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Isochronous Liénard-type nonlinear oscillators of arbitrary dimensions

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Abstract. In this paper, we briefly present an overview of the recent developments made in identifying/generating systems of Liénard-type nonlinear oscillators exhibiting isochronous properties, including linear, quadratic and mixed cases and their higher-order generalizations. There exists several procedures/methods in the literature to identify/generate isochronous systems. The application of local as well as nonlocal transformations and Ω-modified Hamiltonian method in identifying and generating systems exhibiting isochronous properties of arbitrary dimensions is also discussed in detail. The identified oscillators include singular and nonsingular Hamiltonian systems and *PT*-symmetric systems.

Keywords. Isochronous system; Liénard-type system; singular and nonsingular Hamiltonian.

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1. Introduction

The formulation of fundamental natural laws often results in the form of ordinary differential equations (ODEs), especially nonlinear ones. Different aspects/properties of nonlinear ODEs have been studied in order to obtain information about physical systems [1–8]. Isochronocity is one among of them. A dynamical system is called isochronous if it features in its phase space an open, fully-dimensional region where all its solutions are periodic in all its degrees of freedom with the same, fixed period [9]. The linear harmonic oscillator, that is $\ddot{x} + \omega_0^2 x = 0$ is the prototype of an isochronous system and all the solutions

$$x(t) = A\sin(\omega_0 t + \phi) \tag{1}$$

of its dynamics are periodic with angular frequency ω_0. All other isochronous systems are isoperiodic with the harmonic oscillator. In fact, simple harmonic oscillator itself can be used to identify various isochronous systems of different orders, nonlinearity and dimensions. This identification is based on the introduction of certain local/nonlocal transformations that transform the simple harmonic oscillator equation, under appropriate conditions, to an isochronous oscillator equation having different orders, nonlinearity and dimensions [10,11]. Isochronicity phenomena have been widely studied not only for its impact in stability theory, but also for its relationship with bifurcation and boundary value problems. Various approaches/methods have been developed in order to identify such systems. In this connection, Calogero and his coworkers have done significant work and demonstrated the procedures to identify/generate the isochronous cases. Linearization, Hamiltonian method, defining isochronous constant and identifying isochronous centres are some of the procedures that play vital roles in identifying/generating isochronous systems [12–15].

In this paper, we concentrate on the identification of the isochronous cases belonging to two specific classes of Liénard-type equations and their higher-order generalization. To begin with, we consider quadratic Liénard-type equation [16,17],

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{2}$$

where $f(x)$ and $g(x)$ are arbitrary functions of x. Equation (1) is well known in literature for a long time and is important from both physical and mathematical points of view. For example, the Mathews–Lakshmanan (ML) oscillator, i.e.,

$$\ddot{x} - \frac{\lambda x}{1+\lambda x^2}\dot{x}^2 + \frac{\omega^2 x}{1+\lambda x^2} = 0, \tag{3}$$

where λ and ω are constants, belongs to this class of equation [18,19]. Though it exhibits only one Lie point symmetry, it has been proved to be linearizable through nonlocal transformation. Various generalizations of this equation have been made and it has been analysed in detail at classical as well as quantum levels [20–24]. Due to the importance of eq. (2), several attempts have been made to explore the isochronous properties associated with it and to identify the isochronous cases belonging to this class of equation through local as well nonlocal transformations for scalar case and its higher-order generalizations. In fact, several theorems have been established to obtain isochronocity condition for this equation [25–28].

Further, we consider the mixed Liénard-type equation [29,30],

$$\ddot{x} + f(x)\dot{x}^2 + g(x)\dot{x} + h(x) = 0, \tag{4}$$

where $f(x)$, $g(x)$ and $h(x)$ are arbitrary functions of x. This equation is also well known in literature from physical and mathematical points of view. The Lotka–Volterra equation (written as a second-order ODE), second-order Gambier equation, when the coefficients are assumed to be constant parameters, and second-order Ricatti equation are notable examples of the mixed Liénard-type equation [31–33]. In fact, the linear Liénard-type equation,

$$\ddot{x} + g(x)\dot{x} + h(x) = 0, \tag{5}$$

which is a subcase of the mixed-type Liénard equation corresponding to $f = 0$, has been known for more than seven decades now. Initially, it was introduced for modelling electrical circuits. Later on it has been used in various fields of science, mathematics, engineering and biology [34–37]. Many dynamical systems exhibiting interesting properties correspond to this class of equations. The modified Emden equation (MEE) introduced by Chandrasekar, Senthilvelan and Lakshmanan [38–42], namely

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \omega_0^2 x = 0 \tag{6}$$

is an example of this class of equation that corresponds to amplitude-independent frequency of oscillations, which is an interesting fact in view of the nonlinear nature of the oscillator. This has motivated many researchers to identify such nonlinear oscillators exhibiting amplitude-independent frequency of oscillations. The isochronous properties of the mixed-type Liénard equation are also studied in literature with the help of both local and nonlocal transformations but not much progress has been made in this direction through point transformations. However, the nonlocal transformations play vital roles in identifying isochronous cases belonging to this class of equation.

In this paper, we present a brief overview of some of the recent progress in understanding isochronous nonlinear systems. To start with, we consider the scalar quadratic Liénard-type equation and discuss its isochronocity through point transformations. We then move on to identify the singular and nonsingular N-dimensional Hamiltonian systems exhibiting isochronous properties. Further, we discuss the isochronous properties of mixed-type Liénard equation through nonlocal transformation and identify a class of isochronous equations belonging to this case. Next, we consider the two- and N-dimensional generalizations of the MEE equation and show that they correspond to singular isochronous Hamiltonian systems. In addition to this, we also report the method of identification of a generic class of two-dimensional nonstandard Hamiltonian systems which is PT-symmetric in nature and exhibits isochronous behaviour. Finally, we discuss the identification of a class of equations exhibiting isochronous properties through a general nonlocal transformation.

