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Integrability detectors

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Abstract. In this short review, we present some applications and historical facts about the integrability detectors: Painlevé analysis, singularity confinement and algebraic entropy.

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1. Introduction

Around 1860, mathematicians started to investigate differential equations mainly on two aspects. First, on finding solutions of particular equations like Legendre, hypergeometric etc. and solutions in terms of elementary functions and secondly, by giving proofs of the existence of solutions for large class of equations. Cauchy [1,2] in his papers proved theorems for the existence of solutions to analytic ordinary differential equations with given initial conditions. For the real solution, he used linear approximation theory and for solutions in the complex domain he employed the series expansion and majorant method. Here, the solution is analysed in the neighbourhood of a point. The power series solution given in the neighbourhood of a given point (local) is continued analytically along a path (avoiding the set of fixed singularities) from the given point to another point. Riemann studied in detail the hypergeometric equation from the standpoint of complex analysis [3] and classified the nature of the solutions as functions of a complex variable. In particular, his analysis clarified how the solutions behave under analytic continuation. Riemann further investigated the global characterization of the differential equations and solutions. In the year 1850, French mathematicians have investigated the singularities of analytic functions. Briot and Bouquet [4] analysed and found that polynomial autonomous first-order ordinary differential equation of the form

$$\sum_{k=0}^{m}\sum_{j=0}^{2m-2k} h_{jk}y^{j}y'^{k} = 0, \qquad h_{0m} = 1, \tag{1}$$

where m is a positive integer and h_{jk} are constants, admits single-valued general solution. Briot and Bouquet [4] proved that the general solutions of eq. (1) are either elliptic functions or rational functions of e^{rx}, r being some constant, or a rational function of x. In particular, they analysed equations of the form $y'^n = f(y, x)$, and found that for $n > 1$ the following equations have Painlevé property:

$$y'^2 = 4y^3 + \lambda y + 1, \quad \text{elliptic functions,} \tag{2}$$
$$y'^2 = y(q(x)y + r(x))^2, \quad \text{reducible to a Riccati,} \tag{3}$$
$$y'^3 = y^2(y-1)^2, \quad \text{elliptic function,} \tag{4}$$
$$y'^4 = y^3(y-1)^3, \quad \text{elliptic function,} \tag{5}$$
$$y'^6 = y^4(y-1)^3, \quad \text{elliptic function,} \tag{6}$$
$$y'^n = q(x)y^{n-1}, \quad \text{integrable by quadratures.} \tag{7}$$

Based on the lectures of Wierstrass [5] and works of Riemann, Briot and Bouquet [4], Fuch carried out his investigations on the nature of solutions of differential equations. He [6,7] pointed out that the singular points of the solutions of the linear ordinary differential equations are fixed. In 1884, Fuch [6] considered the first-order nonlinear differential equations

$$y' = f(x, y),$$

where f is a rational in y and analytic in x. He has given the necessary and sufficient condition that the most general first-order and first degree equation with fixed critical points (in other words all the movable singularities are poles), is the Riccati equation

$$y' = a_2(x)y^2 + a_1(x)y + a_0(x). \tag{8}$$

Riccati equation can be linearized through the transformation

$$y = -\frac{\phi'}{a_2\phi}, \tag{9}$$

to a linear second-order differential equation

$$\phi'' - \left(\frac{a_2'}{a_2} + a_1\right)\phi' + a_0a_2\phi = 0. \tag{10}$$

As Riccati equation is reduced to a linear equation, it defines no new function. In 1879 Paul Hoyer [8], a student of Weierstrass, in his thesis, considered the following system of nonlinear differential equations:

$$\frac{dx}{dt} = a_1yz + b_1zx + c_1xy,$$
$$\frac{dy}{dt} = a_2yz + b_2zx + c_2xy,$$
$$\frac{dz}{dt} = a_3yz + b_3zx + c_3xy, \tag{11}$$

and looked for solutions in terms of elliptic functions. He carried out his analysis by considering the series expansions of the form:

$$x = A_1u^{-n} + A_2u^{-n+1} + \cdots,$$
$$y = B_1u^{-n} + B_2u^{-n+1} + \cdots,$$
$$z = C_1u^{-n} + C_2u^{-n+1} + \cdots, \tag{12}$$

where $u = (t - t'')^{1/r}$. Kovalevskaya [9] who was also a student of Weierstrass considered the general case of the rigid body motion about fixed points and looked for other integrable cases (other than the known integrable cases: spherical, Euler and Lagrange) whose solutions are meromorphic i.e., have only poles as singularities in the finite complex plane. She considered the series expansion of the form [10]

$$\begin{aligned} x &= t^{-1}(x_0 + x_1 t + \cdots), \\ y &= t^{-1}(y_0 + y_1 t + \cdots), \\ z &= t^{-1}(z_0 + z_1 t + \cdots). \end{aligned} \tag{13}$$

Kovalevskaya's work was two-fold. First, she found a new case for which she constructed general solution in terms of theta functions of two variables and hyperelliptic integrals. Secondly, she showed that apart from the four known cases, there are no other cases for which the solution is single-valued. Kovalevskaya started her studies by noticing that for Euler and Lagrangian cases of rigid body, the solutions are single-valued function of time whose only singularities in complex plane are poles. That means, the general solution has property that it can be expanded in terms of Laurent series with finite principle part so as to exclude essential singularities. That is, the series is self-consistent, single-valued and contains sufficient number of degrees of freedom (arbitrary coefficients). For this remarkable work, Kovalevskaya was awarded the prestigious Bordin prize of the French Academy in 1889. It should be pointed out here that the analysis of Kovalevskaya can detect only logarithmic and algebraic branch points. Hence her method cannot be used to detect the existence of essential singularities. Thus, Kovalevskaya method leads to a necessary condition for the existence of formal Laurent series solution.

In 1885, Poincaré [11] used the theory of automorphic function and showed that when the genus is greater than one, one of the solutions is an algebraic function of z. However, his new differential equation does not lead to a new class of transcendental functions. Painlevé in 1888 [12] observed that the solutions of an autonomous polynomial first-order equations can have only two types of singularities; poles and branch points. Moreover, he confirmed the results of Fuch stated earlier. Inspired by Picard's [13] work, Painlevé in a series of papers [3,12,14] carried out a still broader search for second-order nonlinear ODEs of the form

$$w'' = f(w', w, z),$$

where f is a polynomial in w', rational in w and analytic in z, and can neither be solved by quadratures nor reduced to linear equations, but obtain new transcendent as a solution. His approach was based on the observation that critical singularities (multivalued) of second-order equations can be branch points, algebraic, logarithmic and essential singularities.

Painlevé developed α-method to test the occurrence of branch point solution and to prove single-valuedness. Painlevé [12,14], Gambier [15] and Fuchs [16] found the six Painlevé equations whose general solutions require the introduction of new transcendents called Painlevé transcendents. This classification of second-order equations was completed by Gambier who presented the complete list of 50 equations including the six Painlevé equations that satisfied the requirement of the absence of movable critical singularities. Forty-four of them can be solved by known functions. It is clear from the analysis made by the above-mentioned researchers that the integrability of a differential equation is related to the existence of the general solution which is single-valued and analytic.

