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Generalized Cole–Hopf transformations for generalized Burgers equations

B MAYIL VAGANAN* and E EMILY PRIYA
Department of Applied Mathematics and Statistics, School of Mathematics, Madurai Kamaraj University, Madurai 625 021, India
*Corresponding author. E-mail: vkbmv66@gmail.com

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Abstract. A detailed review of the invention of Cole–Hopf transformations for the Burgers equation and all the subsequent works which include generalizations of the Burgers equation and the corresponding developments in Cole–Hopf transformations are documented.

Keywords. Cole–Hopf transformations; Burgers equation; invariance analysis.

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1. Introduction

The Burgers equation [1]

$$u_t + uu_x = \frac{\delta}{2}u_{xx}, \quad \delta > 0, \tag{1}$$

is the simplest second-order nonlinear equation which balances the effect of nonlinear convection and linear diffusion. It is shown that (1) can be linearized to the heat conduction equation

$$\phi_t = \frac{\delta}{2}\phi_{xx}, \tag{2}$$

via the Cole–Hopf [2,3] transformation

$$u = -\delta\frac{\phi_x}{\phi} \tag{3}$$

and this remarkable result helped to solve the initial and/or boundary value problems of the Burgers equations.

2. Cole–Hopf transformations: Generalizations

After Hopf and Cole introduced the transformation, several attempts have been made to generalize Cole–Hopf transformation. We shall enlist them here.

In 1965, Chu [4] provided Cole–Hopf transformation to a system of quasilinear equations. An inverse approach adopted by Sachdev [5] helped to introduce Cole–Hopf transformations to both the parabolic and hyperbolic equations. Sachdev's approach was to generate nonlinear parabolic equations from a linear parabolic equation via a generalization of the Cole–Hopf transformation, viz.,

$$F(u(x,t)) = k(x,t)[\log\phi(x,t)]_x. \tag{4}$$

Indeed the linear parabolic equation

$$\phi_t + b(x,t)\phi_x + c(x,t)\phi = \epsilon a(x,t)\phi_{xx}, \tag{5}$$

under (4) generates the nonlinear parabolic equation

$$\begin{aligned} u_t &+ \left(b - \epsilon a_x + 2\epsilon a\frac{k_x}{k} - 2\epsilon a\frac{F}{k}\right)u_x - \epsilon a\frac{F''}{F'}u_x^2 \\ &+\frac{F}{F'}\left[\frac{\epsilon a}{k^2}(kk_{xx} - 2k_x^2) - \frac{k_t}{k} - \frac{bk_x}{k} + \epsilon a_x\frac{k_x}{k} + b_x + \frac{kc_x}{F}\right] \\ &+\frac{F^2}{F'}\left(\frac{2\epsilon a k_x}{k^2} - \frac{\epsilon a_x}{k}\right) = \epsilon a u_{xx}. \end{aligned} \tag{6}$$

Evidently, Cole–Hopf transformation (1) is recovered from (4) and (5) if we choose

$$F = u, \quad k = -2a = -\delta, \quad b = c = 0. \tag{7}$$

In this case, (6) coincides with (2).

In a similar manner, the linear hyperbolic equation

$$r_t + b\phi_x + c\phi = \epsilon a\phi_{xt} \tag{8}$$

is used to generate the nonlinear hyperbolic equations

$$\begin{aligned} &\left[\frac{-\epsilon a_x F'}{k - \epsilon a F} + \frac{F'}{k^2(k - \epsilon a F)}(\epsilon a k k_x + (k - \epsilon a F)^2)\right]u_t \\ &\quad + \frac{F'}{k(k - \epsilon a F)}((b + \epsilon a c)k + \epsilon a k_t)u_x \\ &\quad - \frac{\epsilon a}{k(k - \epsilon a F)}(\epsilon a F'^2 + F''(k - \epsilon a F))u_x u_t \\ &\quad - \frac{\epsilon a}{k}F' u_{xt} - \frac{\epsilon a_x}{k(k - \epsilon a F)}(k_t F + bF^2 + ckF) \\ &= \frac{\epsilon a k_x c F}{k(k - \epsilon a F)} + \frac{bFk_x}{k(k - \epsilon a F)} + \frac{Fk_t}{k(k - \epsilon a F)} - b_x\frac{F}{k} - c_x \\ &\quad - \left(\frac{F^2}{k} - \frac{k_x}{k}F\right)\left(\frac{\epsilon a k_t F}{K(k - \epsilon a F)}\right) - \frac{\epsilon a}{k}\left(\left(\frac{k_x}{k}\right)_t F + \frac{F^2}{k^2}k_t\right), \end{aligned} \tag{9}$$

via (4).