The plan of the paper is as follows. In §2, we discuss the isochronocity of quadratic-type Liénard equation through point transformation. The method of Ω-modified Hamiltonian is discussed in §3. Section 4 deals with the method of constructing higher-dimensional isochronous systems. In §5, we discuss the isochronocity of mixed-type Liénard equation through nonlocal transformation. Higher-dimensional coupled integrable versions of MEE are discussed in §6. In §7, we discuss the two-dimensional isochronous PT-symmetric nonstandard Hamiltonian systems. The identification of isochronous cases belonging to N coupled dynamical systems through a generalized nonlocal transformations is discussed in §8. Finally, conclusion is given in §9.

2. Quadratic Liénard-type equation

As mentioned earlier, the quadratic Liénard-type equation is important from both physical and mathematical points of view. Hence, various aspects including isochronous properties have been studied in detail in the literature. In fact, the isochronous properties of this equation have been studied through local as well as nonlocal transformations and several isochronous systems have been identified. However, in this section we discuss the results regarding the isochronous properties associated with the quadratic Liénard-type equation through invertible point transformation.

2.1 *Isochronous Liénard-type oscillator*

To start with, let us consider the dynamical system

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0. \tag{7}$$

The isochronous properties of this equation have been studied in detail and several theorems have also been established [16,25–28]. In fact, it has been proved by many researchers that eq. (7) exhibits isochronous properties provided the arbitrary functions $f(x)$ and $g(x)$ follow the relation

$$g_x + fg = g_1, \tag{8}$$

where g_1 is a constant. The Lie symmetries of eq. (7) have been studied in detail and the equations belonging to one-, two-, three- and eight-parameter Lie point symmetries have been identified. Interestingly, for the linearizable case (eight-parameter symmetry) the form of the function g matches exactly with relation (8). Hence, one can conclude that all the linearizable cases of eq. (7) are isochronous [16].

Starting with the simple harmonic oscillator

$$\ddot{Y} + \omega_0^2 Y = 0, \tag{9}$$

one can identify the isochronous case belonging to eq. (7) through the invertible point transformation

$$Y = h(x), \tag{10}$$

where $h(x)$ is a function of x as

$$\ddot{x} + \frac{h''(x)}{h'(x)}\dot{x}^2 + \omega_0^2\frac{h(x)}{h'(x)} = 0, \tag{11}$$

where $'$ denotes differentiation with respect to x. It is to be noted that the isochronocity condition given by (8) can be obtained with the help of the relations obtained by comparing eqs (7) and (11). The Lagrangian and the corresponding Hamiltonian of this equation are of the form:

$$L = \dot{x}^2 h'^2 - \omega_0^2 h^2, \tag{12a}$$

$$H = \frac{1}{4}\frac{p^2}{h'^2} + \omega_0^2 h^2, \tag{12b}$$

where the canonically conjugate momentum is defined as $p = 2\dot{x}h'^2$. Under the canonical transformation $U = \sqrt{2}h(x)$ and $P = p/\sqrt{2}h'$, Hamiltonian (12b) can be transformed to the Hamiltonian of the simple harmonic oscillator, that is $H = \frac{1}{2}(P^2 + \omega_0^2 U^2)$ and the solution can be written as

$$h(x) = \frac{1}{\sqrt{2}} A \sin(\omega_0 t + \delta), \tag{13}$$

where A and δ are arbitrary parameters. Here, the frequency of oscillations is independent of the amplitude. Thus, all the periodic solutions of eq. (11) are isochronous.

2.2 *Nonisochronous Liénard-type oscillator*

As discussed earlier, eq. (7) contains several equations having physical and mathematical importance. One such example to this class of equation is ML oscillator, that is

$$\ddot{x} - \frac{\lambda x}{1 + \lambda x^2}\dot{x}^2 + \frac{\omega^2 x}{1 + \lambda x^2} = 0. \tag{14}$$

Interestingly, eq. (14) exhibits only one Lie point symmetry but is found to be linearizable through nonlocal transformation [16,18,19]. This rather unusual property leads to the investigations of the properties associated with this equation from classical as well as quantum points of view. The general solution of the above equation can be written as

$$x(t) = A \sin(\Omega t + \delta), \qquad \Omega = \frac{\omega_0}{\sqrt{1 + \lambda A^2}}. \tag{15}$$

The ML oscillator exhibits simple harmonic periodic solutions but with amplitude-dependent frequency. The Hamiltonian corresponding to eq. (14) is of the form

$$H = \frac{1}{2} p^2 (1 + \lambda x^2) + \frac{1}{2} \frac{\omega_0^2 x^2}{(1 + \lambda x^2)}, \tag{16}$$

where the canonical conjugate momentum $p = \dot{x}/(1 + \lambda x^2)$. Note that when λ is negative, $|x| < 1/\sqrt{\lambda}$.

3. Ω-modified Hamiltonian method

In the previous subsection, we discussed the isochronous properties associated with the scalar case (7). However, in this section, we consider higher-dimensional quadratic Liénard-type equation and discuss the procedure to identify a class of singular N-dimensional Hamiltonian system exhibiting isochronous properties [43].