Painlevé property:
A system of ODE is said to have the Painlevé property if its general solution has no movable critical singular point.

In other words, an ODE is said to possess the Painlevé property if all movable singularities of all solutions are poles. For example [17], the solution of

$$w' + w^2 = 0,$$

is $w = (z - z_0)^{-1}$. Note that, if the initial condition is $w(0) = 1$, then $z_0 = -1$. On the other hand, if the initial condition is $w(0) = 2$, then z_0 moves to $z_0 = -1/2$, i.e., the location of the singularity at z_0 moves with the initial condition. Such singularities are called movable singularities. For Painlevé property to hold the only forbidden singularities are the movable critical (multivalued) singularities.

Consider the ODE [17,18]

$$w'' = w'^2 \frac{2w - 1}{w^2 + 1}.$$

The singularities of this equation are $w = \pm i$, $w = \infty$ and $w' = \infty$. Series expansion can be developed for solutions exhibiting each of the above singular behaviour and the equations pass the Painlevé test. This equation, however, has the general solution

$$w = \tan\{\log[k(z - z_0)]\},$$

where k and z_0 are constants. For $k \neq 0$, w has poles at

$$z = z_0 + k^{-1} \exp\left\{-\left(n + \frac{1}{2}\right)\pi\right\},$$

for every integer n. These poles accumulate at the movable point z_0, giving rise to a movable branched nonisolated essential singularity. This example clearly shows that passing the Painlevé test need not guarantee that the equation actually possesses the Painlevé property.

1.1 *List of continuous Painlevé equations*

P_I

$$w'' = 6w^2 + z,$$

P_{II}

$$w'' = 2w^3 + zw + a,$$

P_{III}

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{1}{z}(aw^2 + b) + cw^3 + \frac{d}{w},$$

P_{IV}

$$w'' = \frac{w'^2}{2w} + \frac{3w^3}{2} + 4zw^2 + 2(z^2 - a)w - \frac{b^2}{2w},$$

$\mathrm{P_V}$

$$w'' = w'^2\left(\frac{1}{2w}+\frac{1}{w-1}\right)-\frac{w'}{z}+\frac{(w-1)^2}{2z^2}\left(aw+\frac{b}{w}\right)+\frac{cw}{z}+\frac{dw(w+1)}{(w-1)},$$

$\mathrm{P_{VI}}$

$$\begin{aligned} w'' = {} & \frac{w'^2}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-z}\right)-w'\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{w-z}\right) \\ & +\frac{w(w-1)(w-z)}{2z^2(z-1)^2}\left(a-b\frac{z}{w^2}+c\frac{z-1}{(w-1)^2}-(d-1)\frac{z(z-1)}{(w-z)^2}\right), \end{aligned}$$

where a, b, c, d are arbitrary constants. All the solutions of the Painlevé equations are meromorphic functions and are called Painlevé transcendents.

Painlevé equations have many wonderful properties: By coalescence one can obtain all other Painlevé equations from $\mathrm{P_{VI}}$ by taking special limits on the parameters; the special function solutions exist for specific values of parameters [18–21] etc. Recently, it has been found that Painlevé equations admit Lax pairs, bilinear forms as well. Theory of space of initial conditions was used extensively by Okamoto to find the Hamiltonian structure of the Painlevé equations by the introduction of tau functions [22–27].

Painlevé results were extended to higher-order and higher degree equations particularly by Gambier [15], Chazy (third-order) [28], Garnier [29] and Bureau (fourth-order) [30,31]. Bureau also considered in his investigations the binomial equations of the form

$$w''^2 = f(w', w, z), \tag{14}$$

and recently Cosgrove [32,33] generalized this idea to

$$w''^n = f(w', w, z). \tag{15}$$

Painlevé theory attracted lots of attention after the connection with the theory of integrable systems were made in 1980 [34,35]. In fact, the interest in Painlevé transcendent is due to their appearance in integrable systems [34,35] and statistical mechanics [36]. First in 1981, Ablowitz *et al* [35] realized that all the reduction of IST solvable partial differential equations possess Painlevé property and they proposed what is called the Painlevé test. This procedure is proposed in the spirit of Kovalevskaya and Gambier works to test the integrability of system of ordinary differential equations.

ARS conjucture:
Any ODE which arises as a reduction of an integrable PDE possesses Painlevé property, possibly after a transformation of variable.

The Painlevé test (ARS) has been successfully implemented for many years to identify new integrable systems [37,38]. Grammaticos *et al* [39,40] introduced the concept of weak Painlevé property for identifying certain new classes of integrable nonlinear ODEs. Painlevé test was further generalized to partial differential equations by Weiss *et al* [41], wherein there is no need to reduce the PDE to ODE for testing the Painlevé property. Here, Cauchy–Kovalevskaya theorem and several complex variable theories were used. They looked for the solution of PDE in terms of Laurent expansion over 'non-characteristic' singularity manifold. For linearizable systems some caution should be taken while applying the Painlevé test. In [42], we have shown that Painlevé property need not hold for certain class of linearizable systems/systems that are solvable by quadratures.

2. Discrete Painlevé equations

Space-time continuum hypothesis advanced mathematics of the physical world to a larger extent. However, it is important to recall the quote of Albert Einstein [43] regarding the continuous systems.

"To be sure, it has been pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraical method of description of nature, that is to the elimination of continuous functions from physics. Then, however, we must also give up, by principle, the space-time continuum. It is not unimaginable that human ingenuity will some day find methods which will make it possible to proceed along such a path. At the present time, however, such a program looks like an attempt to breathe in empty space".

It is undoubtedly clear that considering discrete systems is more fundamental than continuous equations. When we look for discrete systems the independent variables assume discrete values. However, it is not an easy task to discretize the given physical model preserving the fundamental properties of the continuous systems. As far as nonlinear systems are concerned, this process is quite difficult because there is no unique way to express in discrete form of a given nonlinear system preserving the essential basic features of the continuous systems. Until now, no algorithmic procedures exists to perform integrable discretization. In recent years, many attempts were made to obtain discrete versions of certain continuous equations. One important advancement made in this direction is the derivation of discrete versions of the Painlevé equations. They are integrable, non-autonomous, discrete equations whose continuous limits go to continuous Painlevé equations.

