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Commutation and Darboux transformation

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Abstract. In this paper we show that the Darboux transformation for a large class of nonlinear evolution equations arises due to factorization and commutation. The factorization and commutation has been pointed out earlier for Schrödinger operator. We show that it extends to a large class of nonlinear differential equations which admit Lax pairs including Boussinesq, Davey–Stewartson, Bogoyavlensky–Schiff and n-wave interaction equation.

Keywords. Darboux transformation; nonlinear evolution equations; exact solutions.

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1. Introduction

Darboux transformations were first introduced [1] to construct new solutions for the Sturm–Liouville equation

$$-y'' + u(x)y = \lambda y \tag{1}$$

from a given solution. It is well known [2] that the Darboux transformation for the Sturm–Liouville equation is a particular application of the commutation method.

Darboux transformations have also been investigated for a large class of nonlinear evolution equations that are solvable by inverse scattering transform. These include, for example, the nonlinear Schrödinger equation, the Korteweg–de Vries (KdV) equation, the AKNS hierarchy, the three-wave interaction equation, sine-Gordon equation, two (space) dimensional equations like Davey–Stewartson equation, and discrete analogues, like the Toda lattice equation, non-Abelian Toda lattice equations and the lattice Silin–Tikhonchuk equation [3–6]. However, it is not clear from the methods introduced so far whether Darboux transformation is in fact based on commutation in all the above cases, just as it is for the Schrödinger equation.

We reinterpret the Darboux transformation for the Sturm–Liouville equation in such a manner that the commutation method has an immediate extension to all the above-mentioned completely integrable nonlinear evolution equations. If $L[1]$ is the Darboux transformation of a differential operator L, then there exists a differential operator A of order one, such that $L[1]A = AL$, i.e., A interwines the operator $L[1]$ and L. The intertwining property has been pointed out for the Schrödinger equation earlier. We show that it also extends to other operators and to the corresponding nonlinear evolution equations. Application of this method to the KdV equation is reviewed and then the extensions to the Boussinesq and Davey–Stewartson equations are given.

We apply this method to the Bogoyavlensky–Schiff (BS) equation which is a particular case of the generalized Calogero–Bogoyavlensky–Schiff (CBS) equation. Exact solutions using Lie symmetries were studied in [7,8]. Exact and analytical solutions for the CBS equation using the genaralized Riccati equation expansion are given in [9]. The application of this method to the Bogoyavlensky–Konopelchenko equation is given in [10]. In §3 exact solutions of the BS equations using our method are given. In §4 n-wave interaction equation is considered.

2. Commutation method

In this section we recall how to look at Darboux transformation as an application of factorization and commutation for the KdV equation.This method is then applied to Boussinesq equation and the extended Davey–Stewartson system.

It is well known that the Schrödinger operator $-(\mathrm{d}^2/\mathrm{d}x^2) + u - \lambda$ can be written as a product of two first-order operators. This is sometimes attributed to d'Almbert and is traditionally referred to as the method of reduction (p. 88 of [2]).

2.1 *Korteweg–de Vries equation*

The Korteweg–de Vries equation is

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2}$$

It arises as the compatibility condition of the equations

$$L\varphi \equiv \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u\right)\varphi = \lambda\varphi \tag{3}$$

and

$$T\varphi \equiv \varphi_t + 4\varphi_{xxx} - 6u\varphi_x - 3u_x\varphi = 0. \tag{4}$$

Let ψ_1 be a fixed solution of (3) for $\lambda = \lambda_1$, i.e., $L\psi_1 = \lambda_1\psi_1$ and let

$$A\varphi = \left(\psi_1\frac{\partial}{\partial x}\psi_1^{-1}\right)\varphi, \tag{5}$$

then $A\varphi = \varphi_x - \sigma\varphi$ with $\sigma = \psi_{1x}\psi^{-1} = (\log\psi_1)_x$. Let

$$B\varphi = \psi_1\int_0^x \psi_1^{-1}\varphi\mathrm{d}x, \tag{6}$$

then $AB\varphi = \varphi$ and $BA\varphi = \varphi - \psi_1\psi_1^{-1}(x=0)\varphi(x=0)$. A version of this operator B is used in [6] to prove covariance theorem. However, it has not been mentioned explicitly that this is a consequence of commutation. Note that B is a right inverse of A. However, A is not a right inverse of B. The Darboux transformation technique depends crucially on the fact that the error $BA - I$ maps into kernel of A, that is $A(BA - I) = 0$. An immediate calculation gives