Let us consider a N-dimensional system with a Hamiltonian of the form involving velocity-dependent potentials [44],

$$\tilde{H}(\underline{u}, \underline{v}; \Omega) = \frac{1}{2} \left[\left\{ \sum_{n=1}^{N} a_n v_n \left(\frac{\partial U(\underline{u})}{\partial u_n} \right)^{-1} \right\}^2 + \Omega^2 U(\underline{u})^2 \right], \tag{17}$$

where a_n and Ω are constants, $\underline{v} = (v_1, v_2, \ldots, v_N)$, $\underline{u} = (u_1, u_2, \ldots, u_N)$ and $U(\underline{u})$ is an arbitrary function of the canonical coordinates u_n's, $n = 1, 2, ..., N$. However, in the present case, Hessian

$$\Delta \equiv \left| \frac{\partial^2 H}{\partial V_i \partial V_j} \right| = 0, \qquad i, j = 1, 2, \ldots, N, \tag{18}$$

so that the Lagrangian is singular. The corresponding $2N$ first-order canonical equations of motion for the canonical coordinates u_n and v_n are of the form

$$\dot{u}_n = a_n \left[\frac{\partial U(\underline{u})}{\partial u_n} \right]^{-1} H, \qquad n = 1, 2, \ldots, N, \tag{19a}$$

$$\dot{v}_n = H \sum_{m=1}^{N} \left\{ a_m v_m \left[\frac{\partial U(\underline{u})}{\partial u_m} \right]^{-2} \left[\frac{\partial^2 U(\underline{u})}{\partial u_m \partial u_n} \right] \right\} - \Omega^2 U(u) \frac{\partial U(\underline{u})}{\partial u_n}. \tag{19b}$$

Now, the Newton's equation of motion corresponding to the singular Hamiltonian system (17) is accompanied by a system of constraint equations. The resulting constrained equation of motion is of the form

$$\frac{\partial U(\underline{u})}{\partial u_n} \ddot{u}_n + \sum_{m=1}^{N} \left(\dot{u}_n \dot{u}_m \frac{\partial^2 U(\underline{u})}{\partial u_n u_m} \right) + \Omega^2 a_n U(\underline{u}) = 0, \qquad n = 1, 2, \ldots, N. \tag{20}$$

However, not all the coordinates u_i, $i = 1, 2, \ldots, N$ are independent. There exists $(N-1)$ holonomic constraints leading to the following set of $(N-1)$ functional relations on the coordinates u_i, $i = 1, 2, \ldots, N$:

$$\int \mathrm{d}u_1 \frac{\partial U}{\partial u_1} - \int \mathrm{d}u_j \frac{\partial U}{\partial u_j} = C_j, \qquad j = 2, 3, \ldots, N, \tag{21}$$

where C_j's are constants. Equation (21) obviously constitutes a set of $(N-1)$ holonomic constraints on the coordinates u_i. However, eq. (20) without the constraints (21) corresponds to the nonsingular Hamiltonian

$$\tilde{H} = \frac{1}{2} \left(\sum_{i=1}^{N} a_i^2 v_i^2 \left[\frac{\partial U(\underline{u})}{\partial u_i} \right]^{-2} + \Omega^2 U(\underline{u})^2 \right), \tag{22}$$

where the associated canonical equations of motion are

$$\dot{u}_i = a_i^2 v_i \left[\frac{\partial U(\underline{u})}{\partial u_i} \right]^{-2}, \tag{23a}$$

$$\dot{v}_i = \sum_{j=1}^{N} a_j^2 v_j^2 \left[\frac{\partial U(\underline{u})}{\partial u_j} \right]^{-3} \frac{\partial^2 U(\underline{u})}{\partial u_j \partial u_i} - \Omega^2 U(\underline{u}) \frac{\partial U}{\partial u_i}. \tag{23b}$$

Let us consider the canonical coordinate $U(\underline{u})$ as

$$U(\underline{u}) = \sum_{m=1}^{N} b_m u_m^{k_m}, \tag{24}$$

where b_m's are arbitrary real parameters and k_m's are such that $(1/k_m)$'s are positive integers (the general solution becomes multivalued/complex if k_m's are positive integers and hence are avoided). Correspondingly, the general solution for the Newton's equation of motion (without the constraints) (20) is unbounded and nonisochronous. The form of the solution is

$$u_n(t) = u_n(0)\left[1 + \frac{a_n}{b_n(u_n(0))^{k_n}} \times \left(\frac{H(0)\sin\Omega t}{\Omega} + U(0)(\cos\Omega t - 1)\right) + C_n(0)t\right]^{1/k_n}, \tag{25}$$

where $u_n(0)$, $H(0)$ and $C_n(0)$, $\sum_{n=1}^{N} C_n = 0$, $n = 1, 2, \ldots, N$, are integration constants fixed by the initial condition and $U(0) = \sum_{m=1}^{N} b_m u_m(0)^{k_m}$. However, subject to the $(N-1)$ constraints (21), the Newton's equation admits the $(N+1)$ parameter bounded, isochronous solution

$$u_n(t) = u_n(0)\left[1 + \frac{a_n}{b_n(u_n(0))^{k_n}} \times \left(\frac{H(0)\sin(\Omega t)}{\Omega} + U(0)(\cos(\Omega t) - 1)\right)\right]^{1/k_n}, \tag{26}$$

which is also the solution of the Hamilton's equations (19). The above analysis clearly shows that for the singular Hamiltonian systems (17), the equivalent Newton's equation is a holonomic constrained system (with $(N-1)$ constraint conditions) admitting isochronous oscillatory solution as the general solution.

4. Systematic method to construct higher-dimensional isochronous systems

In the previous section, we have discussed the case of isochronous singular Hamiltonian system of higher dimensions. Now, in this section, we discuss a method to construct nonsingular higher-dimensional isochronous systems [14]. For this purpose, we begin with identifying a two-dimensional nonsingular Hamiltonian and then we generalize the results for N-dimensional systems.