First appearance of integrable discrete non-autonomous systems was due to Lagurre in the framework of recurrence relations of orthogonal polynomials [44]. In 1939, Shohat [45] derived (using orthogonal polynomials), what we call now d-P_{I} [46]

$$x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n} + a, \tag{16}$$

where $z_n = \alpha n + \beta$. The next example appeared in the work of Jimbo and Miwa on continuous Painlevé equation [47]. They discussed the contiguity relation of the continuous Painlevé equations (a relation between solutions of a given Painlevé equation for different values of some parameter). From the solutions of P_{II} [47]

$$w'' = 2w^3 + zw + \alpha, \tag{17}$$

they obtained the contiguity relation

$$\frac{\alpha_n + \frac{1}{2}}{x_{n+1} + x_n} + \frac{\alpha_n + \frac{1}{2}}{x_n + x_{n-1}} = -(2{x_n}^2 + z), \tag{18}$$

where $x_n = w(z, \alpha_n)$, $\alpha_n = n+\alpha_0$. However, no continuous limit was derived then. While investigating a field-theoretical model of two-dimensional gravity, Brézin and Kazakov [46] derived d-P_{I} mentioned earlier. They clearly identified the recurrence relation (18) with the continuous Painlevé equation. Thus, eq. (16) is called discrete Painlevé equation-I. Shortly afterwards, Periwal and Shevitz [36] discovered d-P_{II}, which assumes the form

$$x_{n+1} + x_{n-1} = \frac{z_n x_n}{1 - x_n^2}. \tag{19}$$

At the same time, Nijhoff and Papageorgiou [48] independently derived d-P$_{\rm II}$ from the similarity from reductions of discrete mKdV equation. Already it becomes clear that discrete Painlevé equation (d-P's) can be derived from different approaches: orthogonal polynomials, reductions and contiguity relations. By now, it is well known that the classification problem of nonlinear ODEs based on the Painlevé property was successful in isolating the integrable equations [37]. In the similar spirit, the necessity to introduce an integrability detector becomes an urgent task. In this context, the singularity confinement criterion [49] for discrete systems played the role of the Painlevé property for continuous case. Discrete Riccati equation

$$x_{n+1} = \frac{a_n x_n + b_n}{c_n x_n + d_n}, \tag{20}$$

is a well-known integrable first-order discrete system. For second-order systems, to start with, precise functional form of the mapping must be known. In this context, it should be mentioned that the elliptic functions obey addition relation which can be interpreted as discrete equation [50]. In fact, the two-point correspondence

$$\begin{aligned} &\alpha x_{n+1}^2 x_n^2 + \beta x_{n+1} x_n (x_{n+1} + x_n) + \gamma (x_{n+1}^2 + x_n^2) \\ &\qquad + \epsilon x_{n+1} x_n + \xi (x_{n+1} + x_n) + \mu = 0, \end{aligned} \tag{21}$$

can be parametrized in terms of elliptic function. This was further generalized by Quispel, Roberts and Thompson (QRT) [51] for three-point mapping. There exist two families of QRT mapping which are symmetric and asymmetric.

The symmetric QRT mapping is of the form [51]

$$x_{n+1} x_{n-1} f_3(x_n) - (x_{n+1} + x_{n-1}) f_2(x_n) + f_1(x_n) = 0, \tag{22}$$

and the asymmetric form is given by

$$\begin{aligned} x_{n+1} x_n f_3(y_n) - (x_{n+1} + x_n) f_2(y_n) + f_1(y_n) &= 0, \\ y_n y_{n-1} g_3(x_n) - (y_n + y_{n-1}) g_2(x_n) + g_1(x_n) &= 0, \end{aligned} \tag{23}$$

where f_i's and g_i's are, in general, specific quartic polynomials in the variables. The number of parameters involved in symmetric QRT mapping (22) is 12 and asymmetric QRT mapping (23) is 18. However, after considering all possible transformations, one can show that the number of genuine parameters in asymmetric QRT mapping is 8 and in symmetric QRT mapping is 5 [52]. The QRT mapping possesses an invariant which is biquadratic in x and y. Invariant for asymmetric case is given by [52]

$$\begin{aligned} &x^2 y^2 (\alpha_0 + K\alpha_1) + x^2 y (\beta_0 + K\beta_1) + x^2 (\gamma_0 + K\gamma_1) \\ &\quad + x y^2 (\delta_0 + K\delta_1) + x y (\epsilon_0 + K\epsilon_1) + y^2 (\kappa_0 + K\kappa_1) \\ &\quad + x (\xi_0 + K\xi_1) + y (\lambda_0 + K\lambda_1) + (\mu_0 + K\mu_1) = 0. \end{aligned} \tag{24}$$

Invariant for symmetric case is given by

$$\begin{aligned} &x^2 y^2 (\alpha_0 + K\alpha_1) + x^2 y (\delta_0 + K\delta_1) + x y^2 (\delta_0 + K\delta_1) \\ &\quad + x y (\epsilon_0 + K\epsilon_1) + x^2 (\kappa_0 + K\kappa_1) + y^2 (\kappa_0 + K\kappa_1) \\ &\quad + x (\lambda_0 + K\lambda_1) + y (\lambda_0 + K\lambda_1) + (\mu_0 + K\mu_1) = 0. \end{aligned} \tag{25}$$

This means that,

$$x_n^2x_{n+1}^2(\alpha_0 + K\alpha_1) + x_n^2x_{n+1}(\delta_0 + K\delta_1) + x_nx_{n+1}^2(\delta_0 + K\delta_1)$$
$$+ x_nx_{n+1}(\epsilon_0 + K\epsilon_1) + x_n^2(\kappa_0 + K\kappa_1) + x_{n+1}^2(\kappa_0 + K\kappa_1)$$
$$+ x_n(\lambda_0 + K\lambda_1) + x_{n+1}(\lambda_0 + K\lambda_1) + (\mu_0 + K\mu_1) = 0, \tag{26}$$

where K plays the role of integration constant. Thus, the solution of the QRT mapping is given in terms of elliptic functions. The integration of symmetric case through invariant goes back to Euler. The integration of asymmetric mapping in terms of elliptic functions was carried out in [53,54].

3. Singularity confinement (SC) [49]

For the continuous systems, the Painlevé test (ARS) has been used successfully for many years and new integrable systems [37,38] were obtained. For discrete systems an analogue test was proposed by Grammaticos *et al* [49]. For the past few years, it has been productively used and has identified many integrable discrete systems [55]. In particular, using singularity confinement method, the discrete Painlevé equations [55,56] and asymmetric discrete Painlevé equations have been derived [55,57–60]. Singularity confinement test is used to analyse the behaviour around movable singularity of the map. Here, it is important to realize that appearance of infinity in the process of iteration of the rational map is not a real singularity. As pointed out by Kruskal, this difficulty can be avoided by considering the map in the projective space, where infinity is not a singularity. However, if one encounters the indeterminate (ambiguous quantity) like $\infty - \infty$, $0/0$ and ∞/∞, the mapping is not well defined. The point at which these values appear are singularities of the map. The way to treat this difficulty is to use an argument of continuity with respect to the initial conditions and to introduce a small parameter. For the mapping $x_{n+1} = f(x_n, x_{n-1})$ at some point it may happen that $(\partial x_{n+1}/\partial x_{n-1}) = 0$ holds. This is a dangerous situation, because the initial condition disappears from the iteration and as a result, the mapping may lose a degree of freedom. After a finite number of iterative steps, the mapping must recover the lost degree of freedom. Hence, we can say that the discrete system passes the singularity confinement test if the singularities appear spontaneously in the mapping and disappears without loss of degree of freedom after a finite number of steps. This idea can be generalized to N-component mapping. Normally, for a general N-component mapping, N free parameters, introduced by the initial conditions, must be present at every step. Now, it may happen that at some iteration, one (or more) degrees of freedom be lost. The condition for this to occur is that the Jacobian of $(x_1', x_2', \ldots, x_N')$ with respect to $(x_1, x_2, \ldots, x_N)$ vanishes. For a general mapping $x_i' = f_i(x_k)$ this reads as

$$J = \begin{vmatrix} \partial x_1'/\partial x_1 & \partial x_1'/\partial x_2 & \ldots & \partial x_1'/\partial x_N \\ \partial x_2'/\partial x_1 & \partial x_2'/\partial x_2 & \ldots & \partial x_2'/\partial x_N \\ \vdots & \vdots & \ddots & \vdots \\ \partial x_N'/\partial x_1 & \partial x_N'/\partial x_2 & \ldots & \partial x_N'/\partial x_N \end{vmatrix} = 0.$$