Theorem 1. *Let ψ_1, A and B be as above and $F\varphi = (L - \lambda_1)B\varphi$, then $F\varphi = -(\varphi_x + \sigma\varphi) = A^*\varphi$ and $AF\varphi = (L[1]-\lambda_1)\varphi$ where $L[1] = \left(-(\mathrm{d}^2/\mathrm{d}x^2) + u[1]\right)$, $u[1] = u - 2\sigma_x$ and A^* is the formal adjoint of A.*

The above theorem shows the factorization and commutation version of the Darboux transformation for the Schrödinger equation, i.e., first formally factorize the operator L as FA (with $F = A^*$) and then commute F and A to get a new operator $L[1]$ with u replaced by $u[1]$.

Now we show that the same method works for the time evolution of the Lax pair, i.e., eq. (4). To prove that a similar result holds for the operator T we use the following result:

Let $L\psi = 0$ and $T\psi = 0$ be the Lax pairs giving rise to a nonlinear evolution equation in 1+1 or 2+1 dimensions. Let ψ_1 be a solution of both $L\psi = 0$ and $T\psi = 0$ equations. Let A and B be as in (5) and (6) respectively. Then we have

Theorem 2. *Let $T\psi_1 = 0$ and $G = TB$. Then $GA - T = K_{\psi_1}$ where the image of K_{ψ_1} is contained in the kernel of A.*

Proof. Formally writing $G = TB$ and using BA value and the fact that $T\psi_1 = 0$, have

$$\begin{aligned}(GA - T)\varphi &= T(BA\varphi) - T\varphi \\ &= T[\varphi - \psi_1\psi_1^{-1}(0,\cdot)\varphi(0,\cdot)] - T\varphi \\ &= -T[\psi_1\psi_1^{-1}(0,\cdot)\varphi(0,\cdot)].\end{aligned}$$

Now for the KdV equation $T\varphi = (\partial_t + \partial_{xxx} - 6u\partial_x - 3u_x)\varphi = (\partial_t + P(\partial_x))\varphi$, where $\partial_t = \partial/\partial t$, $\partial_{xxx} = \partial^3/\partial x^3$ and $\partial_x = \partial/\partial x$, and $P(\partial_x) = \partial_{xxx} - 6u\partial_x - 3u_x$. Therefore

$$\begin{aligned}T[\psi_1\psi_1^{-1}(0,\cdot)\varphi(0,\cdot)] &= (\partial_t\psi_1)\psi_1^{-1}(0,\cdot)\varphi(0,\cdot) + \psi_1\partial_t(\psi_1^{-1}(0,\cdot)\varphi(0,\cdot)) \\ &\quad +\{P(\partial_x)\psi_1\}\psi_1^{-1}(0,\cdot)\varphi(0,\cdot) \\ &= (T\psi_1)\psi_1^{-1}(0,\cdot)\varphi(0,\cdot) + \psi_1\partial_t(\psi_1^{-1}(0,\cdot)\varphi(0,\cdot)) \\ &= \psi_1\partial_t(\psi_1^{-1}(0,\cdot)\varphi(0,\cdot)) \\ &= K_{\psi_1}\varphi.\end{aligned}$$

Thus $AK_{\psi_1}\varphi = A(\psi_1\partial_t(\psi_1^{-1}(0,\cdot)\varphi(0,\cdot))) = \partial_t(\psi_1^{-1}(0,\cdot)\varphi(0,\cdot))A\psi_1 = 0$ □

Suppose that ψ_1 satisfies the Lax pair equations (3) and (4) for potential u and eigenvalue λ_1. Let ψ be any solution of (3) and (4) for potential u and eigenvalue λ. Let A and B be as in (5) and (6), then we have