Let us consider a two-dimensional system having modified Hamiltonian of the form

$$\tilde{H} = \frac{1}{2}\left[\frac{\left(v_1 U_{2u_2} - v_2 U_{2u_1}\right)^2}{\Delta^2} + \frac{\left(v_2 U_{1u_1} - v_1 U_{1u_2}\right)^2}{\Delta^2} + \Omega_1^2 U_1(u_1, u_2)^2 + \Omega_2^2 U_2(u_1, u_2)^2\right]. \tag{27}$$

Consequently, the equation of motion corresponding to the above Hamiltonian is the following system of constraint-free two coupled second-order ODEs,

$$\ddot{u}_i = \frac{1}{\Delta}\left(\sum_{k=1}^{2}\sum_{j=1}^{2} A_{ijk}\dot{u}_j\dot{u}_k + B_i\right), \qquad i = 1, 2, \tag{28}$$

where

$$A_{ijk} = (-1)^{i+1} \begin{vmatrix} U_{1u_{i+1}} & U_{2u_{i+1}} \\ U_{1u_j u_k} & U_{2u_j u_k} \end{vmatrix},$$

$$B_i = (-1)^i \begin{vmatrix} U_1\Omega_1^2 & U_2\Omega_2^2 \\ U_{1u_{i+1}} & U_{2u_{i+1}} \end{vmatrix}, \qquad 2+i=i, \tag{29}$$

and $U_j = U_j(u_1, u_2)$. In order to obtain explicit general solution of (28) one has to fix the form of U_1 and U_2 in the above equation such that the resultant solutions are analytic and single-valued. For example, we can make the choice

$$U_1 = k_1 u_1^{r_1} + k_2 u_2^{r_2}, \qquad U_2 = k_3 u_1^{r_1} + k_4 u_2^{r_2}, \tag{30}$$

where r_1 and r_2 are such that $1/r_1$ and $1/r_2$ are positive integers so that the resultant solution is single-valued and analytic.

The general solution of system (28) can be obtained as

$$u_1 = \left(\frac{(Bk_2/\Omega_2)\sin(\Omega_2 t + \delta_2) - (Ak_4/\Omega_1)\sin(\Omega_1 t + \delta_1)}{k_2k_3 - k_1k_4} \right)^{1/r_1},$$

$$k_2k_3 - k_1k_4 \neq 0, \tag{31a}$$

$$u_2 = \left(\frac{(Ak_3/\Omega_1)\sin(\Omega_1 t + \delta_1) - (Bk_1/\Omega_2)\sin(\Omega_2 t + \delta_2)}{k_2k_3 - k_1k_4} \right)^{1/r_2}. \tag{31b}$$

The obtained solution (31) is analytic and bounded and exhibits oscillatory behaviour for the choice $1/r_1$ and $1/r_2$ which are positive integers.

This procedure can be generalized to construct isochronous Hamiltonian systems to N degrees of freedom system. Consequently, one gets the following system of N-coupled second-order ODEs,

$$\ddot{u}_i = \frac{-1}{\Delta} \left(\sum_{k=1}^{N} \sum_{j=1}^{N} A_{ijk} \dot{u}_j \dot{u}_k + B_i \right), \qquad i = 1, 2, \ldots, N, \ N > 2, \tag{32}$$

where A_{ijk} and B_i, $j, k = 1, 2, \ldots, N$ are determinants of the form

$$A_{ijk} = \begin{vmatrix} U_{1u_{i+1}} & U_{2u_{i+1}} & \ldots & U_{Nu_{i+1}} \\ U_{1u_{i+2}} & U_{2u_{i+2}} & \ldots & U_{Nu_{i+2}} \\ \vdots & \ddots & \ddots & \vdots \\ U_{1u_N} & U_{2u_N} & \ldots & U_{Nu_N} \\ U_{1u_1} & U_{2u_1} & \ldots & U_{Nu_1} \\ \vdots & \ddots & \ddots & \vdots \\ U_{1u_{i-1}} & U_{2u_{i-1}} & \ldots & U_{Nu_{i-1}} \\ U_{1u_j u_k} & U_{2u_j u_k} & \ldots & U_{Nu_j u_k} \end{vmatrix}_{N\times N}, \quad B_i = \begin{vmatrix} U_1\Omega_1^2 & Q_2\Omega_2^2 & \ldots & Q_N\Omega_N^2 \\ U_{1u_{i+1}} & U_{2u_{i+1}} & \ldots & U_{Nu_{i+1}} \\ U_{1u_{i+2}} & U_{2u_{i+2}} & \ldots & U_{Nu_{i+2}} \\ \vdots & \ddots & \ddots & \vdots \\ U_{1u_N} & U_{2u_N} & \ldots & U_{Nu_N} \\ U_{1u_{i-1}} & U_{2u_{i-1}} & \ldots & U_{Nu_{i-1}} \end{vmatrix}_{N\times N}$$

admit isochronous oscillations for appropriate choices of the determinants A and B. For the general solution of (32), canonical variables u_i, $i = 1, 2, \ldots, N$, evolve periodically with a fixed period T when Ω_i's are commensurate, for appropriate forms of $U_i(u_1, u_2, \ldots, u_N)$ such that the resultant solutions u_i's are analytic and single-valued.

5. Mixed-type Liénard equation

In §2, we have seen that the isochronous cases belonging to quadratic Liénard-type equations can be obtained by a simple point transformation except the equations that are isochronous through more general transformations. However, in the case of mixed Liénard-type equation it is difficult to get the general form for the isochronous cases through point transformation. In this direction, the nonlocal transformations play vital roles. Various attempts have been made and several isochronous equations belonging to this class of equations have been identified. Hence, in this section, we discuss some of the results including nonlocal transformation and their role in the identification of the isochronous cases.

The connection between the linear and nonlinear oscillators is obtained with the help of general nonlocal transformations. In nonlocal transformation, $U = G(x(t)) \times \mathrm{e}^{\int F(x(t'))\mathrm{d}t'}$, the simple harmonic oscillator equation becomes mixed-type Liénard equation of the form

$$\ddot{x} + A(x)\dot{x}^2 + B(x)\dot{x} + C(x) = 0, \quad {}' = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{33}$$

where we have considered

$$A(x) = \frac{G''}{G'}, \qquad B(x) = \frac{G}{G'}F' + 2F$$

and

$$C(x) = \left(F^2 + \omega^2\right)\frac{G}{G'}$$

for our convenience. However, it is not possible to write the Lagrangian and Hamiltonian structure for such general class of equation. But, for suitable specific choices one can identify the isochronous cases belonging to this class of equation. In fact, for

