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Soliton solutions of the generalized sinh-Gordon equation by the binary (G'/G)-expansion method

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Abstract. The aim of this paper is to extend the applications of (G'/G)-expansion method to solve a generalized sinh-Gordon equation. In fact, the binary (G'/G)-expansion method is introduced for finding different new exact solutions. It is shown that this method is a powerful mathematical tool for solving nonlinear evolution equations with time-dependent coefficients in mathematical physics.

Keywords. Binary (G'/G)-expansion method; generalized sinh-Gordon equation; exact solutions.

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1. Introduction

Phenomena in physics and other fields are often described by nonlinear evolution equations. When we want to understand the physical mechanism of phenomena in nature, which are described by nonlinear evolution equations, exact solutions for the nonlinear evolution equations have to be explored. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media, and optical fibres, etc. Thus, the methods for deriving exact solutions for the governing equations have to be developed. Recently, many powerful methods have been established and improved. Some of these methods include the inverse scattering method [1], Darboux transform [2], Hirota bilinear method [3], Painlevé expansion method [4], Bäcklund transformation method [5], tanh-function method [6], homogeneous balance method [7], Jacobi elliptic function expansion method [8], sub-ODE method [9], F-expansion method [10], sine–cosine function method [11] and the exp-function method [12]. Wazwaz [13] studied the following generalized sinh-Gordon equation:

$$u_{tt} - au_{xx} + b\sinh(nu) = 0, \tag{1}$$

where a, b are two constants and n is a positive integer. In this work, the aim is to augment the previous works in [13,14], and obtain new exact solutions of eq. (1), including double periodic wave solutions.

The sinh-Gordon equation is one of the essential nonlinear equations in mathematics and physics. Therefore, it is important to find solutions for this equation. This equation arises as a special case of the Toda lattice equation, a well-known soliton equation in one space and one time dimension, which can be used to model the interaction of neighbouring particles of equal mass in a lattice formation with a crystal. The sinh-Gordon equation has many applications in other branches of nonlinear science. One application of the sinh-Gordon equation is that it can be used to describe the generic properties of string dynamics for strings and multistrings in constant curvature space. Another application of the sinh-Gordon equation is in the field of thermodynamics, where it can be applied to calculate partition and correlation functions precisely.

2. Algorithm of the binary (G'/G)-expansion method

For a given nonlinear partial differential equation

$$\varphi(g(u), u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0, \tag{2}$$

where $g(u)$ is a composite function which is similar to $\sin(nu)$ or $\sinh(nu)$ etc. The (G'/G)-expansion method is simply represented as follows:

We make a transformation

$$u = \phi\left(\frac{U(\xi)}{V(\eta)}\right), \tag{3}$$

where $\xi = \lambda_1(x+c_1t)$, $\eta = \lambda_2(x+c_2t)$, and $\lambda_1, \lambda_2, c_1, c_2$ are unknown parameters which are to be determined. Substituting (3) into (2), yields

$$\varphi(U, U', U'', ..., V, V', V'', ...) = 0. \tag{4}$$

On some constraint conditions, if eq. (4) can be differentiated as

$$U'^2 = P_1 + Q_1U^2 + R_1U^4, \tag{5}$$

$$V'^2 = P_2 + Q_2V^2 + R_2V^4, \tag{6}$$

where $P_1, Q_1, R_1, P_2, Q_2, R_2$ are some parameters. Then, suppose that the solution of ODEs (5) and (6) can be expressed by a polynomial in G'/G as follows:

$$u(\xi) = \alpha_m\left(\frac{G'}{G}\right)^m + \alpha_{m-1}\left(\frac{G'}{G}\right)^{m-1} + \cdots + \alpha_0, \tag{7}$$

where $G = G(\xi)$ satisfies the second-order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{8}$$

$\alpha_m, \ldots, \lambda$ and μ are constants to be determined and $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in ODE (3).

By substituting (7) into eqs (5) and (6) and using the second-order linear ODE (8), collecting all terms with the same order of G'/G together, the left-hand side of eqs (5) and (6) are converted into another polynomial in G'/G. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $\alpha_m, \ldots, \lambda$ and μ.

Assume that the constants $\alpha_m, \ldots, \lambda$ and μ can be obtained by solving the above algebraic equations. As the general solutions of the second-order LODE (8) are well known, substituting $\alpha_m, \ldots, \upsilon$ and the general solutions of eq. (8) into (7) we have more travelling wave solutions of the nonlinear evolution eq. (1).

