k\} = 0$, $j \neq k$, demanding independence of all conserved charges. Similarly, for quantum models, the quantum integrability requires the commutativity of all conserved operators: $[C_j, C_k] = 0$.

1.3 *Yang–Baxter equation*

A class of integrable models, known as the ultralocal models, may have a deep underlying algebraic relation, called the Yang–Baxter equation (YBE), validity of which would guarantee the integrability of a system in the Liouville sense. For quantum and classical systems, the YBE takes different form, though the classical and the quantum cases are related through proper transition limit. Since the basic structure of the Liouville integrability criteria and the YBE involve the Poisson brackets (PB) in the classical or the commutator relations (CR) in the quantum case, for highlighting these concepts we need to define such structures along with a local object called the space-Lax operator $U(\lambda, x)$ and a space-independent matrix, known as the $R(\lambda - \mu)$-matrix, both however are dependent on the additional spectral parameter λ, μ.

The classical Yang–Baxter equation (CYBE) through the above objects may be expressed as

$$\{U_1(\lambda, x), U_2(\mu, x')\} = i[r(\lambda - \mu), U_1(\lambda, x) + U_2(\mu, x)]\delta(x - x'), \tag{3}$$

where $U_1 \equiv U \otimes I$, $U_2 \equiv I \otimes U$, and r is the classical limit of the quantum R-matrix: $R \to r + O(\hbar)$.

Similarly, the quantum Yang–Baxter equation (QYBE) for the discretized Lax operator $U^j(\lambda)$ may be given by

$$R(\lambda-\mu)U_1^j(\lambda)U_2^j(\mu)=U_2^j(\mu)U_1^j(\lambda)R(\lambda-\mu), \qquad j=1,2,\ldots. \tag{4}$$

It can be shown, that from the local Lax operator $U(\lambda, x)$, the generator for all higher conserved charges of the model can be constructed and at the same time the Liouville integrability criteria can also follow, provided the Lax operator satisfies the YBE together with a nonoverlap condition at different space points: $[U_1(\lambda, x), U_2(\mu, y)] = 0$, for $x \neq y$, known as the untralocality condition.

2. Generation of conserved charges and integrable hierarchy

As mentioned above, the integrable systems admit N-number of conserved charges, matching with the degrees of freedom of the system, which can be generated using the space-Lax operator $U(\lambda, x)$ or its discrete analogue, representing the integrable system. Let us demonstrate this important construction on the example of the NLS field equation, though the generality of this approach allows similar construction for any other integrable field model at the classical level.

The space-Lax equation in its well-known form is given as

$$\Phi_x^i=\sum_j U^{ij}(\lambda,x)\Phi^j, \tag{5}$$

where we consider U as a 2×2 matrix and the Jost function Φ^i as a two-component vector, where Φ^1 can be shown to be linked directly to the generator of the conserved quantities as

$$\Phi^1(x,\lambda)=\mathrm{e}^{\int^x \rho(\lambda,x')\mathrm{d}x'+U_\infty^{11}(\lambda)x}. \tag{6}$$

Using the Lax equation (5), the charge density generator may be expressed as

$$\rho(\lambda,x)+U_\infty^{11}(\lambda)=(\ln\Phi^1)_x=\frac{\Phi_x^1}{\Phi^1}=U^{11}+U^{12}\Gamma(\lambda) \tag{7}$$

through a generating function $\Gamma(\lambda) = \Phi^2(\Phi^1)^{-1}$. It is not difficult to check whether the determining equation for $\Gamma(\lambda)$ can be derived from the Lax equation (5) in the form of the Riccati equation:

$$\Gamma_x=U^{21}+(U^{22}-U^{11})\Gamma-U^{12}\Gamma^2. \tag{8}$$

Therefore, we can obtain the set of conserved quantities $C_j=\int_{-\infty}^{\infty}\mathrm{d}x\rho_j, j=1,2,\ldots$, by expanding the generalized density function (7) as $\rho(\lambda,x)+i\lambda x=\sum_{j=1}^{\infty}\rho_j(x)\lambda^{-j}$. The charge densities $\rho_j(x)$ can be determined from the Riccati equation (8) for each model, by knowing the explicit form of the Lax operator matrix U^{ij}, $i, j = 1, 2$ and by expanding eq. (7) in the powers of parameter λ.

2.1 *Integrable hierarchy*

It is interesting to note that the infinite set of conserved quantities associated with an integrable system can be used for generating a hierarchy of higher integrable equations. By considering any of the conserved charges $C_j, j = 1, 2, \ldots$, as the Hamiltonian of the system, we can derive the corresponding Hamilton equation for the field $q_t = \{q, C_j\}$, $t = t_j, j = 1, 2, \ldots$, by defining a proper PB structure. The infinite set of conserved quantities therefore can yield an integrable hierarchy of PDEs, one of which coincides usually with the well-known integrable equation and the rest of the higher-order equations are called the integrable hierarchy of this representative equation. Given a Lax operator $U(\lambda)$, representing an integrable system, we can follow the above prescription to find the whole set of conserved charges and the hierarchy of integrable equations, for all classical integrable models. For quantum models however such a procedure cannot be adopted, though we can still find the conserved operators for exact discrete models, while for field models we have to rely mostly on their classical analogs. We provide now a list of integrable models, which satisfy the YBE.