$$A(x) = \frac{f_x}{f}, \qquad B(x) = \frac{(r+2)h_x}{f(r+1)}$$

and

$$C(x) = \frac{hh_x}{f^2(r+1)},$$

the system admits a nonstandard Lagrangian of the form

$$L = \frac{1}{(f(x)\dot{x} + h(x))^r}, \tag{34}$$

where $f(x)$ and $h(x)$ are arbitrary functions of x, r is a real positive parameter and the Hamiltonian as

$$H = \frac{p}{f}\left(\left(-\frac{rf}{p}\right)^{1/(r+1)} - h\right) - \left(-\frac{rf}{p}\right)^{-r/(r+1)}, \tag{35}$$

with the conjugate momentum

$$p = -rf(f\dot{x} + h)^{-(r+1)}. \tag{36}$$

Though the equation of motion corresponding to arbitrary forms of f, h and r is not isochronous, for specific choices the resultant equation of motion exhibits isochronous properties. For example, when $f = 1, r = 1$ and $h = kx^2$, eq. (33) becomes modified Emden equation, i.e.,

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \omega_0^2 x = 0. \tag{37}$$

Equation (37) is an interesting example belonging to the linear Liénard-type equation. It exhibits eight-parameter Lie point symmetries and so it is linearizable under the point transformation [41]

$$X = \left(\frac{k}{3\lambda} - \frac{1}{\sqrt{-\lambda}x}\right)\mathrm{e}^{-\sqrt{-\lambda}t}, \quad \tau = \left(\frac{k}{3\sqrt{-\lambda}} - \frac{1}{x}\right)\mathrm{e}^{\sqrt{-\lambda}t}, \tag{38}$$

where $\omega_0^2 = \lambda$. Under this transformation, eq. (37) gets modified to the form $X'' = 0$, where prime denotes differentiation with respect to a new variable τ. Interestingly, it is also linearizable under the nonlocal transformation [41]

$$U = x\mathrm{e}^{(k/3)\int x(t')\mathrm{d}t}. \tag{39}$$

The above transformation brings eq. (37) to the simple harmonic equation form, i.e., $\ddot{U} + \lambda U = 0$. The general solution of MEE is of the form

$$x(t) = \frac{A\sin(\omega_0 t + \delta)}{1 - (kA/3\omega_0)\cos(\omega_0 t + \delta)}, \qquad 0 \le A < \frac{3\omega_0}{k}. \tag{40}$$

Interestingly the frequency is independent of the amplitude of oscillations, if the solution given by eq. (40) is isochronous. Equation (37) exhibits two explicit time-dependent integrals of motion from which a time-independent integral can be identified. Consequently, a nonstandard Lagrangian and Hamiltonian description can be obtained

$$L = \frac{1}{k\dot{x} + (k^2x^2/3) + 3\lambda_1}, \tag{41a}$$

$$H = \left(\frac{k}{3}x^2 + \frac{3\lambda_1}{k}\right)p - \sqrt{\frac{-p}{k}}, \tag{41b}$$

with the conjugate momentum

$$p = -\frac{k}{(k\dot{x} + (k^2x^2/3) + 3\lambda_1)^2}.$$

6. Higher-dimensional coupled integrable versions of MEE

In §5, we discussed the isochronous cases for a class of equations belonging to the scalar mixed Liénard-type equation and discussed an example of MEE of this class. In this section we discuss its generalization to two and N dimensions.

A two-dimensional generalization of MEE can be written as [45]

$$\ddot{x} = -2(k_1x + k_2y)\dot{x} - (k_1\dot{x} + k_2\dot{y})x - (k_1x + k_2y)^2x - \lambda_1 x,$$
$$\ddot{y} = -2(k_1x + k_2y)\dot{y} - (k_1\dot{x} + k_2\dot{y})y - (k_1x + k_2y)^2y - \lambda_2 y. \tag{42}$$

Equation (42) can be linearized under the nonlocal transformation

$$X = x\mathrm{e}^{\int(k_1x+k_2y)\mathrm{d}t}, \quad Y = y\mathrm{e}^{\int(k_1x+k_2y)\mathrm{d}t}, \tag{43}$$

so that

$$\ddot{X} + \lambda_1 X = 0, \quad \ddot{Y} + \lambda_2 Y = 0. \tag{44}$$

From (43), one can also identify a set of coupled Riccati equations,

$$\dot{x} = \frac{\dot{X}}{X}x - k_1x^2 - k_2xy, \qquad \dot{y} = \frac{\dot{Y}}{Y}y - k_1xy - k_2y^2. \tag{45}$$

By solving eqs (45) one can obtain the explicit oscillatory solutions,

$$x(t) = \frac{A\sin(\omega_1 t + \delta_1)}{1 - (Ak_1/\omega_1)\cos(\omega_1 t + \delta_1) - (Bk_2/\omega_2)\cos(\omega_2 t + \delta_2)},$$
$$y(t) = \frac{B\sin(\omega_2 t + \delta_2)}{1 - (Ak_1/\omega_1)\cos(\omega_1 t + \delta_1) - (Bk_2/\omega_2)\cos(\omega_2 t + \delta_2)},$$

where $\omega_j = \sqrt{\lambda_j}$, $j = 1, 2$, $\left|\frac{Ak_1}{\omega_1} + \frac{Bk_2}{\omega_2}\right| < 1$. Note that the solution may be periodic or quasiperiodic depending on the value of the ratio ω_1/ω_2, i.e., whether it is rational or irrational.