Over the years, singularity confinement test was found out to be a very convenient discrete integrability detector. For discrete systems which are integrable through inverse scattering transform (IST) methods, singularity confinement holds. However, there exists a rich class of systems [61], the ones integrable by linearization, for which singularity confinement may not pass (just as in continuous case) [42]. Also Hietarinta and Viallet [62] have shown that non-integrable mapping

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{{x_n}^2},$$

also pass the singularity confinement test with singularity pattern $\{0, \infty, \infty, 0\}$ but having exponential degree growth.

Now, we apply the singularity confinement test to the well-known McMillan mapping [56],

$$x_{n+1} + x_{n-1} = \frac{2\mu x_n}{1 - x_n^2}. \tag{27}$$

Singularities are $x_n \pm 1$. Assume that x_0 is finite and $x_1 = 1 + \epsilon$ and we find that $x_2 = -\mu/\epsilon - (x_0 + \mu/2) + \mathcal{O}(\epsilon)$, $x_3 = -1 + \epsilon + \mathcal{O}(\epsilon^2)$ and $x_4 = x_0 + \mathcal{O}(\epsilon)$. Thus, singularity is confined and mapping recovered the memory of the initial conditions through x_0.

One of the greatest successes of SC is the non-autonomization of discrete systems. Here, it should be remarked that the procedure is based on the assumption that singularity pattern for autonomous and non-autonomous systems are exactly the same. We can de-autonomize the McMillan mapping by considering that the parameters depend on n. Hence, we have

$$x_{n+1} + x_{n-1} = \frac{a(n) + b(n)x_n}{1 - x_n^2}. \tag{28}$$

Now assume that x_{n-1} is finite and $x_n = \sigma + \epsilon$, where $\sigma = \pm 1$, then we can find that

$$x_{n+2} = -\frac{b_{n+1} + \sigma a_{n+1}}{2\epsilon} + \frac{a_{n+1} - \sigma b_{n+1}}{4} - x_n + \mathcal{O}(\epsilon), \tag{29}$$

$$x_{n+3} = -\sigma + \frac{2b_{n+2} - b_{n+1} - \sigma a_{n+1}}{b_{n+1} + \sigma a_{n+1}}\epsilon + \mathcal{O}(\epsilon^2). \tag{30}$$

For x_{n+4} to be finite, one has to impose the condition

$$b_{n+1} - 2b_{n+2} + b_{n+3} + \sigma(a_{n+1} - a_{n+3}) = 0, \tag{31}$$

to hold. Since $\sigma = \pm 1$, we have two equations for a_n and b_n. By adding and subtracting we get

$$b_{n+3} - 2b_{n+2} + b_{n+1} = 0,$$

$$a_{n+3} - a_{n+1} = 0.$$

Solving them, we get

$$b_n(\equiv z_n) = \alpha n + \beta$$

and

$$a_n = \delta + \gamma(-1)^n.$$

Ignoring even–odd dependence ($a =$ constant)

$$x_{n+1} + x_{n-1} = \frac{a + z_n x_n}{1 - x_n^2}, \tag{19}$$

which is the discrete form of P_{II}.

Discrete forms do exist for all Painlevé equations. This means that there exist discrete systems having each of the Painlevé equations as continuous limit. These discrete equations can be of either d- or q-type, depending on whether the independent variable is of additive ($z_n = \alpha_n + \beta$) or multiplicative ($q_n = q_0\lambda^n$) type and may belong to either the symmetric or asymmetric form. It must be stressed here that more than one discrete analogues exist for each of the continuous P's. This is already true for symmetric mappings of single variable but all the more so for asymmetric mappings involving two or more dependent variables. The name of a discrete P is based on its continuous limit. The application of singularity confinement method quite often leads to terms of the form $(-1)^n$, but also j^n, where $j^3 = 1$, i^n and even k^n, where $k^5 = 1$. The existence of binary, ternary, etc. symmetry indicates that the equation is better written as a system of two or three equations. Here, we present the list of discrete Painlevé equations in the symmetric form only.

3.1 *List of discrete Painlevé equations*

d-P_I

$$x_{n+1} + x_{n-1} = -x_n + \frac{z_n}{x_n} + 1,$$

d-P_{II}

$$x_{n+1} + x_{n-1} = \frac{z_n x_n + a}{1 - x_n^2},$$

q-P_{III}

$$x_{n+1}x_{n-1} = \frac{(x_n - aq_n)(x_n - bq_n)}{(1 - cx_n)(1 - x_n/c)},$$

d-P_{IV}

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - z_n)^2 - c^2},$$

q-P_V

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(1 - cx_n q_n)(1 - x_n q_n/c)},$$

d-P_V

$$\frac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1})} = \frac{((x_n - z_n)^2 - a^2)((x_n - z_n)^2 - b^2)}{(x_n^2 - c^2)(x_n^2 - d^2)}.$$

q-P_{VI}

$$\frac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{(x_n - a q_n)(x_n - q_n/a)(x_n - b q_n)(x_n - q_n/b)}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)},$$

where $z_n = \alpha n + \beta$, $q_n = q_0 \lambda^n$ and a, b, c, d are constants. However, it should be remarked that the members of discrete Painlevé equations are infinite. The classification of discrete Painlevé equations based on geometrical description was undertaken in [52,63–65]. Inspired by the work of Okamoto, Sakai [59] classified d-P's in studying rational surfaces in connection to extended Weyl groups and gave a complete classification of the d-P's. However, it was noticed that d-P's can involve more than one dependent variables. The systematic derivation of the asymmetric form of discrete equations was carried out by Kruskal *et al* [58].