Yet another indirect generalization of the Cole–Hopf transformation was derived by Tasso and Teichmann [6]. Their transfornation is

$$F(u(x,t),x,t) = [\log v(x,t)]_x, \quad G(u(x,t),x,t) = [\log v(x,t)]_t, \tag{10}$$

where $u(x,t)$, $v(x,t)$ satisfy the nonlinear evolution equation and the linear parabolic equation respectively. Indeed, if the $v(x,t)$ equation is of the form

$$v_t = \sum_{n=1}^{N} \phi_n(x,t) \frac{\partial^n v}{\partial x^n}, \tag{11}$$

then the nonlinear evolution equation to be satisfied by $u(x,t)$ is of the form

$$G = \sum_{n=1}^{N} \alpha_n(x,t) G_n, \tag{12}$$

where

$$\begin{aligned} v_x &= Fv \equiv G_1 v, \\ v_{xx} &= \left(F^2 + F_x\right) v \equiv G_2 v, \end{aligned}$$

and in general

$$v_x^{(n)} = (G_1 G_{n-1} + (G_{n-1})_x) v \equiv G_n v.$$

In search of linearizable Bäcklund transformations for nonlinear parabolic equations, [7] extracted only the Burgers equation (1) and its inhomogeneous version, viz.,

$$u_t + uu_x + a(x,t) = \frac{\delta}{2} u_{xx}, \tag{13}$$

from

$$u_t - u_{xx} + H(t,x,u,u_x) = 0. \tag{14}$$

For, the governing equations are assumed to be in the form

$$r + q + H(p,z,x,y) = 0 \quad \text{and} \quad r' + q' + G(p',z',x,y) = 0, \tag{15}$$

where

$$z = z(x,y), \quad p = z_x, \quad q = z_y, \quad r = z_{xx}, \quad s = z_{xy}, \quad t = z_{yy}.$$

And Bäcklund transformations were sought in the form

$$p' = f(x,y,z,z',p), \quad q' = \psi(x,y,z,z',p,q). \tag{16}$$

The significant outcome was that if z and z' satisfy the equations

$$r + q + 2pz + h_{oo}(x,y) = 0, \quad r' + q' + 2p'z' + g_{oo}(x,y) = 0, \tag{17}$$

then the Bäcklund transformation between solutions of these equations is

$$p' = (z' - \mu)(z - z') + \mu_x, \tag{18}$$

$$q' = -[p'(z+\mu) + p(z'-\mu)] + \mu_x(z - z') - \mu_{xx} - g_{oo}, \tag{19}$$

where μ, h_{oo} and g_{oo} satisfy

$$2\mu_{xx} + h_{oo} - g_{oo} = 0, \quad \mu_{xx} + \mu_y + 2\mu\mu_x + g_{oo} = 0. \tag{20}$$

In [8] necessary and sufficient conditions are derived under which a system of nonlinear equations may be linearized. The algorithm of Bluman [9] maps linear partial differential equations with variable coefficients to linear partial differential equations with constant coefficients.

Sachdev and Mayil Vaganan [10], using Clairin's method [11], derived generalized Bäcklund transformations involving derivatives upto any finite order for linear parabolic and hyperbolic partial differential equations with variable coefficients. For instance, the Bäcklund transformation of the equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{x}{1-x^2}\frac{\partial f}{\partial x} + \frac{k^2}{1-x^2} f = a\frac{\partial f}{\partial t} + b\frac{\partial^2 f}{\partial t^2}, \tag{21}$$

may be sought in the form $f = f_0(x)F(x,t) + f_1(x)F_x(x,t)$, where $F(x,t)$ satisfies

$$\frac{\partial^2 F}{\partial x^2} = a\frac{\partial F}{\partial t} + b\frac{\partial^2 F}{\partial t^2}. \tag{22}$$

The system of ordinary differential equations satisfied by $f_0(x)$, $f_1(x)$ are

$$\begin{aligned} f_0''(x) - \frac{-x}{1-x^2} f_0'(x) + \frac{k^2}{1-x^2} f_0(x) &= 0, \\ f_1''(x) - \frac{-x}{1-x^2} f_1'(x) + \frac{k^2}{1-x^2} f_1(x) &= -2f_0'(x) - \alpha(x) f_0(x), \end{aligned} \tag{23}$$

whose solution is

$$\begin{aligned} f &= [\lambda_1 T_k(x) + \lambda_2 U_k(x)]\left[(x+1)\mathrm{e}^{-(a/b)t} + x\right] \\ &\quad + \left[x + \lambda_3 \int \frac{\mathrm{d}x}{T_k^2(x)\sqrt{1-x^2}}\right] T_k(x)(1 + \mathrm{e}^{-(a/b)t}) \\ &\quad + \left[x + \lambda_4 \int \frac{\mathrm{d}x}{U_k^2(x)\sqrt{1-x^2}}\right] U_k(x)(1 + \mathrm{e}^{-(a/b)t}), \end{aligned} \tag{24}$$

where $T_k(x)$ and $U_k(x)$ are Chebyshev polynomials.