Lemma 1. *Let $G = TB$ then*

$$\begin{aligned}G\varphi &= \psi_1\int_0^x \psi_1^{-1}\varphi_t\mathrm{d}x - \psi_1\int_0^x \psi_1^{-1}\psi_{1t}\psi_1^{-1}\varphi\mathrm{d}x \\ &\quad +4(\varphi_{xx} + \sigma^2\varphi + \sigma\varphi_x + 2\sigma_x\varphi) - 6u\varphi\end{aligned}$$

and

$$AG\varphi = \varphi_t + 4\varphi_{xxx} - 6u[1]\varphi_x - 3u_x[1]\varphi = T[1]\varphi$$

Proof. A straightforward calculation gives

$$\begin{aligned} AG\varphi &= \psi_1 \frac{\mathrm{d}}{\mathrm{d}x}(\psi_1^{-1} G\varphi) = (G\varphi)_x - \sigma(G\varphi) \\ &= \varphi_t + 4\varphi_{xxx} - 6u[1]\varphi_x - 3u_x[1]\varphi. \end{aligned}$$ □

Finally we have the following theorem:

Theorem 3. *Let* ψ_1 *satisfy* (3) *and* (4) *with eigenvalue* $\lambda = \lambda_1$. *Let* ψ *satisfy* (3) *and* (4) *for* $\lambda (\neq \lambda_1)$. *Then* $\psi[1] = A\psi$ *is a solution of*

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u[1]\right)\varphi = \lambda\varphi \tag{7}$$

and

$$\varphi_t + 4\varphi_{xxx} - 6u[1]\varphi_x - 3u_x[1]\varphi = 0 \tag{8}$$

with $u[1] = u - 2\sigma_x$.

Proof. We only have to verify that $\psi[1] = A\psi$ satisfies (8). Now by the previous lemma,

$$T[1]\varphi = AG\varphi = \varphi_t + 4\varphi_{xxx} - 6u[1]\varphi_x - 3u_x[1]\varphi,$$

which implies that

$$T[1]A\psi = AGA\psi = A(TB)(A\psi) = A(T - K_{\psi_1})\psi$$

Therefore

$$T[1]A\psi = AT\psi = 0.$$ □

Note that if $AG = T[1]$, we have just verified that $T[1]A = AT$, i.e., A interwines $T[1]$ and T. This shows that if u satisfies the KdV equation, so does $u[1]$.

The map $u \mapsto u[1]$, $\psi \mapsto \psi[1]$ is called the Darboux transformation for the KdV equation. Iterations of this result gives rise to the Crum theorem for solutions of the Schrödinger operator. Even though this result is well known, the reason for writing it in this manner is that it has an immediate extension to a wide class of nonlinear evolution equations.

We now give the application of the above method to the Boussinesq equation.

2.2 *The Boussinesq equation*

The Boussinesq equation

$$w_{tt} - w_{xx} + (w^2)_{xx} + \frac{1}{3}w_{xxxx} = 0 \tag{9}$$

arises as the compatibility condition of

$$L\varphi \equiv \left[\partial_x^3 + \left(\frac{3w}{2} - \frac{3}{4}\right)\partial_x + u\right]\varphi = \lambda\varphi \tag{10}$$

$$T\varphi \equiv (\partial_t + \partial_{xx} + w)\varphi = 0, \tag{11}$$

where u and w are scalar functions. The compatibility condition of (10) and (11) gives rise to the following equations:

$$u_x = \frac{3w_{xx}}{4} - \frac{3w_t}{4}, \tag{12}$$

$$u_t = \frac{w_{xxx}}{4} + \frac{3(w^2)_x}{4} - \frac{3w_x}{4} + \frac{3w_{xt}}{4}. \tag{13}$$

Eliminating u from these two systems gives (9). Again, let ψ_1 be a fixed invertible solution of (10) for $\lambda = \lambda_1$. A and B be as in (5) and (6), then we have the following lemma:

Lemma 2. *If* $F = LB$, *then*

$$F\varphi = 2\psi_{1xx}\psi_1^{-1}\varphi - \sigma^2\varphi + \sigma\varphi_x + \varphi_{xx} + \left(\frac{3w}{2} - \frac{3}{4}\right)\varphi + \lambda_1 B\varphi \tag{14}$$

and

$$L[1]\varphi = AF\varphi = \left(\partial_x^3 + \left(\frac{3w[1]}{2} - \frac{3}{4}\right)\partial_x + u[1]\right)\varphi \tag{15}$$

and $L[1]A\varphi = AL\varphi$ *with* $w[1] = w + 2\sigma_x$, *and* $u[1]$ *equal to a polynomial in* ψ_1, ψ_{1x}, ψ_{1xx}, w *and* w_x.