The system (42) does admit a singular Lagrangian for the case $\lambda_1 = \lambda_2 = \lambda$ as

$$L = \frac{1}{\left[k_1(\dot{x} + (k_1x + k_2y)x) + k_2(\dot{y} + (k_1x + k_2y)y) + \lambda\right]}. \tag{46}$$

The problem of constructing appropriate Lagrangian and Hamiltonian of (42) still remains an open problem. One can generalize the above results to a system of N-coupled MEEs,

$$\ddot{x}_i + 2\sum_{j=1}^{N} k_jx_j\dot{x}_i + \sum_{j=1}^{N} k_j\dot{x}_jx_i + \left(\sum_{j=1}^{N} k_jx_j\right)^2 x_i + \lambda_i x_i = 0, \quad i = 1, 2, ..., N, \tag{47}$$

and obtain the explicit periodic solutions. The associated $2N$ integrals turn out to be

$$I_{1i} = \frac{(\dot{x}_i + \sum_{j=1}^{N}(k_jx_j)x_i)^2 + \lambda_i x_i^2}{\left[\sum_{j=1}^{N}[(k_j/\lambda_j)(\dot{x}_j + \sum_{n=1}^{N}(k_nx_n)x_j)] + 1\right]^2}, \tag{48}$$

$$I_{2i} = \tan^{-1}\left[\frac{\sqrt{\lambda_i}x_i}{\dot{x}_i + \sum_{j=1}^{N}(k_jx_j)x_i}\right] - \sqrt{\lambda_i}t, \qquad \lambda_i > 0. \tag{49}$$

We also note here that one can make appropriate contact-type transformations to (47) which maps them onto a system of N uncoupled harmonic oscillators.

7. Two-dimensional isochronous *PT*-symmetric nonstandard Hamiltonian systems

In §6, we discussed the higher-dimensional generalization of MEE equation, where the Lagrangian of the system was singular. However, in this section, we start with a nonsingular Lagrangian and identify a generic class of two-dimensional nonstandard Hamiltonian systems which is *PT*-symmetric in nature and exhibits isochronous behaviour.

As noted in §5, eq. (33) exhibits a nonstandard *PT*-symmetric Hamiltonian and admits isochronous solutions for certain parametric choices. The basic idea of *PT*-symmetric Hamiltonian is to replace the condition that the Hamiltonian of a quantum theory be Hermitian with the weaker condition that it possess space-time reflection symmetry (*PT*-symmetry). This gives freedom to construct and study new kinds of Hamiltonians without giving up any of the key physical properties of quantum theory. These new Hamiltonians exhibit various important mathematical properties that may be useful in understanding various physical phenomena. The operator P is called parity (space reflection) whereas T represents time reversal. These two operators affect the dynamical variables, i.e., the position operator ($\hat{x}$) and the momentum operator ($\hat{p}$) by changing their sign. Their operation can be given as

$$\begin{aligned} &P\colon\ x \to -x, \quad p \to -p, \\ &T\colon\ x \to x, \quad p \to -p, \quad i \to -i. \end{aligned} \tag{50}$$

If the Hamiltonian remains invariant under the operation of the operators P and T then it is called *PT*-symmetric [46].

Due to its importance, this property can be used to identify such Hamiltonian systems by a suitable generalization of the PT-symmetric Lagrangian. For this purpose, one can define the nonstandard Lagrangian as [47]

$$L = \sum_{i=1}^{2} \frac{1}{(f_i \dot{x} + g_i \dot{y} + h_i)^{r_i}}, \tag{51}$$

where $f_i = f_i(x, y)$, $g_i = g_i(x, y)$, $h_i = h_i(x, y)$, r_i = arbitrary parameter, $i = 1, 2$. With this choice of the Lagrangian we can show that Hessian in the present case

$$\Delta \equiv \begin{vmatrix} \partial^2 L/\partial x^2 & \partial^2 L/\partial x \partial y \\ \partial^2 L/\partial y \partial x & \partial^2 L/\partial y^2 \end{vmatrix} \neq 0. \tag{52}$$

Hence, the Lagrangian is nonsingular. From the above Lagrangian, the equation of motion, for the restriction $f_1 = g_1 h_{1x}/h_{1y}$, $f_{1y} = g_{1x}$, $f_2 = g_2 h_{2x}/h_{2y}$ and $f_{2y} = g_{2x}$, turns out to be a system of coupled mixed-type Liénard equation of the form

$$\begin{aligned} \ddot{x} = &-\frac{1}{\hat{\Delta} g_1 g_2 r_{12}} [g_1 g_2((1+r_1)(1+r_2)((f_{2x} g_1 - f_{1x} g_2)\dot{x}^2 - (g_{1y} g_2 - g_1 g_{2y})\dot{y}^2 \\ &- 2(g_{1x} g_2 - g_1 g_{2x})\dot{x}\dot{y}) + (r_1 + 1)(r_2 + 2)(h_{2x}\dot{x} + h_{2y}\dot{y})g_1 \\ &- (r_1 + 2)(r_2 + 1)(h_{1x}\dot{x} + h_{1y}\dot{y})g_2) - g_2^2 h_1 h_{1y}(r_2 + 1) \\ &+ g_1^2 h_2 h_{2y}(r_1 + 1)], \end{aligned} \tag{53a}$$

$$\ddot{y} = \frac{1}{\hat{\Delta} g_1 g_2 r_{12} h_{1y} h_{2y}} [g_1 g_2((r_1+1)(r_2+1)((f_{2x} g_1 h_{1x} h_{2y} - f_{1x} g_2 h_{1y} h_{2x})\dot{x}^2 + (g_{2y} g_1 h_{1x} h_{2y} - g_{1y} g_2 h_{1y} h_{2x})\dot{y}^2 + 2(g_1 g_{2x} h_{1x} h_{2y} - g_{1x} g_2 h_{1y} h_{2x})\dot{x}\dot{y}) - (r_1+2)(r_2+1) g_2 h_{1y} h_{2x}(h_{1x}\dot{x} + h_{1y}\dot{y}) + (r_1+1)(r_2+2) g_1 h_{2y} h_{1x}(h_{2x}\dot{x} + h_{2y}\dot{y})) - g_2^2 h_1 h_{1y}(r_2+1) + g_1^2 h_2 h_{2y}(r_1+1)], \tag{53b}$$

where

$$\hat{\Delta} = g_1 g_2 (h_{1y} h_{2x} - h_{1x} h_{2y})(h_{1y} h_{2y})^{-1} r_{12}, \qquad r_{12} = [(r_1+1)(r_2+1)]^{1/2}.$$