2.2 *List of integrable models satisfying the YBE*

(I) *Discrete models*
(i) The following models are integrable both as classical and quantum systems.

- (1) Toda chain
- (2) Relativistic Toda chain
- (3) Exact lattice NLS models
- (4) Ablowitz–Ladik model

(ii) Models integrable only at the quantum level.

- (5) Isotropic XXX spin-$\frac{1}{2}$ chain
- (6) Anisotropic XXZ spin-$\frac{1}{2}$ chain (related to the 6-vertex model)
- (7) t–J model
- (8) 1D Hubbard model

(II) *Field models*
(i) The following models are integrable both as classical and quantum systems

- (9) NLS model
- (10) Derivative NLS model
- (11) Massive Thirring model (bosonic)
- (12) Sine-Gordon model
- (13) Liouville model

(ii) Equation integrable as a classical system

- (14) Landau–Lifshitz equation

(iii) Models integrable as a quantum system

- (15) δ-function Bose gas
- (16) δ'-function Bose gas

All the models listed above (see [1] and [2] for more details on these models and introduction to classical and quantum integrable systems, respectively) have spectral parameter-dependent Lax operators associated with them together with an R-matrix (classical or quantum), which satisfy the YBE. Note however, that while the R-matrix for the above models takes only two different forms: rational or trigonometric depending on the class to which the models belong, the Lax operators of different models turn out to be different. However, interestingly, as we elaborate below, all different Lax operators are in fact reducible from the same Ancestor model [3].

3. The NLS model

Let us focus now exclusively on the NLS model as a concrete example for demonstrating all the important features of an integrable system presented above in general terms.

3.1 *Conserved charges, integrable hierarchy and integrability for the NLS model*

We can derive explicit expressions for the conserved charges using the example of the well-known NLS model, by following the steps formulated above. First we need the Lax operator for the model, which can be given in the well-known AKNS form [1]

$$U = i\begin{pmatrix} \lambda & q \\ q^* & -\lambda \end{pmatrix}. \tag{9}$$

Now, using explicit matrix elements of the Lax operator (9), in the Riccati equation (8) and expanding $\Gamma(\lambda) = \sum_{j=1}^{\infty} \Gamma_j \lambda^{-j}$, in powers of the spectral parameter λ^{-j} (with the assumption that the series is convergent at $\lambda^{-1} \to 0$), we can derive a recurrence relation for the coefficient of the expansion Γ_j as

$$\Gamma_{jx} = -2i\Gamma_{j+1} - q\sum_{k=1}^{j-1}\Gamma_k\Gamma_{j-k}, \tag{10}$$

for $j > 1$, $\Gamma_1 = -(i/2)q^*$, which can systematically extract all solutions of Γ_j, $j = 1, 2, \ldots$. Therefore, by expanding (7) also in the powers of λ^{-1} we get the relation $\rho_j = q\,\Gamma_j, j = 1, 2, \ldots$, and obtain the densities of all the conserved charges $C_j = \int_{-\infty}^{\infty} \mathrm{d}x\ \rho_j$, $j = 1, 2, \ldots$, deriving thus the infinite set of conserved quantities for the NLS system in the explicit form as

$$\begin{aligned} C_1 &= \int \mathrm{d}x\ |q|^2, \\ C_2 &= i\int \mathrm{d}x\ (q_x^* q - q^* q_x), \\ C_3 &= \int \mathrm{d}x\ (q_x^* q_x + |q|^4), \end{aligned} \tag{11}$$

and so on.

For constructing integrable hierarchy for the NLS equation, following the steps mentioned above, we define first an equal-time PB relation in the form

$$\{q(x,t), q^*(x',t)\} = i\delta(x - x') \tag{12}$$

for the scalar field q and its conjugate q^*. Note that this fundamental PB must be consistent with the integrability criteria guaranteed by YBE (3). It can be checked easily, that using (12) and taking the Hamiltonian as $H = C_3$, we can derive the NLS equation (1) as a Hamilton equation $iq_t = \{q, H\}$, with a time $t = t_2$, having scaling dimension 2 (see [4] for more on the scaling dimension).

However, if we take $H = C_4$ as the Hamiltonian, using the same PB, we would arrive at a higher NLS equation with third-order dispersion:

$$iq_t - (q_{xxx} + 6|q|^2 q_x) = 0, \quad \text{c.c.} = 0, \tag{13}$$

for time $t = t_3$ with scaling dimension 3.

Similarly with higher and higher $H = C_j$, $j > 5$, using the same PB (12) we can generate the whole integrable NLS hierarchy $iq_{t_j} = \{q, C_j\}$, $j = 2, 3, \ldots$, having increasingly higher-order dispersion and nonlinearity related to higher-order scaling.