Proof. The expression for $F\varphi$ is immediate. The expression for $L[1]\varphi = AF\varphi = (F\varphi)_x - \sigma(F\varphi)$ then follows by using the equation satisfied by ψ_1. □

We now show that this result extends to the time evolution as well. We have the following lemma:

Lemma 3. *If* $G = TB$, *then*

$$G\varphi = -\psi_1 \int_0^x \psi_1^{-1}\psi_{1t}\psi_1^{-1}\varphi \mathrm{d}x + \psi_1 \int_0^x \psi_1\varphi_t \mathrm{d}x + \sigma\varphi + \varphi_x, \tag{16}$$

$$AG\varphi = (\partial_t + \partial_{xx} + w[1])\varphi = T[1]\varphi, \tag{17}$$

and

$$T[1]A\varphi = AT\varphi.$$

Proof. The expression for $G\varphi$ and $AG\varphi = T[1]\varphi$ are straightforward, though involved computations. $T[1]A\varphi = AGA\varphi = ATBA\varphi = A(T - K_{\psi_1})\varphi = AT\varphi$. □

Thus, if ψ is any solution of (10) and (11), then $A\psi$ satisfies $L[1]\varphi = \lambda\varphi$ and $T[1]\varphi = 0$. This shows that $w[1] = w + 2\sigma_x$ is also a solution of the Boussinesq equation.

2.3 *The extended Davey–Stewartson system*

The extended Davey–Stewartson system is [3]

$$\psi_y = J\psi_x + U\psi, \tag{18}$$

$$\psi_t = 2i\alpha^{-1}J\psi_{xx} + 2i\alpha^{-1}U\psi_x + V\psi, \tag{19}$$

where

$$U = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix};\ J = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix};\ V = \begin{pmatrix} (w+iQ)/2 & i\alpha^{-2}(\alpha u_x + u_y) \\ i\alpha^{-2}(\alpha v_x - v_y) & (w-iQ)/2 \end{pmatrix}.$$

Let ψ_1 be a fixed matrix valued invertible solution of (18). Let A and B be as before. Let $L = (\partial_y - J\partial_x - U)$.

Lemma 4. *If $F = LB$ then*

$$F\varphi = \psi_1\int_0^x \psi_1^{-1}\varphi_y \mathrm{d}x - \psi_1\int_0^x \psi_1^{-1}\psi_{1y}\psi_1^{-1}\varphi \mathrm{d}x - J\varphi$$

and

$$AF\varphi = \varphi_y - J\varphi_x - \{U + [J,\sigma]\}\varphi$$

Proof.

$$\begin{aligned} AF\varphi &= \psi_1\partial_x\left(\int_0^x \psi_1^{-1}\varphi_y \mathrm{d}x - \int_0^x \psi_1^{-1}\psi_{1y}\psi_1^{-1}\varphi \mathrm{d}x - \psi_1^{-1}J\varphi\right) \\ &= \varphi_y - J\varphi_x - \{U + (J\sigma - \sigma J)\}\varphi, \end{aligned} \qquad \square$$

as required. Thus, we have

Lemma 5. *If ψ is a solution of $L\psi = 0$, then $A\psi = \psi_x - \sigma\psi$ is a solution of $L[1]\varphi = 0$.*

It is easy to see that if ψ_1 and ψ satisfy (18) then $A\psi$ satisfies $L[1]\varphi = 0$. We now show that this extends to the time evolution equation as well. Let

$$T\varphi \equiv (\partial_t - 2i\alpha^{-1}J\partial_{xx} - 2i\alpha^{-1}U\partial_x - V)\varphi = 0.$$

Let ψ_1 be an invertible solution of $L\varphi = 0$ and $T\varphi = 0$ and let A and B be as before.