In (14) and (13), the viscosity is 1 and $\delta/2$ respectively. But, Lighthill [12] showed that the viscosity was in fact a function of t. Taking this fact into consideration, Doyle and Englefield [13] found similarity solutions of the Burgers equation with variable viscosity $\Delta(t)$

$$u_t + uu_x = \frac{\Delta(t)}{2} u_{xx}. \tag{25}$$

Motivated by the works of Lighthill and Doyle and Englefield, Mayil Vaganan and Senthilkumaran [14,15] applied Lie's classical method [16] to the Burgers equation with algebraic and exponential viscosity

$$u_t + u^n u_x = \frac{(a+bt)^m}{2} u_{xx}. \tag{26}$$

$$u_t + uu_x + \alpha u = \frac{\mathrm{e}^{-\alpha t}}{2} u_{xx}, \tag{27}$$

and these equations are reduced to Bernoulli's and Kummer's equations, respectively.

Motivated by the similarity reductions of the generalized Burgers equations (GBEs) to linear ordinary differential equations, Mayil Vaganan and Jeyalakshmi [17], by combining the methods of [18] and [19], derived the generalized Cole–Hopf transformation

$$u(x,t)=\left[-[G(z,\tau)]^{-1}\frac{\partial G(z,\tau)}{\partial z}-\alpha e^{-\alpha t}z\right]\left(e^{2\alpha t}-\frac{2}{\alpha}e^{\alpha t}+1\right)^{-1/2}, \quad (28)$$

$$z(x,t)=x\frac{e^{\alpha t}}{e^{\alpha t}\pm 1}, \quad (29)$$

$$\tau(t)=\frac{a}{4}\int_0^t \frac{ds}{1-\alpha\cosh(\alpha s)}, \quad (30)$$

for the Burgers equation with algebraic viscosity

$$u_t+u^n u_x=\frac{(a+bt)^m}{2}u_{xx}, \quad (31)$$

where the function $G(z,\tau)$ is any solution of the linear parabolic equation

$$\frac{\partial^2 G}{\partial z^2}+2z\frac{\partial G}{\partial z}+\frac{a}{\alpha}\frac{\partial G}{\partial \tau}+\alpha^2 z^2 G=0. \quad (32)$$

In 2012, Mayil Vaganan [20] sought a Cole–Hopf transformation for two-dimensional Burgers equations with a variable coefficient

$$u_t+\frac{2}{F(y,t)}uu_x=\gamma\left(u_{xx}+u_{yy}\right), \quad (33)$$

in the form

$$u(x,y,t)=-\gamma F(y,t)\frac{1}{\phi(x,t)}\frac{\partial}{\partial x}\phi(x,t), \quad (34)$$

where the functions $F(y,t), \phi(x,t)$ are shown to satisfy the linear diffusion equations

$$F_t = \gamma F_{yy}, \quad (35)$$

$$\phi_t = \gamma\phi_{xx}. \quad (36)$$

Further, Mayil Vaganan [20] derived another Cole–Hopf transformation for a three-dimensional Burgers equation with variable coefficient, viz.,

$$u_t+\frac{1}{\xi(y,t)\eta(z,t)}uu_x=\gamma\left(u_{xx}+u_{yy}+u_{zz}\right), \quad (37)$$

in the form

$$u(x,y,z,t)=-2\gamma\xi(y,t)\eta(z,t)\frac{1}{\phi(x,t)}\frac{\partial}{\partial x}\phi(x,t), \quad (38)$$

where $\phi(x,t), \xi(y,t), \eta(z,t)$ are respectively shown to satisfy the linear diffusion equations

$$\phi_t = \gamma\phi_{xx}, \quad (39)$$

$$\xi_t = \gamma\xi_{yy}, \quad (40)$$

$$\eta_t = \gamma\eta_{zz}. \quad (41)$$

In 2013, Humi [21] found Cole–Hopf transformations in the form

$$\psi(x) = P(x) + Q(x)\frac{\phi'(x)}{\phi(x)} \tag{42}$$

between the nonlinear ordinary differential equation

$$\psi''(x) = S(x) + V(x)\psi(x) + W(x)\psi^2 + R(x)\psi^3 + \lambda\psi(x) \tag{43}$$

and the linear equation

$$\phi''(x) = U(x)\phi(x) + K(x)\phi'(x) + \lambda\phi(x). \tag{44}$$

3. New Cole–Hopf transformations

Consider the two-dimensional Burgers equations with algebraic viscosity and variable convection, viz.,

$$u_t + \frac{1}{F(y,t)}uu_x + t^m u_{xx} + t^n u_{yy} = 0, \tag{45}$$

which we transform via

$$u(x,y,t) = \frac{\partial}{\partial x}v(x,y,t) \tag{46}$$

to

$$v_t + \frac{1}{F(y,t)}\frac{v_x^2}{2} + t^m v_{xx} + t^n v_{yy} = 0. \tag{47}$$