Now, we consider asymmetric d-P_{II} and take its continuous limit. Consider d-P with even and odd terms

$$x_{n+1} + x_{n-1} + x_n = \frac{\alpha n + \beta + \gamma(-1)^n}{x_n} + a.$$

We can consider even and odd term separately and write

$$x_{2m+1} + x_{2m} + x_{2m-1} = \frac{z_{2m} + \gamma}{x_{2m}} + a,$$

$$x_{2m+2} + x_{2m+1} + x_{2m} = \frac{z_{2m+1} - \gamma}{x_{2m+1}} + a,$$

where $z_m = \alpha m + \beta$. Introducing variables $X_m = x_{2m}$ and $Y_m = x_{2m+1}$, we arrive at the asymmetric form

$$Y_m + X_m + Y_{m-1} = \frac{Z_m + \gamma}{X_m} + a,$$

$$X_{m+1} + X_m + X_m = \frac{Z_m + \alpha - \gamma}{Y_m} + a,$$

with $Z_m = 2\alpha m + \beta$.

We perform continuous limit of the above equation with $X = 1 + \epsilon w + \epsilon^2 u$, $Y = 1 - \epsilon w + \epsilon^2 u$, $Z = 1 - \epsilon^3 m$, $a = 2$, $\gamma = -\epsilon^3 c/4 \rightarrow u = \frac{1}{4}(w^2 - w' + t)$, with $t = \epsilon m$, leading to $w'' = 2w^3 + 2tw + c$, i.e. P_{II}. Thus, it is clear that by choosing appropriate continuous limit, one could get continuous Painlevé equations.

3.2 *Properties of discrete Painlevé equations*

Analogues of the properties of the continuous Painlevé equations, the discrete Painlevé equations also have a number of properties. To mention a few:

(A) Coalescence limits,
(B) Linearization,
(C) Bilinear formalism.

(A) *Coalescence limits*

Now, we briefly discuss about coalescence limits. Let us recall that from the continuous Painlevé equation six (P_{VI}), one can obtain other Painlevé equations by taking coalescence limits (through adequate limiting procedure involving the dependent and independent variables and the free constants entering the equation). This process of step-by-step degeneration can be carried out easily. For example, consider P_{II}

$$w'' = 2w^3 + zw + a.$$

It is well known [18] that by replacing z by $\epsilon^2 z - (6/\epsilon^{10})$, w by $\epsilon w + (1/\epsilon^5)$, a by $(4/\epsilon^{15})$ in P_{II} and taking the limit $\epsilon \to 0$, P_{II} degenerates into P_I.

Now, we discuss the degeneration through coalescence limits of the discrete Painlevé equations. As an example, we consider coalescence limits of d-P_{II} $\to$ d-P_I. Start with d-P_{II} equation

$$X_{n+1} + X_{n-1} = \frac{Z_n X_n + A}{1 - X_n^2}.$$

Put $X = 1 + \delta x$:

$$4 + 2\delta(x_{n+1} + x_{n-1} + x_n) + \mathcal{O}(\delta^2) = -\frac{Z_n(1 + \delta x_n) + A}{\delta x_n},$$

where

$$Z_n = -A - 2\delta^2 z_n, \quad A = 4 + 2\delta a \text{ at } \delta \to 0,$$

we get

$$x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n} + a,$$

i.e., precisely d-P_I. For more details see [55,66].

(B) *Linearization*

Special function solutions of continuous Painlevé equations have been studied extensively in [20]. The study of the special function solutions can be done through coalescence limits framework [21,66] for both continuous and discrete Painlevé equations. Here, we give only the linearization and special function solution of P_{II} and d-P_{II}. We consider the continuous P_{II} equation:

$$w'' = 2w^3 + wt + \alpha. \tag{32}$$

Assume that the solutions of the Riccati equation and Painlevé equation are the same. This means that these two equations must be compatible with each other. So, consider the Riccati equation

$$w' = a(t)w^2 + b(t)w + c(t). \tag{33}$$

Take the first derivative of the Riccati equation and compare the coefficients of the resulting equation with P_{II} equation and solve for the unknowns. We get the linearizability condition

$$\alpha = \epsilon_1/2. \tag{34}$$

The Riccati equation now becomes

$$\epsilon_1 w' = w^2 + t/2. \tag{35}$$

Use the Cole–Hopf transformation,

$$w = -\epsilon_1 u'/u, \tag{36}$$

to reduce the Riccati equation (35) into the Airy equation

$$u'' + \frac{t}{2}u = 0. \tag{37}$$

As α is the unique parameter of P_{II}, it is not possible to impose further constraints. Similar situation arises in the discrete P_{II} as well. Here, d-P_{II} equation should be consistent with the discrete Riccati equation

$$x_{n+1} = \frac{a_n x_n + b_n}{c_n x_n + d_n}. \tag{38}$$

We start with the discrete P_{II}

$$x_{n+1} + x_{n-1} = \frac{z_n x_n + a}{1 - x_n^2}, \tag{19}$$

where $z_n = \delta n + z_0$ and a is a constant.

The compatibility of discrete P_{II} and discrete Riccati equations leads to the following linearizability condition: $a = \delta/2$. Thus, we obtain the homographic mapping (discrete Riccati) associated with discrete P_{II}:

$$x_{n+1} + 1 = \frac{a + z_n}{2(1 - x_n)}. \tag{39}$$

Putting $x_n = P_n/Q_n$ into (39), we obtain $Q_{n+1} = 2Q_n - 2P_n$ and $P_{n+1} = (a - 2 + z_n)Q_n + 2P_n$. Eliminating P_n between these two equations, we arrive at the discrete form of the Airy equation

$$Q_{n+1} - 4Q_n + 2(a + z_{n-1})Q_{n-1} = 0. \tag{40}$$

Elaborate work has been done on finding special function solutions of discrete Painlevé equations in [21,66–71]. It is worth mentioning that by imposing further restrictions on the parameters, one can obtain other types of solutions such as elementary functions, discrete error, beta and gamma functions etc.

(C) *The bilinear formalism for d-P's*

Introduction of tau (τ) function for the continuous Painlevé equations was initiated by Okamoto [22–27], in the context of studying the Hamiltonian structures. Hietarinta and Kruskal [72] wrote the Hirota bilinear forms for continuous Painlevé equations. The Hirota bilinear equations were derived for discrete Painlevé equations by Grammaticos *et al* [73,74]. We present here just d-P_{II} as an example.

Consider the symmetric d-P_{II} equation:

$$x_{n+1} + x_{n-1} = \frac{z x_n + a}{1 - x_n^2}, \tag{19}$$

with $z = \alpha n + \beta$. A detailed study of the d-P$_{\text{II}}$ singularities leads to the singularity patterns $\{-1, \infty, +1\}$ and $\{+1, \infty, -1\}$. The assumption made in [73,74] was that there exists a relationship between the number of singularity patterns of a d-P and the number of the τ-functions necessary for its description. Here, we must introduce two τ-functions F and G. As the τ-functions are entire, x must involve ratios of such functions. With the appropriate choice of gauge, the ansatz for x turns out to be

$$x_n = -1 + \frac{F_{n+1}G_{n-1}}{F_nG_n} = 1 - \frac{F_{n-1}G_{n+1}}{F_nG_n}. \tag{41}$$