Lemma 6. *If $G\varphi = TB\varphi$ then*

$$\begin{aligned} G\varphi = &-\psi_1\int_0^x \psi_1^{-1}\psi_{1t}\psi_1^{-1}\varphi \mathrm{d}x + \psi_1\int_0^x \psi_1^{-1}\varphi_t \mathrm{d}x \\ &-2i\alpha^{-1}J\psi_{1x}\psi_1^{-1}\varphi - 2i\alpha^{-1}J\varphi_x - 2i\alpha^{-1}\varphi \end{aligned}$$

and $AG = T[1]$, where

$$\begin{aligned} T[1]\varphi = \varphi_t - 2i\alpha^{-1}J\varphi_{xx} - 2i\alpha^{-1}(U + [J,\sigma])\varphi_x \\ - \{V + 2i\alpha^{-1}(\sigma_y + J\sigma_x)\}\varphi \end{aligned}$$

Proof.

$$\begin{aligned} AG\varphi &= (G\varphi)_x - \sigma(G\varphi) \\ &= \varphi_t - 2i\alpha^{-1}(J\varphi_{xx} + U\varphi_x + J\sigma\varphi_x - \sigma J\varphi_x) \\ &\quad -V\varphi - 2i\alpha^{-1}(\sigma_y + J\sigma_x)\varphi. \end{aligned}$$

Therefore, $AG\varphi = T[1]\varphi$. □

Theorem 4. *$T[1]A\varphi = AGA\varphi = AT\varphi$. Thus if ψ satisfies* (18) *and* (19), *then $T[1]A\psi = 0$.*

Proof.

$$T[1]A\varphi = AGA\varphi = A\{T - K_{\psi_1}\}\varphi = AT\varphi.$$

Thus

$$T[1]A\psi = AT\psi = 0.$$ □

This shows that $A\psi$ is a solution of $T[1]\varphi = 0$ with

$$T[1] = (\partial_t - 2i\alpha^{-1}J\partial_{xx} - 2i\alpha^{-1}U[1]\partial_x - V[1])$$

where $U[1] = U + [J, \sigma]$, $V[1] = V + 2i\alpha^{-1}(\sigma_y + J\sigma_x)$ and $\sigma = \psi_{1x}\psi_1^{-1}$.

The map $U \mapsto U[1]$, $V \mapsto V[1]$ is the Darboux transformation for the Davey–Stewartson system.

3. Exact solutions of the Bogoyavlensky–Schiff equation

The generalized Calogero–Bogoyavlenskii–Schiff (GCBS) equation is given by

$$\alpha u_{xt} + \beta u_x u_{xy} + \delta u_y u_{xx} + u_{xxxy} = 0, \tag{20}$$

where α, β and δ are constants. Some special cases of eq. (20) have been studied by many authors. When $\alpha = 4$, $\beta = 4$, $\delta = 2$, eq. (20) becomes the Bogoyavlenskii–Schiff equation [11,12]:

$$4u_{xt} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xxxy} = 0. \tag{21}$$

By using the field strength $v(=u_x)$ this equation can be represented as

$$v_t + \frac{1}{4}v_{xxy} + vv_y + \frac{1}{2}v_x\partial_x^{-1}v_y = 0, \tag{22}$$

where

$$\partial_x^{-1}v_y = \int_0^x v_y(x, y, t)\mathrm{d}x.$$

The Lax pair for the above equation is

$$L\psi \equiv (\partial_x^2 + v(x,y,t) - \lambda)\psi = 0 \tag{23}$$

$$T\psi \equiv \left(\partial_t + \partial_y\partial_x^2 + v\partial_y + \frac{1}{2}\partial_x^{-1}v_y\partial_x + \frac{3}{4}v_y\right)\psi = 0. \tag{24}$$

The compatibility condition of (23) and (24), i.e., $[L, T] = 0$, if and only if, v satisfies (22).

3.1 *Darboux transformation*

We now prove the Darboux transformation for the BS equation. The following lemma is a consequence of Theorem 1.

Lemma 7. *Let* ψ_1 *be a solution of* $L_1\psi \equiv (\partial_x^2 + v - \lambda_1)\psi = 0$ *and* φ *be a nontrivial solution of* $L\psi \equiv (\partial_x^2 + v - \lambda)\psi = 0$, *then* $A\varphi = \varphi_x - \sigma\phi$, *where* $\sigma = \psi_{1x}\psi_1^{-1}$, *is a solution of*

$$L[1]\psi \equiv (\partial_x^2 + v[1] - \lambda)\psi = 0$$

with $v[1] = v + 2(\log\psi_1)_{xx}$.