Note however, that the existence of infinite conserved quantities alone cannot confirm the integrability of the model. The integrability can be guaranteed only when these conserved charges commute among themselves, which in turn could follow from the YBE. Therefore, for proving the integrability of the NLS equation, one has to prove the validity of the YBE (3) (similarly (4) in the quantum case). Notice that, for the NLS model with the Lax operator (9) in the AKNS form, the YBE fortunately yields only a few nontrivial relations because of the simple structure of the NLS Lax operator. In fact, among a total of 16 relations, only a single one together with its conjugate, related to the PB between the matrix elements $\{U_{12}(\lambda, x), U_{21}(\mu, x')\}$ remains nontrivial, which can be solved easily using the canonical PB (12). Similarly, the quantum YBE for the NLS field model can also be solved (up to the required order: $O(\Delta)$ due to discretization, needed in the quantum case). We do not provide the details here, because they are available in [2].

Let us pause here and note an important issue, not emphasized much in the literation and exploit it to arrive at our result presented below. This is the appearance of a multitime picture: $q(x, t_1, t_2, \ldots, t_j, \ldots)$ in describing the higher-order equations in the integrable hierarchy (see for example [5] illuminating this concept for the Toda chain). Our intention is to revert the concept of multitime to multispace for our construction of higher-dimensional integrable models. Though we find that such a construction does not yield genuine 2D models, the $(2 + 1)$-dimensional NLS model we construct exhibits interesting properties and applications in quasi-two dimensions.

4. Alternative Lax operator for generating integrable systems

As pointed out above, in spite of explicit differences between the Lax operators of integrable models satisfying the YBE, there is a beautiful universality among them, in the sense that all of them are reducible from the same model, the so-called Ancestor model [3], and basically have the same structure with a linear dependence in the spectral parameter or its q-deformation, as observed also in all descendant models. In fact all of them are linked to the AKNS form of the Lax operator having a scaling dimension of 1 [4].

Therefore, as it stands, all integrable models satisfying the YBE are likely to be generated from the Lax operators which have the simplest AKNS-type dependence on the spectral parameter (note that the models belonging to the Kaup–Newell spectral problem with a different spectral parameter dependence can either be represented as the q-deformation of the AKNS-type Lax operators [6] or do not satisfy the YBE and therefore go beyond the scope of the present investigation).

A few natural questions arise at this point.

(1) Should we have to confine always to the AKNS (or its deformation) type Lax operators with scaling dimension 1 (with linear dependence on the spectral parameter, or its q-deformation), satisfying the YBE, as happens for all known integrable models we have listed above?
(2) Can one use any alternative Lax operator, perhaps generated by an Ancestor model of higher scaling dimension?
(3) Apart from the simple canonical or bosonic PB (or commutator) or their q-deformation, do other forms of the PB compatible with the YBE exist?

4.1 *Our approach*

Motivated by the above queries, we intend to look for and use an alternative Lax operator, different from the AKNS type, limiting ourselves to the NLS-type model (though our approach is universal enough to be applicable to other systems). In search of new ideas we turn to the known integrable hierarchy of the NLS equation and notice that each equation in the hierarchy describes the evolution along a new higher time $t = t_j$, $j = 1, 2, \ldots$, with an additional Lax equation $\Phi_{t_j} = U^{(j)}$ with Lax operator $U^{(j)}$, representing an infinitesimal translation operator along time direction t_j and having higher nonlinear dependence on the spectral parameter λ. This multitime dimension hidden in $(1 + 1)$-dimensional integrable systems inspire us to explore Lax operator $U^{(2)}$ with λ^2 dependence. We therefore focus on the Lax operator of scaling dimension 2, involving a complex scalar field $q(x, y, t)$ by renaming $t_2 = y$, $t_3 = t$. This defines $U^{(2)} \equiv \mathcal{U}(\lambda)$ in quasi-$(2 + 1)$ space-time, having the form

$$\mathcal{U}(\lambda) = i \begin{pmatrix} 2\lambda^2 - q^* q & 2\lambda q - i q_x, \\ -2\lambda q^* + i q_x^*, & -2\lambda^2 + q^* q \end{pmatrix}, \tag{14}$$

and associated with the Lax equation

$$\Phi_y = \mathcal{U}(\lambda, x, y)\Phi. \tag{15}$$

Note the appearance of higher powers of the spectral parameter up to λ^2 and the derivative of the field: q_x in (14). We hope to derive a novel integrable higher-dimensional NLS model using the alternative Lax operator (14) and shall see below how much we can achieve through such an approach and what are its limitations.