The Hamiltonian associated with (53) corresponding to the Lagrangian (51) is

$$H = \frac{r_{12}}{\hat{\Delta}} \left[\frac{g_1}{h_{1y}}(p_2 h_{1x} - p_1 h_{1y}) \left(h_2 - \left(\frac{g_1(h_{1x} p_2 - h_{1y} p_1) r_{12}}{h_{1y} r_2 \hat{\Delta}} \right)^{-1/(r_2+1)} \right) + \frac{g_2}{h_{2y}}(p_1 h_{2y} - p_2 h_{2x}) \left(h_1 - \left(\frac{g_2(h_{2y} p_1 - h_{2x} p_2) r_{12}}{h_{2y} r_1 \hat{\Delta}} \right)^{-1/(r_1+1)} \right) \right] - \left(\frac{g_1(h_{1x} p_2 - h_{1y} p_1) r_{12}}{h_{1y} r_2 \hat{\Delta}} \right)^{r_2/(r_2+1)} - \left(\frac{g_2(h_{2y} p_1 - h_{2x} p_2) r_{12}}{h_{2y} r_1 \hat{\Delta}} \right)^{r_1/(r_1+1)}, \tag{54}$$

and the corresponding canonical conjugate momentum is

$$p_1 = -\frac{f_1 r_1}{(f_1\dot{x} + g_1\dot{y} + h_1)^{r_1+1}} - \frac{f_2 r_2}{(f_2\dot{x} + g_2\dot{y} + h_2)^{r_2+1}}, \tag{55a}$$

$$p_2 = -\frac{g_1 r_1}{(f_1\dot{x} + g_1\dot{y} + h_1)^{r_1+1}} - \frac{g_2 r_2}{(f_2\dot{x} + g_2\dot{y} + h_2)^{r_2+1}}. \tag{55b}$$

However, the sytem (53) cannot be solved for the arbitrary functions and hence one needs to consider specific forms for the arbitrary functions in order to obtain *PT*-symmetric isochronous systems.

For example, by suitably choosing the parameters one can obtain the following coupled MEE equation:

$$\ddot{x} = \frac{-1}{\hat{\delta}_1}[(3k_1(\alpha_1\dot{x} + \alpha_2\dot{y}) + k_1^2(\alpha_1 x + \alpha_2 y)^2 + \lambda_1)(\alpha_3(\alpha_1 x + \alpha_2 y)) - (3k_2(\alpha_3\dot{y} + \alpha_4\dot{x}) + k_2^2(\alpha_3 y + \alpha_4 x)^2 + \lambda_2)(\alpha_2(\alpha_3 y + \alpha_4 x))], \tag{56a}$$

$$\ddot{y} = \frac{1}{\hat{\delta}_1}[(3k_1(\alpha_1\dot{x} + \alpha_2\dot{y}) + k_1^2(\alpha_1 x + \alpha_2 y)^2 + \lambda_1)(\alpha_4(\alpha_1 x + \alpha_2 y)) - (3k_2(\alpha_3\dot{y} + \alpha_4\dot{x}) + k_2^2(\alpha_3 y + \alpha_4 x)^2 + \lambda_2)(\alpha_1(\alpha_3 y + \alpha_4 x))], \tag{56b}$$

where $\hat{\delta}_1 = (\alpha_1\alpha_3 - \alpha_2\alpha_4)$. There are other ways of generalizing the modified Emden equation to two dimensions. Even though the generalization made about the coupled modified Emden equation is isochronous, the system lacks a Hamiltonian description in order to quantize it. However, we find that the system (56) has the Hamiltonian

$$H = \frac{1}{k_1 k_2 \hat{\delta}_1}[k_2(p_2\alpha_4 - p_1\alpha_3)(k_1^2 X^2 + \lambda_1) + k_1(p_1\alpha_2 - p_2\alpha_1)(k_2^2 Y^2 + \lambda_2)$$
$$-2\sqrt{\hat{\delta}_1}(k_1\sqrt{k_2(p_1\alpha_2 - p_2\alpha_1)} + k_2\sqrt{k_1(p_2\alpha_4 - p_1\alpha_3)})], \qquad (57)$$

where

$$X = (\alpha_1 x + \alpha_2 y) \quad \text{and} \quad Y = (\alpha_3 y + \alpha_4 x).$$

The general solutions of eqs (56a) and (56b) are

$$x = \frac{A\alpha_3\omega_1 \sin(\omega_1 t + \delta_1)}{k_1\hat{\delta}_1(\omega_1 - A\cos(\omega_1 t + \delta_1))} - \frac{B\alpha_2\omega_2 \sin(\omega_2 t + \delta_2)}{k_2\hat{\delta}_1(\omega_2 - B\cos(\omega_2 t + \delta_2))}, \qquad (58a)$$

$$y = -\frac{A\alpha_4\omega_1 \sin(\omega_1 t + \delta_1)}{k_1\hat{\delta}_1(\omega_1 - A\cos(\omega_1 t + \delta_1))} + \frac{B\alpha_1\omega_2 \sin(\omega_2 t + \delta_2)}{k_2\hat{\delta}_1(\omega_2 - B\cos(\omega_2 t + \delta_2))}, \qquad (58b)$$

where $A,\ B,\ \hat{\delta}_1,\ \delta_1$ and δ_2 are arbitrary constants.

8. Generalized nonlocal transformations and integrable N-coupled dynamical systems

In §7, we have discussed how a nonsingular isochronous solutions can be obtained by suitably choosing the form of the Lagrangian. Now, in this section we consider a suitable nonlocal transformation that can be used to identify isochronous systems belonging to N-dimensional coupled dynamical systems exhibiting nonsingular solutions.