Equating the two rightmost sides of this relation leads to the first bilinear equation for d-P$_{\text{II}}$:

$$F_{n+1}G_{n-1} + F_{n-1}G_{n+1} - 2F_nG_n = 0, \tag{42}$$

and substituting the ansatz into (19) yields the second equation:

$$F_{n+2}G_{n-2} - F_{n-2}G_{n+2} = z(F_{n+1}G_{n-1} - F_{n-1}G_{n+1}) + 2aF_nG_n. \tag{43}$$

3.3 *Singularity confinement method for differential-difference equation*

Singularity confinement method has been extended to differential-difference equations [60,75,76]. Consider the Toda lattice

$$\ddot{x}_n = \mathrm{e}^{x_{n+1}-x_n} - \mathrm{e}^{x_n-x_{n-1}}. \tag{44}$$

Introduce $a_n = \mathrm{e}^{x_{n+1}-x_n}$, $b_n = \dot{x}_n$, and rearrange

$$a_n = a_{n-1} + \dot{b}_n, \tag{45}$$

$$b_{n+1} = b_n + \frac{\dot{a}_n}{a_n}. \tag{46}$$

Let us start with case of single zero. Consider

$$a_n = \alpha\tau, \tag{47}$$

where $\tau = t - t_0$ and $\alpha = \alpha(t)$ with $\alpha(t_0) \neq 0$.

Substituting in the equations, we find that

$$b_{n+1} = \frac{1}{\tau} + b_n + \frac{\dot{\alpha}}{\alpha}, \tag{48}$$

$$a_{n+1} = -\frac{1}{\tau^2} + \dot{b}_n + \frac{\ddot{\alpha}}{\alpha} - \left(\frac{\dot{\alpha}}{\alpha}\right)^2 + \alpha\tau. \tag{49}$$

Iterating further, we obtain

$$\begin{aligned} b_{n+2} &= -\frac{1}{\tau} + b_n + \frac{\dot{\alpha}}{\alpha} - 2\tau\left(\dot{b}_n + \frac{\ddot{\alpha}}{\alpha} - \left(\frac{\dot{\alpha}}{\alpha}\right)^2\right) - A\tau^2 + \mathcal{O}, \\ a_{n+2} &= (4A - 7\alpha)\tau + \mathcal{O}(\tau^2), \end{aligned} \tag{50}$$

where A is a quantity which depends on α and b_n. On further iteration, we obtain a finite result for b_{n+3}. Thus, the singularity that appeared at b_{n+1} due to the simple root in a_n is confined after two steps. Clearly the vanishing a_n behaviour examined above which induces the divergence of b_{n+1} is not the only one. One can imagine higher-order zeros of the type $a_n = \alpha\tau^k$. Depending on the value of k, more and more intermediate steps are necessary for the confinement of singularity. In principle, the case $a_n \propto \tau^k$ would necessitate $k+1$ steps. However, the simplest singular behaviour is also the most generic one and its study yields the most important integrability constraints for the system. That is, first, the singularities that appear do have the Painlevé property. Secondly, they do not propagate *ad infinitum* under the recursion but they are confined to a few steps.

3.4 *Pre-image non-proliferation*

It is important to note that singularity confinement test is only a necessary condition for the mapping to be integrable. In this context, another crucial aspect of the mapping should be examined. That is, whether the backward evolution of the map introduces any singularity or not. For this purpose, it is important to introduce a notion called pre-image non-proliferation criterian [77]. This means that at each backward evolution there must be a unique pre-image for integrability. Let us start with a very simple example of rational mapping, in which the growth of the number of pre-images must be invoked. In [77,78], we studied the one-component, two-point mapping of the form:

$$x_{n+1} = f(x_n), \tag{51}$$

where f is rational. Singularity confinement considerations lead to

$$f(x_n) = \alpha + \sum_k \frac{1}{(x_n - \beta_k)^{\nu_k}}, \tag{52}$$

with integer ν_k, provided that for all k, $\beta_k \neq \alpha$. Indeed, if $x_n = \beta_k$ at some step, then x_{n+1} diverges, $x_{n+2} = \alpha$ and x_{n+3} is finite. So, the mapping propagates without any further difficulty. However, if we consider the 'backward' evolution, then (52) solved for x_n in terms of x_{n+1} leads to multideterminacy and the number of pre-images grows exponentially with the number of 'backward' iterations. Indeed, the only mapping of the form (51)–(52) with no growth is just the homographic:

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \tag{53}$$

which is the discrete form of the Riccati equation. Thus, in this case, the argument of slow growth of the number of pre-images of x_n is essential in deriving the form of the discrete Riccati equation. Veselov [79] has studied the integrability of polynomial mappings and has shown that the mapping $x_{n+1} = P(x_n, y_n)$, $y_{n+1} = Q(x_n, y_n)$ is integrable (in the sense that it possesses a non-constant polynomial integral $\Phi(x_n, y_n)$), if there exists a polynomial change of coordinate variables transforming the mapping to triangular form:

$$x_{n+1} = \alpha x_n + P(y_n), \tag{54}$$

$$y_{n+1} = \beta y_n + \gamma, \tag{55}$$

for polynomial P. Moreover, he has shown that in this case the complexity of the mapping is bounded. The important feature in (55) is the fact that the equation for y_{n+1} is linear. Thus, the inversion of (55) is straightforward and hence it has a unique pre-image.

4. Algebraic entropy

The measure of the complexity of the discrete dynamical systems is an interesting area of research [60,79–87]. In recent years, this topic assumes importance in the study of integrability of the dynamics of rational maps [55,62,78,85–87]. It has been observed that the dynamical complexity and degree of the composed map has a close link. One can easily define the algebraic entropy of a map from the growth of the degrees of its iterates. Knowing the degree of the first few iterates, generally, it is possible to find a finite recurrence relation between the degrees which could be solved exactly and obtain a closed form expression. In addition, it is also possible to find the generating function for the degree sequence. Moreover, from the generating function, one can easily find the exact value of the algebraic entropy [62]. First point to remember is that, we should consider the birational map in the projective space. This means that the map is written in terms of the projective coordinates (see Appendix A). By this process the map must have homogeneous degree in both the numerator and the denominator. After cancelling the common factors between the numerator and the denominator, the degree of the map is computed. If birational map is denoted by ϕ, then we can define the sequence d_n of the degrees of the successive iterates ϕ^n of ϕ. Here, d_n denotes the number of intersection of the nth image of ϕ generic line with a fixed hyperplane. Then growth of the degree sequence d_n is a measure of the complexity of ϕ. If there is no factorization of the polynomial, the degree sequence $d_n = d^n$ holds. On the other hand, if some factorization appears, then it induces a drop of the degree and thus the degree sequence should satisfy $d_n \leq d^n$. The drop may even be so important that the growth of d_n becomes polynomial and not exponential. For the convenience of the reader, we give a necessary background from algebraic geometry [88] about degree. An algebraic curve is a curve which is determined by some polynomial equation

$$f(x, y) = \sum a_{ij} x^i y^j = 0. \tag{56}$$

The degree of the curve is the degree of the polynomial $f(x, y)$. For computing the degree, we need an important theorem from algebraic geometry, the Etienne Bézout (1779) theorem.