4.2 *Conserved charges of the new system*

We focus first on the derivation of the infinite set of conserved quantities associated with our new integrable system and the related hierarchy of integrable equations, because such an information is crucial for an integrable model and is a strong indication of its integrability, at least at the classical level. We have already elaborated on the procedure for extracting the full set of conserved charges, when the Lax operator $U(\lambda)$ for the model is given, on the example of the well-known NLS equation in §3. Therefore, we shall follow the same steps, replacing only the Lax operator with (14) and x-integration by the integration along y-direction, generating the conserved charges as

$$\mathcal{C}_n = \int_{-\infty}^{\infty} \mathrm{d}y\ \rho_n, \quad n = 1, 2, \ldots. \tag{16}$$

Note however that due to the more complicated structure of the Lax operator (14) in comparison with the NLS case, the relations presented for the NLS model become more involved in the present case. For example, for the charge density we get

$$\rho_n = -iq_x\ \Gamma_n + \ q\ \Gamma_{n+1}, \qquad n = 1, 2, \ldots \tag{17}$$

while the recurrence relations derived from the corresponding Riccati equation gives

$$-i\Gamma_{ny} = -\Gamma_{j+2} + 2|q|^2\Gamma_j + iq_x \sum_{k=1}^{n-1} \Gamma_k\Gamma_{n-k} - q\sum_{k=1}^{n-1}\Gamma_k\Gamma_{n+1-k}, \tag{18}$$

for $j > 2$, $\Gamma_1 = q^*$, $\Gamma_2 = iq_x^*$, which can extract all solutions of Γ_n, $n = 1, 2, \ldots$, systematically. Therefore, inserting in (17) the solution of the recurrence relation (18), we can derive finally the explicit expressions for all conserved charges for our model in the form

$$\begin{aligned}
\mathcal{C}_1 &= i\int \mathrm{d}y(q_x^*q - q^*q_x),\\
\mathcal{C}_2 &= \int \mathrm{d}y(iq_y^*q + q_x^*q_x + (q^*q)^2 + cc),\\
\mathcal{C}_3 &= \int \mathrm{d}y\, q_y^*q_x,\\
\mathcal{C}_4 &= \int \mathrm{d}y(iq_{xy}^*q_x + q_y^*q_y - i|q|^2(q^*q_y - q_y^*q)\\
&\quad -2|q|^2\, q_x^*q_x + (q^{*2}q_x^2 + q_x^{*2}q^2)),
\end{aligned} \tag{19}$$

and so on.

4.3 *Classical integrability of the model*

Recall that for showing the Liouville integrability for this system, we have to show $\{\mathcal{C}_j, \mathcal{C}_k\}_2 = 0$, which may follow from the YBE. On the other hand, for obtaining the integrable hierarchy for our system one should use the Hamilton equations $iq_{t_n} = \{q, \mathcal{C}_n\}_2$, $n = 1, 2, \ldots$. Consequently, for establishing both these relations we need to discover a suitable PB structure $\{,\}_2$, consistent with the YBE (3), because the known PB (12) for the NLS equation will not work in this case due to more complicated form of the Lax operator having a scaling dimension of 2.

First let us concentrate on the classical YBE, which for our model will take the form

$$\{\mathcal{U}_1(y,\lambda), \mathcal{U}_2(y',\mu)\}_2 = \delta(y-y')[r(\lambda-\mu), \mathcal{U}_1(y,\lambda) + \mathcal{U}_2(y,\mu)], \tag{20}$$

with the same rational 4×4, r-matrix as for the well-known 1D NLS model: $r(\lambda-\mu) = (P/2(\lambda-\mu))$ with P as the permutation operator. Recall that the present YBE with 4×4,

r-matrix represents in total 16 algebraic relations involving Poisson brackets and matrix multiplications. It is important to note that out of these relations only six relations remain trivial (four diagonal plus two extreme off-diagonal terms), while it gives ten nontrivial relations which need to be proved. Comparing with the standard NLS case discussed above, where only two relations are nontrivial, the complexity of the present case due to its association with a higher-order Lax operator is eminent.

Interestingly, however, we find that if we define a novel PB structure in the form

$$\{q(y), q_x^*(y')\}_2 = \delta(y - y'), \ \{q(y), q^*(y')\}_2 = 0, \tag{21}$$

all the ten nontrivial YBE relations of the model are satisfied exactly, proving the integrability of the present 2D NLS model. Note also that the presence of the derivative term along x direction in the Lax operator (14) and PB (21) does not violate the ultralocality condition necessary for proving the integrability of the model, because the ultralocality is needed here only along the y direction which is indeed valid. We skip the details of the proof of the YBE and switch in §5 to our $(2+1)$-dimensional integrable NLS model and its application.

4.4 *Quantum integrability of the model*

To show that the NLS model in 2D, we are considering, is also integrable at the quantum level, we have to prove the validity of the quantum YBE, which in the present case may be expressed as

$$R(\lambda - \mu)\mathcal{U}_1^j(x, \lambda)\mathcal{U}_2^j(x, \mu) = \mathcal{U}_2^j(x, \mu)\mathcal{U}_1^j(x, \lambda)R(\lambda - \mu), \tag{22}$$