To begin with, consider the set of uncoupled linear harmonic oscillators [48]

$$\ddot{Y}_i + \omega_i^2 Y_i = 0, \qquad i = 1, 2, \ldots, N. \qquad (59)$$

Under the nonlocal transformation

$$Y_i = x_i^l e^{\int f_i(x_1, x_2, \ldots, x_N)\mathrm{d}t}, \qquad (60)$$

where l is an arbitrary parameter, eq. (59) becomes

$$\ddot{x}_i + (l-1)\frac{\dot{x}_i^2}{x_i} + \frac{1}{l}\sum_{j=1}^{N} f_i^{(j)}(\bar{x})x_i\dot{x}_j + 2f_i(\bar{x})\dot{x}_i + \frac{1}{l}f_i^2(\bar{x})x_i + \frac{1}{l}\omega_i^2 x_i = 0, \qquad (61)$$

where $i = 1, 2, \ldots, N,\ f_i^{(j)} = (\mathrm{d}f_i/\mathrm{d}x_j)$ and $\bar{x} = (x_1, x_2, \ldots, x_N)$. Then using the Riccati connection, we can obtain the set of first-order ODEs,

$$\dot{x}_i = \frac{1}{l}\left[\frac{\omega_i}{\tan(\omega_i t + \delta_i)}x_i - f_i(x_1, x_2, \ldots, x_N)x_i\right], \qquad i = 1, 2, \ldots, N. \qquad (62)$$

(δ_i's: N-integration constants). For the special choice

$$f_k = f_N(\bar{x}) = f(\bar{x}) = f(x_1, x_2, \ldots, x_N), \qquad k = 1, 2, \ldots, N-1, \qquad (63)$$

$(N-1)$ integrals can be obtained from the relations

$$\left(\frac{x_k}{x_N}\right)^l = I_k \frac{\sin(\omega_k t + \delta_k)}{\sin(\omega_N t + \delta_N)} = h_k(t), \qquad k = 1, 2, \ldots, N-1, \tag{64}$$

where I_k's are the $(N-1)$ independent integrals.

Then the problem (61) reduces to the problem of solving a single first-order ODE:

$$\dot{x}_N = \frac{1}{l}\left[\frac{\omega_N x_N}{\tan(\omega_N t + \delta_N)} - f(\bar{h}(t), x_N)x_N\right]. \tag{65}$$

The associated integrals can be explicitly given as

$$I_k^2 = \left(\frac{x_N}{x_k}\right)^{2(l-1)} \times \left[\frac{\omega_k^2 x_k^2 + (\dot{x}_k + lf(\bar{x})x_k)^2}{\omega_N^2 x_N^2 + (\dot{x}_N + lf(\bar{x})x_N)^2}\right], \qquad k = 1, 2, \ldots, N-1. \tag{66}$$

The remaining integral I_N can be obtained by solving Riccati equation (65) for appropriate forms of f. Consider f as a homogeneous polynomial of the form

$$f = \sum_{p=1}^{N} \mu_p(t) x_p^q, \tag{67}$$

where $\mu_p(t)$'s are arbitrary functions of t. However, we consider μ_p's to be constants only for simplicity, in the present case. Under these assumptions eq. (61) reduces to the system of coupled nonlinear oscillator equations,

$$\ddot{x}_i + (l-1)\frac{\dot{x}_i^2}{x_i} + \frac{q}{l}\sum_{p=1}^{N}\mu_p x_i x_p^{q-1}\dot{x}_p + 2\sum_{p=1}^{N}\mu_p x_p^q \dot{x}_i + \frac{1}{l}\left(\sum_{p=1}^{N}\mu_p x_p^q\right)^2 x_i + \frac{1}{l}\omega_i^2 x_i = 0, \qquad i = 1, 2, \ldots, N. \tag{68}$$

The solution of this system of coupled nonlinear oscillators can be obtained by solving the following first-order nonlinear differential equation obtained by substituting (67) into (65) along with (64):

$$\dot{x}_N = \frac{1}{l}\left[\omega_N \cot(\omega_N t + \delta_N)x_N - x_N^{q+1}\sum_{p=1}^{N}\bar{\mu}_p \frac{\sin^{q/l}(\omega_p t + \delta_p)}{\sin^{q/l}(\omega_N t + \delta_N)}\right], \tag{69}$$

where $\bar{\mu}_p = \mu_p I_p^{q/l}$, $p = 1, 2, \ldots, N-1$ and $\bar{\mu}_N = \mu_N$. The above equation is of the first-order Bernoulli equation type. Hence, by solving the above equation, we get

$$x_N(t) = \frac{\sin^{1/l}(\omega_N t + \delta_N)}{[I_N + (q/l)\int \sum_{p=1}^{N}\bar{\mu}_p \sin^{q/l}(\omega_p t + \delta_p)]^{1/q}}, \tag{70}$$

where I_N is the $2N$th integration constant. The integral appearing in the denominator of the above expression can be integrated explicitly for arbitrary values of q. However, the system admits oscillatory solutions only when l/q is a positive integer.

9. Conclusion

The identification and classification of isochronous systems are of much interest due to its application in various fields of science and technology. In this direction Calogero and his coworkers [9] have introduced a number of systematic procedures for generating isochronous oscillator systems. On the other hand, the introduction of various nonlocal transformations plays a vital role in identifying nonlinear systems exhibiting amplitude-independent frequency of oscillations. In this context, we have presented a brief overview of some of the recent developments made in this direction. Specifically, we have focussed our attention on the progress made in the identification/generation of nonlinear isochronous systems belonging to certain Liénard-type nonlinear systems and their higher-order generalizations. We have shown that the quadratic Liénard-type equation is linearizable through invertible point transformation. Further, we discussed the higher-dimensional singular and nonsingular Hamiltonian systems exhibiting isochronous properties. In addition to this, we have shown that nonlocal transformation plays a vital role in identifying the isochronous cases belonging to the mixed Liénard-type systems. MEE is an interesting example belonging to this class of equation that admits isochronous properties. We have also shown that the higher-dimensional generalization of MEE exhibits isochronocity. The singular and nonsingular nature of the Lagrangian is also discussed in this regard.

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