Bézout theorem is concerned with the number of intersection points of two curves in the projective plane.

Claim: [88]

The number of intersection of two algebraic curves is equal to the product of their degrees, if they have no common components.

If X and Y are two algebraic curves without any non-constant common point in P^2 of degree m and n respectively, then the number of intersection points does not exceed mn provided the curves are not tangent to each other at any of their intersection points. Bézout theorem is used by many authors in different contexts. In 1965, Artin and Mazur [89] used Bézout's theorem to prove that the number of isolated periodic points of a map grows at most exponentially. It is interesting to note that topological entropy was introduced in [90]. This idea can be explained as follows. Let ϕ be a continuous mapping of X into itself. The topological entropy $h(\phi, U)$ of a mapping ϕ with respect to a cover U is defined as

$$h(\phi, U) = \lim_{n\to\infty} \frac{H_n}{n},$$

where H_n satisfies $H_{m+n} \leq H_m + H_n$ and $H_n \geq 0$, for all positive integers m, n. (for any open cover U of a compact topological space X, $N(U)$ denotes the member of sets in a subcover of minimal cardinality. Define $M(U) = \log N(U)$, the entropy of U. For any two covers U and W, $U \vee W = \{A \cap B \mid A \in U, B \in W\}$ defines their joint. Now H_n is defined as $H_n = H(U \vee \phi^{-1}U \vee \cdots \vee \phi^{-n+1}U))$. The above limit exists and is finite (for more details, see [90]). In 1990, Arnold [80] introduced the concept of topological complexity of the intersection of a submanifold, moved by a dynamical system with a given submanifold of the phase space, that can increase with time and proved that the Morse and Betti numbers of the transversal intersection 'generally' grow atmost exponentially. Bellon and Viallet [81,82] introduced the concept of algebraic entropy (which is in the same spirit of topological entropy) for studying iterates of rational maps, measuring the rate at which the algebraic degree of the n-iterate of the map grows as a function of n, which is defined as

$$\epsilon(\phi) = \lim_{n\to\infty} \frac{1}{n} \log(d_n). \tag{57}$$

Here again d_n satisfies $d_{m+n} \leq d_m + d_n$. Using certain basic facts from the analysis, we can prove that the limit exists and is finite [91]. The main idea is that there exists a close link between the dynamical complexity of a mapping and degree of the iterates. Here, it is very important to note the comments of Veselov [79]. He has pointed out that "integrability has an essential correlation with the weak growth of certain characteristics". The concept of algebraic entropy has been used by Heitarinta and Viallet [62] to study the integrability of rational maps. By blending the singularity confinement method and algebraic entropy, integrability of many autonomous and non-autonomous maps and differential-difference equations have been studied in a series of papers [85,86,92]. It has been observed that

(1) The degree growth of a generic map is exponential,
(2) exponential degree growth indicates non-integrability of the map,
(3) polynomial degree growth indicates that the map is integrable and
(4) linear degree growth indicates that the map is linearizable.

As an application of the method of algebraic entropy, we consider the map [86]

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}.$$

Let us introduce the homogeneous coordinates $x_0 = p$, $x_1 = q/r$, where we assume that $\deg p = 0$, $\deg q = 1$ and $\deg r = 1$. Now, in terms of homogeneous coordinates the first few iterates of the mapping can be written as

$$x_2 = \frac{r^2 + aqr - pq^2}{q^2},$$

$$x_3 = \frac{q\mathbf{p}_4}{r(r^2+aqr-pq^2)^2},$$

$$x_4 = \frac{(r^2+aqr-pq^2)\mathbf{p}_6}{\mathbf{p}_4^2},$$

$$x_5 = \frac{\mathbf{p}_4\mathbf{p}_9}{r\mathbf{p}_6^2},$$

where $\mathbf{p}_k$'s are polynomials in $p,\ q$ of degree k. It is clear that the degree of each iterate can be computed as follows: 0, 1, 2, 5, 8, 13, 18, 25, 32, 41, … . From the above sequence of the degree, we can arrive at the general formulae: $d_{2n} = 2n^2$ and $d_{2n+1} = 2n^2+2n+1$. As the degree growth obeys polynomial expression, by using (57) one can easily show that the algebraic entropy $\epsilon(\phi)$ is vanishing. This clearly indicates the integrability of the mapping.

Most important application of the algebraic entropy is that this method can be used in combination with singularity confinement to find non-autonomous forms of the integrable mapping. For this purpose, one of the basic assumptions one can make is that the degree growth of the autonomous and non-autonomous mapping should be the same. The first step in the de-autonomization procedure is to assume that the parameters in the mapping depend on n. As the generic mapping is of exponential growth, at some stage of the iteration the degree might be different from that of the autonomous case. At this stage, we need to impose certain restrictions on the parameters to bring down the degree. Solving the constraints of the unknown parameters, we get the non-autonomous integrable mapping. As an application, again we consider the same mapping mentioned earlier but now with $a = a(n) = a_n$ [86]

$$x_{n+1} + x_{n-1} = \frac{a_n}{x_n} + \frac{1}{x_n^2}.$$

Again, by using the projective coordinates, we arrive at each iteration:

$$x_2 = \frac{r^2+a_1qr-pq^2}{q^2},$$

$$x_3 = \frac{q\mathbf{Q}_4}{r(r^2+a_1qr-pq^2)^2},$$

$$x_4 = \frac{(r^2+a_1qr-pq^2)\mathbf{Q}_7}{q\mathbf{Q}_4^2},$$

$$x_5 = \frac{q\mathbf{Q}_4\mathbf{Q}_{12}}{r(r^2+a_1qr-pq^2)\mathbf{Q}_7^2},$$

where $\mathbf{Q}_k$'s are polynomials in $p,\ q$ of degree k.

Therefore, we obtain the degrees as follows: 0, 1, 2, 5, 9, 17, … .

Observe that the degree d_5 in the autonomous case is 8 whereas it is 9 in the non-autonomous case. As per our assumption, the degree growth should be the same for both autonomous and non-autonomous cases. So, at the iteration x_5, we should impose restrictions on the parameter a_n to bring down the degree from 9 to 8. We obtain the constraints

$$a_{n+1} - 2a_n + a_{n-1} = 0. \tag{58}$$

Solving (58), we get $a_n = \alpha n + \beta$. Hence, we obtain the non-autonomous mapping

$$x_{n+1} + x_{n-1} = \frac{\alpha n + \beta}{x_n} + \frac{1}{x_n^2}.$$

The degree sequence of this mapping is exactly the same as the autonomous mapping discussed earlier. In that case the degree growth obeys polynomial expression. Hence we conclude that for the non-autonomous mapping the algebraic entropy also vanishes. Thus, we conclude the non-autonomous mapping as well integrable.

4.1 *Algebraic entropy for linearizable mapping*

Consider the discrete Riccati equation

$$x_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \delta}.$$

By applying the algebraic entropy analysis one can find

$$\{d_n\} = \{1, 1, \ldots\}.$$

Obviously, the degree growth is linear and hence the given map is linearizable.