$j = 1, 2, \dots N$, where we have taken the quantum R-matrix in the same rational form as is well known for the standard NLS model. As discretization is needed in the quantum case to avoid short distance singularities, we have to make a naive discretization of the Lax operator $\mathcal{U}(x, y, \lambda)$ given in (14) along the $y \to j$-direction: $\mathcal{U}^j(x, \lambda) = I + \Delta\mathcal{U}(x, y = j, \lambda)$, Δ being the lattice constant along the discretized lattice $j = 1, 2, \dots$. Due to the complicated form of the quantum Lax operator $\mathcal{U}^j(x, \lambda)$ with higher power of λ and the quantum YBE (22) it gives (unlike the standard NLS case with only two nontrivial relations) ten nontrivial commutation relations. However, miraculously, all the ten YBE relations are satisfied when we quantize the PB relation with the CR

$$[q_j, \ (q_k^\dagger)_x] = \frac{1}{\Delta}\, \delta_{jk}. \tag{23}$$

Note that in spite of the presence of the x-derivative term in (23), the necessary ultralocality condition along $y \equiv j$ direction remains valid. It should also be noted that unlike in the classical case, the quantum YBE relations are satisfied here not exactly but up to the order $O(\Delta)$, which however is enough for a quantum field model like in the present 2D NLS field model, obtained at $\Delta \to 0$.

We briefly present the main steps of the algebraic Bethe ansatz for exact solution of the eigenvalue problem for all conserved operators including the Hamiltonian for our quantum integrable model, while a detailed analysis will be given elsewhere [12].

It can be shown that (thanks to the underlying Hopf algebra), when an ultralocal Lax operator $U^j(\lambda)$ satisfies the QYBE, a global operator defined as $T(\lambda) = \prod_{j=1}^M U^j(\lambda)$, (called monodromy matrix) also satisfies the same QYBE, with its trace generating all conserved operators

$$\text{trace } T(\lambda) = \tau(\lambda) = A(\lambda) + A^{\dagger}(\lambda), \qquad \ln \tau(\lambda) = \sum_j C_j \lambda^{-j},$$

including the Hamiltonian. Note that, apart from the trace identity $[\tau(\lambda), \tau(\mu)] = 0$ derivable from the QYBE and confirming the quantum integrability of the system, one can also obtain the CR between the diagonal ($T_{11}(\lambda) = A(\lambda)$) and the off-diagonal ($T_{12}(\lambda) = B(\lambda)$, $T_{21}(\lambda) = B^{\dagger}(\lambda)$) operator elements of the monodromy matrix from the same QYBE. For our quantum 2D NLS model satisfying the QYBE with the known rational R-matrix the CR takes a simpler form as

$$A(\lambda)B(\mu_j) = f_j(\lambda - \mu_j)B(\mu_j)A(\lambda),$$
$$f_j(\lambda - \mu_j) \equiv \frac{\lambda - \mu_j + i(\alpha/2)}{\lambda - \mu_j - i(\alpha/2)} \tag{24}$$

and we can define a pseudovacuum state $|0\rangle$: $B^{\dagger}(\mu_j)|0\rangle = 0$, $A(\lambda)|0\rangle = |0\rangle$ with $B(\mu_j)$, $B^{\dagger}(\mu_j)$ as the generalized creation and annihilation operators, respectively. The required relation (24) also indicates that we can construct an exact M-particle eigenstate in the form $|M\rangle = B(\mu_1)B(\mu_2)\cdots B(\mu_M)|0\rangle$, and solve exactly the EVP as

$$A(\lambda)|M\rangle = F_M(\lambda, \mu_1, \mu_2, \ldots, \mu_M)|M\rangle, \qquad F_M = \prod_j^M f_j(\lambda - \mu_j), \tag{25}$$

using repeatedly CR (24), which consequently solves also the EVP for all the conserved operators C_j through some algebraic steps using the expansion in λ. For our model Hamiltonian $H = C_4$, for the M-particle scattering state, in the case of repulsive coupling $\alpha > 0$, following the above steps we can derive the energy eigenvalue as $E_M = \sum_{j=1}^{M} \mu_j^4$. On the other hand, for attractive interaction: $\alpha = -\gamma < 0$, we can similarly obtain M-particle bound states (called quantum solitons).

We shall not discuss further on the quantum aspects of our model, which has been introduced in [7] and will be detailed elsewhere [12]. In the following sections we focus on the classical aspects of our model together with its possible application.

5. Integrable higher-dimensional NLS equation

Let us consider $\mathcal{C}_4$ given in (19) as the Hamiltonian of the model, which with the use of the PB (21) gives a 2D NLS equation for the field $q(x, y, t)$ as

$$iq_t + q_{xy} + 2iq(q_x^* q - q^* q_x) = 0, \tag{26}$$

which with a rotation in the (x, y) plane can be written in a more traditional $(2+1)$-dimensional form

$$iq_t + q_{xx} - q_{yy} + 2iq(j^x - j^y) = 0, \tag{27}$$

where $j^a = q_a^* q - q^* q_a,\ a = x, y$ are the field current components. It is interesting to compare eq. (27) with the well-known $(2+1)$-dimensional NLS equation [8]

$$iq_t + q_{xx} - q_{yy} + 2q|q|^2 = 0, \tag{28}$$

which however, unlike our model (27), is a nonintegrable equation without having stable soliton solutions. On the other hand, the integrability of the present model has been proved above, both at the classical and the quantum levels with infinite number of conserved quantities $\mathcal{C}_j,\ j = 1, 2, \ldots$ (19) in involution, each of which taken as Hamiltonian leads to the hierarchy of integrable equations evolving along time t_j. It is interesting to note that taking $\mathcal{C}_2$ as the Hamiltonian would lead to a NLS-type equation

$$iq_y + q_{xx} + 2q|q|^2 = 0, \tag{29}$$

with t replaced by y in its known form (1).