The same result holds for the time evolution as well.

Theorem 5. *Let* ψ_1 *be a solution of* $L\psi_1 = \lambda_1\psi_1$. *Let A and B be defined as in* (5) *and* (6). *Further, let* $T\psi_1 = 0$. *Let* $G = TB$, *then* $AG\varphi = T[1]\varphi$ *where* $T[1]$ *is of the form* (24) *with* v *replaced by* $v[1]$.

Proof.

$$\begin{aligned} G\varphi &= TB\varphi \\ G\varphi &= (\psi_{1xx} + v\psi_1)\int_0^x (\psi_1^{-1}\varphi)_y \mathrm{d}x + \sigma_y\varphi + \sigma\varphi_y \\ &\quad +\varphi_{xy} + \frac{1}{2}\partial_x^{-1}v_y\varphi + \psi_1\int_0^x (\psi_1^{-1}\varphi)_t \mathrm{d}x. \end{aligned}$$

Now

$$\begin{aligned} AG\varphi &= (G\varphi)_x - \sigma G\varphi \\ &= \varphi_t + \varphi_{xxy} + \frac{1}{2}\partial_x^{-1}v[1]_y\varphi_x + v[1]\varphi_y + \frac{3}{4}v[1]_y\varphi. \end{aligned}$$

Hence $AG\varphi = T[1]\varphi$. □

3.2 *Exact solutions*

Let $v = 0$. We know that $\exp(\pm kx)$ is a solution of $L\varphi = \lambda_1\varphi$ where $k = \lambda_1^2$. If $v = 0$ then $T\varphi = 0$ becomes

$$(\partial_x^2\partial_y + \partial_t)\varphi = 0. \tag{25}$$

Let $\exp(kx + h(k)y + g(k)t)$ be a solution of (25), then h and g must satisfy

$$k^2 h(k) + g(k) = 0.$$

Therefore, $\theta(k) = \exp(kx + h(k)(y - k^2 t))$ satisfies (23) and (24) for $v = 0$. Further, if $h(k)$ is an odd function of k then $\psi_1 = \theta(k) + \theta(-k)$ is a solution of $L\varphi = 0$ and $T\varphi = 0$ for $v = 0$. Hence $v[1] = 2(\mathrm{d}^2/\mathrm{d}x^2)\log(\psi_1)$ is a solution of the BS equation. Clearly, this process can be iterated. If $\psi_1, \psi_2, \ldots, \psi_n$ are solutions of (23) and (24) for $v = 0$ and $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$ using Crums theorem [3]

$$v[n] = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log W[\psi_1, \psi_2, \ldots, \psi_n]$$

is a solution of the BS equation. The above process gives exponential solutions to the BS equation. A more general solution can be obtained by taking some arbitrary function of $(y - k^2 t)$ (see [13]) which solves (25). For example

$$\psi_1 = \exp(kx) f(y - k^2 t) + \exp(-kx) g(y - k^2 t),$$

where f and g are arbitrary functions. If we choose $f = \sin(y - k^2 t)$ and $g = \cos(y - k^2 t)$ we get

$$v[1] = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\psi_1 = \frac{4k^2 \sin 2(y - k^2 t)}{\mathrm{e}^{kx}\sin(y - k^2 t) + \mathrm{e}^{-kx}\cos(y - k^2 t)}.$$

Iterations of this solution can also be calculated for finding solutions of the BS equation.

4. Exact solutions of the (2+1)-dimensional *n*-wave interaction

Let J and C be constant, real, diagonal matrices with distinct entries $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ respectively. If A is an $n \times n$ matrix, define $\hat{J}A = JA - AJ$, to be the commutator of J and A. For an offdiagonal matrix B, let $\hat{J}^{-1}B = D$, where $\hat{J}D = B$, and D is offdiagonal. Let $Q = Q(x, y)$, with $(x, y) \in \mathbb{R}^2$ be a $n \times n$ matrix valued function whose diagonal entries are zero and the offdiagonal entries Q_{lm} are Schwarz functions. We consider the $n \times n$ hyperbolic system in the plane given by [14]

$$\phi_x - J\phi_y = Q\phi, \tag{26}$$

where ϕ is an $n \times n$ matrix valued function defined on the xy-plane.