We seek Cole–Hopf transformations of (47) in the form

$$v = \beta(y,t)\log(\phi(x,t)). \tag{48}$$

Equations (47) and (48) yield

$$2\phi^2\log[\phi]\beta_t + 2\beta\phi\phi_t - 2t^m\beta\phi_x^2 + \frac{1}{F(y,t)}\beta^2\phi_x^2 + 2t^n\phi^2\log[\phi]\beta_{y,y} + 2t^m\beta\phi\phi_{x,x} = 0. \tag{49}$$

If we set the coefficient of $\log[\phi]$ and ϕ_x^2 in (49) to zero, then

$$\beta_t + t^n\beta_{yy} = 0, \tag{50}$$

$$-2t^m + \frac{1}{F(y,t)}\beta(y,t) = 0. \tag{51}$$

In view of (50) and (51), eq. (49) reduces to the linear equation with time-dependent coefficient for $\phi(x,t)$:

$$\phi_t + t^m\phi_{xx} = 0. \tag{52}$$

Further, eqs (50) and (51) together yield another linear equation with time-dependent coefficient for $F(y,t)$:

$$Fm + tF_t + t^{1+n}F_{yy} = 0. \tag{53}$$

We thus conclude that Cole–Hopf transformation of (45) is

$$u(x,y,t) = \beta(y,t)\frac{\phi_x(x,t)}{\phi(x,t)}, \tag{54}$$

[recall (46) and (48)], where $\phi(x,t)$ and $F(y,t)$ are governed respectively by (52) and (53). Equation (51) connects $\beta(y,t)$ to $F(y,t)$.

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References

[1] J M Burgers, *Adv. Appl. Mech.* **1**, 171 (1948)
[2] E Hopf, *Commun. Pure Appl. Math.* **3**, 201 (1950)
[3] J D Cole, *Quart. Appl. Math.* **9**, 225 (1951)
[4] C W Chu, *Quart. Appl. Math.* **23**, 275 (1965)
[5] P L Sachdev, *ZAMP* **29**, 963 (1978)
[6] H Tasso and J Teichmann, *ZAMP* **30**, 1023 (1979)
[7] J J C Nimmo and D G Crighton, *Proc. R. Soc. London A* **384**, 381 (1982)
[8] S Kumei and G W Bluman, *SIAM J. Appl. Math.* **42**, 1157 (1982)
[9] G W Bluman, *SIAM J. Appl. Math.* **43**, 1259 (1983)
[10] P L Sachdev and B Mayil Vaganan, *Stud. Appl. Math.* **87**, 213 (1992)
[11] J Clairin, *Ann. Sci. Ecole Norm. Sup.* 3^{e} *Ser. Suppl.* **19**, s1 (1902)
J Clairin, *Ann. Fac. Sci. Univ. Toulouse* 2^{e} *Ser.* **5**, 437 (1903)
[12] M J Lighthill, *Surveys in mechanics* edited by G K Batchelor and R M Davies (Cambridge University Press, Cambridge, 1956) p. 250
[13] J Doyle and J Englefield, *IMA J. Appl. Math.* **44**, 145 (1990)
[14] B Mayil Vaganan and M Senthil Kumaran, *Stud. Appl. Math.* **117**, 95 (2006)
[15] B Mayil Vaganana and M Senthil Kumaran, *Nonlinear analysis: Real world applications* **9**, 2222 (2008)
[16] P J Olver, *Applications of Lie Groups to differential equations*, Graduate Text in Mathematics, No. 107 (Springer-Verlag, New York, 1986)
[17] B Mayil Vaganan and T Jeyalakshmi, *Stud. Appl. Math.* **127**, 211 (2011)
[18] S Lie, *Math. Ann.* **32**, 213 (1988)
[19] A V Bäcklund, *Math. Ann.* **9**, 197 (1876)
[20] B Mayil Vaganan, *Stud. Appl. Math.* **129**, 300 (2012)
[21] Mayer Humi, *J. Phys. A: Math. Theor.* **46**, 325201 (2013)