For obtaining stable soliton solutions for our (2 + 1)-dimensional NLS equation (26) we can follow the well-formulated Hirota's bilinear method by expressing the field as

$$q(x, y, t) = \frac{G(x, y, t)}{F(x, y, t)}, \tag{30}$$

where $G(x, y, t)$ and $F(x, y, t)$ are complex and real functions, respectively. Inserting this ansatz in eq. (26) one derives the pair of bilinear equations:

$$i(FG_t - GF_t) + (FG_{xy} + GF_{xy} - G_x F_y - G_y F_x) = 0, \tag{31}$$

$$2i(GG^*_x - G^* G_x) + 2(F_x F_y - FF_{xy}) = 0. \tag{32}$$

Now following the standard steps with a formal expansion

$$F = 1 + \epsilon^2 F_2 + \epsilon^4 F_4 + \cdots, \qquad G = \epsilon G_1 + \epsilon^3 G_3 + \cdots, \tag{33}$$

we obtain different equations at different orders in ϵ, using which recurrently, we can systematically derive N-soliton solutions. Referring for details to [9] we present here only the exact 1-soliton solution for eq. (26)

$$\begin{aligned} q(x, y, t) &= \frac{G_1}{1 + F_2} = \frac{\mathrm{e}^{\eta_1}}{1 + \alpha \mathrm{e}^{(\eta_1 + \eta_1^*)}} \\ &= \operatorname{sech}(x - v_1 y - v_2 t)\mathrm{e}^{i(k_x x + k_y y + \omega t)}, \\ \eta_1 &= k_1 x + p_1 y - w_1 t + \eta_1^0, \end{aligned} \tag{34}$$

where the relationship between the phase modulation and the soliton velocity components along the x and the y directions can be given exactly. A diagrammatic picture of this stable soliton solution is shown in figure 1. Similarly, for finding 2-soliton solution, one has to go up to G_3, F_4 and extend the starting solutions with a second soliton argument $\eta_2 = k_2 x + p_2 y - w_2 t + \eta_2^0$. Details of this and higher soliton solutions are presented in [9].

We focus now on a possible application of our 2D NLS equation (26) in modelling the ocean rogue waves.

5.1 *Ocean rogue wave model*

Ocean rogue wave is one of the mysteries of nature, which has not been understood fully and modelled satisfactorily. Rogue waves are isolated surface waves, which may appear suddenly in the deep sea, can attain enormous heights and disappear again without a trace,

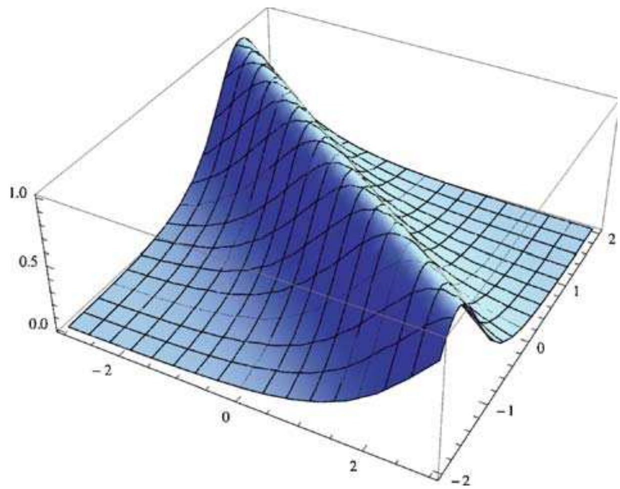

Figure 1. The localized amplitude profile of an exact line soliton of the 2D NLS model on the x, y plane at $t = 0.0$.

preceded usually by strange hole waves in the sea water. Nonlinearity and modulation instability coupled with ocean currents are supposed to be crucial factors in the formation of rogue waves.

Most of the available models of the rogue waves, in spite of the ocean rogue waves being 2D surface waves, are based on the well-known 1D NLS equation, with a 1D rogue wave-like solution known as the Peregrine–Akhmadev breathers [11]. Moreover, such exact solutions do not contain any free parameters and as a result can model only waves of fixed amplitude and steepness. Usually, there is also no scope for considering the influence of the crucial ocean currents in these models.

Based on our 2D NLS equation, we propose a 2D surface wave model for the rogue waves, where we also include the effect of a nonconstant ocean current term $I(x, y, t) = -iU_c q_x$ to get the governing equation in the form

$$iq_t + q_{xy} + 2iq(qq_x^* - q^*q_x) = iU_c q_x, \tag{35}$$

where we take the ocean current as $U_c = \mu t/\alpha x$. It can be shown by direct insertion that a 2D dynamical lump soliton given by

$$q_{(2d)}(x, y, t) = \mathrm{e}^{4iy}\left(-1 + \frac{1 - i4y}{\alpha x^2 + \mu t^2 + 4y^2 + c}\right) \tag{36}$$

is an exact solution of the 2D NLS equation with an ocean term (35).