The (2+1)-dimensional n-wave interaction equations [14] arise as the compatibility condition of (26) together with the time evolution equation:

$$\phi_t - C\phi_y - C(\hat{J}^{-1}Q)\phi + (\hat{J}^{-1}Q)C\phi = -ik\phi C, \tag{27}$$

where now the potential $Q = Q(x, y, t)$ is also time-dependent.

Explicitly, this equation can be written as

$$Q_t = CQ_y + \hat{C}\hat{J}^{-1}(Q_x - JQ_y) + [\hat{C}\,\hat{J}^{-1}Q, Q]. \tag{28}$$

The direct and inverse scattering transforms for (26) and (27) have been developed in [14]. We now give a Darboux transformation for these equations and use it to obtain explicit solution.

Define the operator L by

$$L\varphi = (J^{-1}D_x - D_y - J^{-1}Q)\varphi. \tag{29}$$

Let ψ_1 be a fixed invertible solution of $L\varphi = 0$, A and B be as in (5) and (6). We have the following theorem:

Theorem 6. *Let $F = LB$, then $AF = J^{-1}D_x - D_y - J^{-1}Q[1] = L[1]$ where $Q[1] = Q + (\hat{J}\sigma)J^{-1}$ and $L[1]A = AL$. The map $Q \mapsto Q[1] = Q + (\hat{J}\sigma)J^{-1}$ is called the Darboux transformation for* (26).

Proof. Let $F\varphi = LB\varphi$. Then

$$\begin{aligned} F\varphi &= (J^{-1}D_x - D_y - J^{-1}Q)\psi_1 \int_0^x \psi_1^{-1}\varphi \mathrm{d}x \\ &= J^{-1}\varphi + \psi_1 \int_0^x \psi_1^{-1}\psi_{1y}\psi_1^{-1}\varphi \mathrm{d}x - \psi_1 \int_0^x \psi_1^{-1}\varphi_y \mathrm{d}x. \end{aligned}$$

Thus,

$$\begin{aligned} A(F\varphi) &= (F\varphi)_x - \sigma(F\varphi) \\ &= J^{-1}\varphi_x - \varphi_y - J^{-1}\{Q + (J\sigma - \sigma J)J^{-1}\}\varphi \\ &= L[1]\varphi. \end{aligned}$$

Therefore,

$$\begin{aligned} L[1]A\varphi &= AFA\varphi = ALBA\varphi \\ &= AL\{\varphi - \psi_1\psi_1^{-1}(0,\cdot)\varphi(0,\cdot)\}. \end{aligned}$$

Using the fact that $AL(\psi_1\psi_1^{-1}(0,\cdot)\varphi(0,\cdot)) = 0$ we get $L[1]A\varphi = AL\varphi$ □

We now come to the time evolution. Let

$$T\varphi = [D_t - CJ^{-1}D_x + CJ^{-1}Q - C(\hat{J}^{-1}Q) + (\hat{J}^{-1}Q)C]\varphi. \tag{30}$$

Let ψ_1 be a fixed solution of (26) and (27) and $Q[1] = Q + (\hat{J}\sigma)J^{-1}$. We have the following theorem:

Theorem 7. *If $G = TB$ then $AG = T[1]$ where*

$$T[1]\varphi = [D_t - CJ^{-1}D_x + CJ^{-1}Q[1] - C(\hat{J}^{-1}Q[1]) + (\hat{J}^{-1}Q[1])C]\varphi,$$

further $T[1]A = AT$.