Our second example is

$$x_{n+1} = a x_{n-1}\left(\frac{x_n - a}{x_n - 1}\right).$$

The degrees of this map is 0, 1, 2, 3, 4, … . Again the degree growth is linear and hence the mapping is linearizable.

Our final example is a non-integrable map

$$x_{n+1} = a x_{n-1}\left(x_n + \frac{1}{x_n}\right).$$

Here, we can find that the degree sequence is 0, 1, 2, 4, 8, 14, 24, 40, 66, 108, … . Notice that the degree growth is exponential. Therefore, the above map is non-integrable.

4.2 *Algebraic entropy for differential-difference systems*

In this section, we extend the algebraic entropy method to differential-difference equations. We consider the semidiscrete KdV equation [93]:

$$u_{n+1} = u_{n-1} + \frac{u_n'}{u_n}, \tag{59}$$

As before, we start with $u_0 = p$, $u_1 = q/r$, where $\deg p = 0$, $\deg q = 1$, $\deg r = 1$ and $\deg t = 1$ and compute the first few iterates of (59). We thus obtain

$$u_2 = \frac{pqr - q'r - qr'}{qr}, \tag{60}$$

$$u_3 = \frac{q^2(pqr - q'r - qr' + r'^2 - rr'') + r^2(p'q^2 + qq'' - q'^2)}{(pqr - q'r - qr')qr}, \tag{61}$$

and so on. Computing the degree of the successive iterates, we find $d_n = 0, 1, 2, 4, 7, 11, 16, 22, \ldots$ i.e., given by $d_n = (n^2 - n + 2)/2$ for $n > 0$. The fact is that the degree growth is polynomial and hence the system (59) is integrable.

5. Conclusion

In this short review, we briefly pointed out the historical background (certainly not a complete one) of considering the differential equations over a complex domain. The main theme is to search for differential equations whose general solution is single-valued and analytic. This led to the discovery of six Painlevé transcendents. The discovery of singularity confinement and introduction of algebraic entropy have been discussed and their effectiveness with examples from maps and differential-difference equations was demonstrated. Certain properties of d-P's have also been discussed. However, in this short review, we did not consider many other important developments in this domain. To mention a few:

(1) Nevanlinna theory: Growth of solutions near infinity [85,94–99].
(2) Ultra discrete Painlevé equations [17,61,100–103].
(3) Singularity confinement for lattice systems, cellular automation [104].
(4) Algebraic entropy for lattice system [105].
(5) Linearizable equations [106,107].
(6) Discrete Painlevé equations over finite field [108].

Interested readers may consult [42, 51, 52, 55–71, 73–79, 81–86, 98–101, 104, 106, 107, 109–124] for further reading.

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Appendix A [88]

DEFINITION
An affine space over the field C is a vector space $C^n = A^n$. Affine variety $X \subset A^n$ is the common zero locus of a collection of polynomials $f_k \in C[x_0, x_1, \ldots, x_n]$.

DEFINITION
Projective n-space P^n can be interpreted as

$$P^n = \frac{C^{n+1} \setminus \{0\}}{\sim},$$

where $\sim$ denotes the equivalence relation of points lying on the same line through the origin. If $(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n) \Leftrightarrow$ there exists a complex number $\lambda \neq 0$ such that $(y_0, y_1, \ldots, y_n) \sim (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$. A point in projective space P^n can be thought of as an equivalence class

$$[(x_0, x_1, \ldots, x_n)] = \{(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) | \lambda \in C\}, \tag{A.1}$$

in which at least one of the coordinates $x_0, x_1, \ldots, x_n$ must be non-zero.

Homogeneous coordinates of the point in P^n can be written as

$$[x_0 : x_1 : \ldots : x_n] \in P^n, \tag{A.2}$$

$$P^1 = C \cup \{\infty\} = C \cup \{[1 : 0]\}, \tag{A.3}$$

$$[x_0 : x_1] \to \begin{cases} x_1/x_0, & \text{for } x_0 \neq 0; \\ \infty, & \text{for } x_0 = 0. \end{cases} \tag{A.4}$$

where $[1 : 0]$ is the point at infinity. The line $x_0 = 0$, labelled ∞ is also a point at infinity. Note that

$$P^2 = C^2 \cup P^1. \tag{A.5}$$

Example

We choose the coordinates $(x_0, x_1, x_2) \in C^3$, so that our reference plane is given by $x_0 = 1$. This identification takes the point

$$[x_0 : x_1 : x_2] \to \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right), \quad \text{for } x_0 \neq 0, \tag{A.6}$$

in the complex plane. When $x_0 = 0$, $[x_0 : x_1 : x_2] \to [x_1 : x_2]$ is a point in the projective line.

P^n is the set of all complex lines through the origin in C^{n+1}.

Projective variety

A projective algebraic variety in P^n is a common zero set of an arbitrary collection of homogeneous polynomials in $n + 1$ variables

$$V = V(\{F_i\}_{i \in I}) \subset P^n. \tag{A.7}$$

Numerical invariant

Degree: The degree of the projective variety V in P^n is the greatest possible finite number of intersection points of V with a linear subvariety $L \subset P^n$ such that

$$d = \deg\ V = \max\{\sharp(V \cap L) < \infty \mid L \text{ linear in } P^n, \dim L + \dim V = n\}. \tag{A.8}$$

Let $\phi = [f_0, f_1, \ldots, f_n]$: $P^N \to P^N$ be a canonical map, when $f_0, \ldots, f_N$ are homogeneous polynomials of degree d with no common factors. Then ϕ is defined at all points.

Homogenization of a curve

Consider the cubic curve

$$y^2 - x^3 - x^2 = 0. \tag{A.9}$$

Introduce homogeneous coordinates

$$(x, y) \to [X : Y : Z], \tag{A.10}$$

$$(x, y) \to \left(\frac{X}{Z} : \frac{Y}{Z}\right). \tag{A.11}$$

On substitution in (A.9), we get

$$\frac{Y^2}{Z^2} - \frac{X^3}{Z^3} - \frac{X^2}{Z^2} = 0. \tag{A.12}$$

This implies

$$ZY^2 - X^3 - ZX^2 = 0. \tag{A.13}$$

Let $F(X, Y, Z) = ZY^2 - X^3 - ZX^2$, then the solution of $F_X = F_Y = F_Z = 0$ are singular.

$$F_X = -3X^2 - 2XZ = 0, \tag{A.14}$$

$$F_Y = 2ZY = 0, \tag{A.15}$$

$$F_Z = Y^2 - X^2 = 0. \tag{A.16}$$

If $Y = 0$, then

$$F_Z = 0 \Rightarrow X = 0. \tag{A.17}$$

Therefore, $[0 : 0 : Z] \sim [0 : 0 : 1] \in C$.

If $Z = 0$, then

$$F_X = 0 \Rightarrow X = 0, F_Z = 0 \Rightarrow Y = 0. \tag{A.18}$$

Therefore,

$$[0 : 0 : 0] \text{ is not in } P^2. \tag{A.19}$$

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