Note that the lump soliton (36) possesses many features essential for an ocean rogue wave model. Most important of them is the 2D nature of the solution, which reduces the rogue wave solution, as can be seen from (35), to a background plane-wave configuration, in the distant past ($t \to -\infty$). At $t = 0$, a high and steep 2D lump peaked at $x = y = 0$, appears together with the hole waves. This sudden high wave disappears again to a plane wave in the future ($t \to +\infty$), as the time passes (see figures 2–5 [10]), exhibiting properties close to a realistic ocean rough wave.

Another desirable feature of our rough wave model (36) is the presence of several free parameters c, α, μ, which for different continuum choices for these parameters allows to

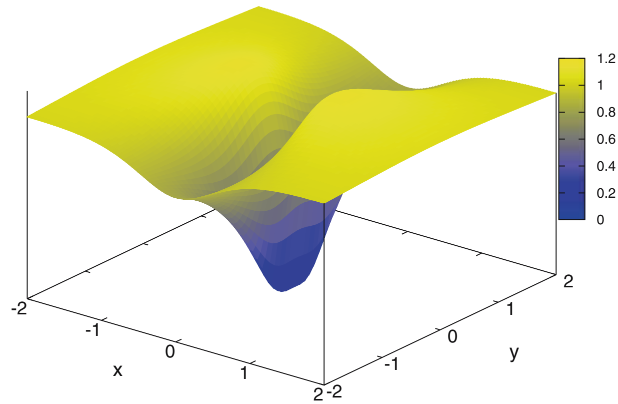

Figure 2. Creation of 2D hole at a certain time, as told in marine-lores.

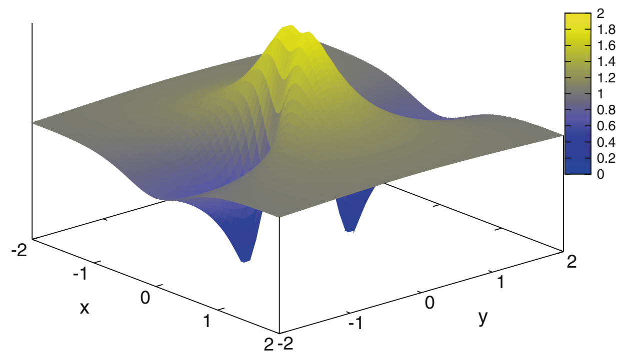

Figure 3. The hole wave splits into two and drifted away from the centre as the time progresses.

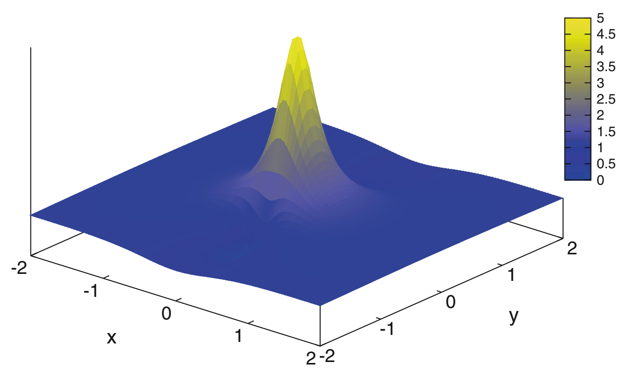

Figure 4. The full grown RW at $t = 0.0$ formed at the centre.

tune the amplitude and steepness of the wave solutions to fit different forms and shapes of the ocean rogue waves observed in nature and in laboratory experiments.

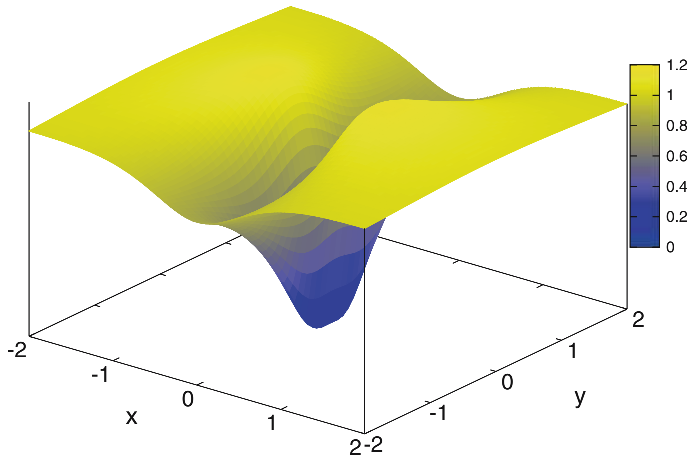

Figure 5. Repeating a similar picture as the time passes, with the reappearance of a hole wave and the decreasing wave amplitude as we proceed towards the future.

Note also that in our rogue wave model, the ocean current term plays a crucial role in controlling the dynamics of the rogue wave solution represented by parameter μ, by determining the speed of appearance of the rough wave, duration of its stay etc. Recall that in the formation of real ocean rough wave, the ocean currents also play important roles, which however could not be considered in most of the rough wave models suggested earlier in the literature.