Proof. Let ψ_1 be a solution of (26) and (27). Let $G\varphi$ be defined as

$$G\varphi = TB\varphi = (D_t - CJ^{-1}D_x + CJ^{-1}Q - C(\hat{J}^{-1}Q) + (\hat{J}^{-1}Q)C)\psi_1 \int_0^x \psi_1^{-1}\varphi \mathrm{d}x.$$

This expression can be simplified, using the fact that ψ_1 satisfies eqs (26) and (27) to obtain

$$\begin{aligned} G\varphi &= -ik\psi_1 C\int_0^x \psi_1^{-1}\varphi \mathrm{d}x + \psi_1\int_0^x \psi_1^{-1}\varphi_t \mathrm{d}x \\ &\quad -\psi_1\int_0^x \psi_1^{-1}\psi_{1t}\psi_1^{-1}\varphi \mathrm{d}x - CJ^{-1}\varphi. \end{aligned}$$

Also $AG\varphi = (G\varphi)_x - \sigma(G\varphi)$ can be simplified to

$$AG\varphi = \varphi_t - CJ^{-1}\varphi_x + K\varphi$$

with

$$K = -CJ^{-1}\sigma + CJ^{-1}Q - C\hat{J}^{-1}Q + (\hat{J}^{-1}Q)C + \sigma CJ^{-1}.$$

We now use the expression for $Q[1] = Q + (\hat{J}\sigma)J^{-1}$, which implies that $\hat{J}^{-1}Q[1] = \hat{J}^{-1}Q + \sigma^{\mathrm{off}}J^{-1}$, where σ^{off} is the offdiagonal part of σ. Also the diagonal part of $\sigma, \sigma^{\mathrm{diag}}$, J and C are diagonal matrices and they commute. Therefore, $AG\varphi = T[1]\varphi$. As a consequence

$$\begin{aligned} T[1]A\varphi &= AGA\varphi = ATBA\varphi \\ &= AT(BA\varphi) = A(T - K_{\psi_1})\varphi. \end{aligned}$$

Therefore, $T[1]A\varphi = AT\varphi$. □

Theorem 8. *If Q satisfies the* (2+1)*-dimensional n-wave interaction equation, so does* $Q[1] = Q + (\hat{J}\sigma)J^{-1}$, *i.e.*, $Q \mapsto Q[1]$ *is a Darboux transformation for this equation.*

Proof. Let $Q(x, y, t)$ satisfy the (2+1)-dimensional n-wave interaction equation. Let $\psi_1(x, y, t, k)$ be a fixed invertible solution of (26) and (27) for $k = k_0$. Let $\varphi(x, y, t, \mu)$ also solve (26) and (27) with $k = \mu$ where $\mu \neq k_0$ be such that $A\varphi = \varphi_x - \psi_{1x}\psi_1^{-1}\varphi \neq 0$. It follows from [14] that such a φ exists. Then $L[1]A = AL$ so that in particular $L[1](A\varphi) = A(L\varphi) = 0$, and so $A\varphi$ is a nontrivial solution of $L[1]\eta = 0$. Also $T[1](A\varphi) = AT\varphi = A(-i\mu\varphi C) = -i\mu A\varphi C$. This shows that $A\varphi$ is a nontrivial solution of $L[1]\eta = 0$ and $T[1]\eta = -i\mu\eta C$. As before, this implies that $Q[1]$ satisfies the (2+1)-dimensional n-wave interaction equations. □

An explicit solution of (26) and (27) can be found in terms of exponentials when the potential Q vanishes. This gives rise to a nontrivial solution of the n-wave interaction.

Theorem 9. *Let $\psi = (\psi_{lm})_{1\leq l,m\leq n}$ be a matrix with entries*

$$\psi_{lm}(x, y, t) = A_{lm}\exp\{i\mu[(\lambda_l x + y) + (\mu_l - \mu_m)t]\},$$

where A_{lm} and μ are constants, and $\det\psi \neq 0$. *Then* $[\hat{J}(\psi_x\psi^{-1})]J^{-1}$ *is a solution of the* (2+1)*-dimensional n-wave interaction.*

Proof. Note that ψ satisfies (26) and (27) with the trivial potential $Q = 0$. Then Theorem 8 gives that $[J, \psi_x\psi^{-1}]J^{-1}$ satisfies the (2+1)-dimensional n-wave interaction equation. □

5. Concluding remarks

The commutation method can be applied to a large class of (1+1) and (2+1)-dimensional nonlinear evolution equations. It can also be applied to discrete differential-difference and difference-difference equations. Using Crums theorem [3] it is possible to construct a large family of solutions. This has been illustrated for the BS equation and the n-wave interaction equation.

While the Darboux method does not solve the initial value problem, it does give a large class of explicit solutions by an elementary method. Why both the equations in the Lax pairs always transform in the same manner has to be investigated further.

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