6. Concluding remarks

Our idea is to use an alternative to the universally considered AKNS-type Lax operator (or its trigonometric deformations) and construct a novel integrable system with a new hierarchy. At the same time, we exploit the concept of multitime hidden in integrable systems to construct multispace models. We have demonstrated this approach using the example of the NLS equation and derived a $(2+1)$-dimensional NLS field model, classical and quantum Liouville integrability of which is proved by the Yang–Baxter equation, by defining a novel commutation relation (and PB) for the basic field. This higher-dimensional NLS model admits exact line soliton and a lump soliton, which together with an ocean current term gives a realistic model for the 2D ocean rough waves.

However, one should note at the same time, that though this class of models can be defined in multidimensional space giving a novel set of conserved quantities, which we have derived explicitly for our $(2+1)$D NLS model, such conserved charges, unlike in a genuine 2D model, are integrated only along the y direction. This quasi-2D nature of the model also indicates that the integrable hierarchy obtained for the model is not an independent set and linked in fact in an involved nonlinear way to the known NLS hierarchy. In fact, it can be shown that combining eqs (26) and (29) of our hierarchy, one can derive the higher NLS equation (13). Nevertheless, it is interesting to study such quasi-multispace models with a novel PB and CR structure, in their own right both at the classical and the quantum levels with possible applications in higher dimensions, as we have demonstrated by a realistic ocean rough wave model in 2D.

Though we have focussed here on the construction of a 2D NLS model, due to the model-independent nature of our approach, it is equally applicable to other integrable models like the Landau–Lifshitz equation, Toda chain etc., the possibility of which is under investigation.

An exciting challenge will be to construct quantum integrable lattice spin and other discrete models on a plane, using our alternative Lax operator approach. The main problem in realizing this scheme however, is the nonavailability of interesting alternative Lax operators in the discrete case. Another challenge is to find an Ancestor model together with its q-deformation, which can generate higher-order Lax operators representing novel higher-dimensional models, along with higher-dimensional quantum algebras, extending the existing knowledge.

References

[1] S Novikov, V Manakov, L Pitaevskii and V Zakharov, *Theory of solitons* (Plenum, New York, 1984)
L D Faddeev and Leon A Takhtajan, *Hamiltonian methods in the theory of solitons* (Springer, Berlin, 2007)
P Kulish and E K Sklyanin, *Lecture Notes in Physics* edited by J Hietarinta *et al* (Springer, Berlin, 1982) Vol. 151, p. 61
M Lakshmanan and S Rajasekar, *Nonlinear dynamics: Integrability, chaos and patterns* (Springer, Berlin, 2003)
M Lakshmanan, in: *Classical and quantum nonlinear integrable systems* edited by A Kundu (IOP Publishing, Bristol, UK, 2003) 3

[2] L D Faddeev, *Sov. Sci. Rev. C* **1**, 107 (1980)
Michio Jimbo (Ed), *Yang-Baxter equation in integrable systems* (World Scientific, Singapore, 1993)
V E Korepin, N M Bogolubov and A G Izergin, *Quantum inverse scattering method and correlation functions* (Cambridge Univ Press, 1997)
Anjan Kundu, in: *Classical and quantum nonlinear integrable systems* (IOP Publishing, Bristol, UK, 2003) 147

[3] B Basumallick and A Kundu, *Mod. Phys. Lett. A* **10**, 2955 (1995)
A Kundu, *Phys. Rev. Lett.* **82**, 3936 (1999)
A Kundu, *Rep. Math. Phys.* **46**, 125 (2000)

[4] A Kundu, *J. Math. Phys.* **50**, 102702 (2009)
A Kundu, in: *Nonlinear dynamics* edited by M Daniel and S Rajasekar (Narora Publication, New Delhi, 2009) p. 25

[5] Yu B Suris, *Variational formulation of commuting hamiltonian flows: Multi-time lagrangian 1-forms*, preprint: arXiv:1212.3314 [math-ph], v2 (2012)

[6] A Kundu and B Basu-mallick, *J. Math. Phys.* **34**, 1052 (1993)

[7] A Kundu, *J. Phys. C* **S 482**, 012022 (2014)
A Kundu, *Unraveling hidden hierarchies and dual structures in an integrable field model*, arXiv:1201.0627 [nlin.SI] (2012)

[8] A R Osborne, M Onorato and M Serio, *Phys. Lett. A* **275**, 386 (2000)

[9] A Kundu and A Mukherjee, *Novel integrable higher-dimensional nonlinear Schroedinger equation: Properties, solutions, applications*, arXiv:1305.4023 [nlin.SI] (2013)

[10] A Kundu, A Mukherjee and T Naskar, *Proc. R. Soc. A* **470**, 20130576 (2014)

[11] D H Peregrine, *Austral. Math. Soc. B* **25**, 16 (1983)
N N Akhmediev, V M Eleonskii and N E Kulagin, *Theor. Math. Phys.* **72**, 809 (1987)

[12] A Kundu, *submitted for publication*