3. Application to the generalized sinh-Gordon equation

First, consider the following transformation:

$$\xi = \lambda(x + ct), \quad \eta = \lambda\left(x + \frac{a}{c}t\right), \quad a \neq c^2, \tag{9}$$

where λ, c are two parameters to be determined. Under the transformation (9) eq. (1) can be rewritten as

$$\lambda^2 c^2 (c^2 - a) u_{\xi\xi} + \lambda^2 a (a - c^2) u_{\eta\eta} + bc^2 \sinh(nu) = 0. \tag{10}$$

By means of a similar ansatz as given in [15,16], letting

$$u = \frac{4}{n} \tanh^{-1}\left(\frac{U(\xi)}{V(\eta)}\right) \tag{11}$$

for eq. (10), yields

$$\begin{aligned} &\lambda^2 c^2 (c^2 - a)(U^2 - V^2)\frac{U_{\xi\xi}}{U} - 2U_\xi^2 + \lambda^2 a (a - c^2)(V^2 - U^2)\frac{U_{\eta\eta}}{U} \\ &-2U_\eta^2 u_{\eta\eta} = nbc^2 (U^2 + V^2). \end{aligned} \tag{12}$$

Successive differentiation of the above equation with respect to both ξ and η results in

$$\frac{c^2}{UU_\xi}\left(\frac{U_{\xi\xi}}{U}\right)_\xi - \frac{a}{VV_\eta}\left(\frac{V_{\eta\eta}}{V}\right)_\eta = 0, \tag{13}$$

and from (13), one has

$$\frac{c^2}{UU_\xi}\left(\frac{U_{\xi\xi}}{U}\right)_\xi = \frac{a}{VV_\eta}\left(\frac{V_{\eta\eta}}{V}\right)_\eta = \omega, \tag{14}$$

where ω is a parameter to be determined and by integrating eq. (14) we have

$$U_\xi^2 = \frac{\omega}{4c^2} U^4 + \mu_1 U^2 + \upsilon_1, \tag{15}$$

$$V_\xi^2 = \frac{\omega}{4a} V^4 + \mu_2 V^2 + \upsilon_2, \tag{16}$$

where $\mu_1, \nu_1, \mu_2, \nu_2$ are integral constants. Considering (12), (15) and (16) we have the corresponding constraint conditions:

$$\lambda^2 c^2(c^2-a)\mu_1 + \lambda^2 a(a-c^2)\mu_2 + nbc^2 = 0, \quad c^2\nu_1 - a\nu_2 = 0. \tag{17}$$

Next, we study eqs (15) and (16) under the conditions of (G'/G)-expansion method [17].

Considering the homogeneous balance between highest-order derivatives and nonlinear terms in (15) and (16) we get $n = 1$ and $m = 1$. Consequently, we have

$$U(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0, \tag{18}$$

$$V(\xi) = \beta_1\left(\frac{G'}{G}\right) + \beta_0, \quad \beta_1 \neq 0. \tag{19}$$

On substituting (18) and (19) into (15) and (16), collecting all terms with the same powers of (G'/G) and setting each coefficient to zero, we obtain the following system of algebraic equations:

$$\begin{aligned}
&\frac{\omega}{4c^2}(\alpha_0^4) + \mu_1\alpha_0^2 - \mu^2\alpha_1^2 + \nu_1 = 0,\\
&\frac{\omega}{4c^2}(4\alpha_1\alpha_0^3) + 2\mu_1\alpha_1\alpha_0 - 2\alpha_1^2\lambda\mu = 0,\\
&\frac{\omega}{4c^2}(6\alpha_1^2\alpha_0^2) + \mu_1\alpha_1^2 - (\alpha_1^2\lambda^2 + 2\mu\alpha_1^2) = 0,\\
&\frac{\omega}{4c^2}(4\alpha_1^3\alpha_0) - 2\alpha_1^2\lambda = 0,\\
&\frac{\omega}{4c^2}\alpha_1^4 - \alpha_1^2 = 0
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\frac{\omega}{4a}(\beta_0^4) + \mu_2\beta_0^2 - \mu^2\beta_1^2 + \nu_2 = 0,\\
&\frac{\omega}{4a}(4\beta_1\beta_0^3) + 2\mu_2\beta_1\beta_0 - 2\beta_1^2\lambda\mu = 0,\\
&\frac{\omega}{4a}(6\beta_1^2\beta_0^2) + \mu_2\beta_1^2 - (\beta_1^2\lambda^2 + 2\mu\beta_1^2) = 0,\\
&\frac{\omega}{4a}(4\beta_1^3\alpha_0) - 2\beta_1^2\lambda = 0,\\
&\frac{\omega}{4a}\beta_1^4 - \beta_1^2 = 0.
\end{aligned} \tag{21}$$

On solving the algebraic eqs (20) and (21) by using the *Maple*, we get

$$\alpha_1 = \pm\frac{2c\sqrt{\omega}}{\omega}, \quad \alpha_0 = \pm\frac{\lambda c\sqrt{\omega}}{\omega}, \quad c = \pm\sqrt{\frac{4\omega\nu_1}{16\mu^2 - 4\mu_1\lambda^2 - \lambda^2}}, \tag{22}$$

$$\beta_1 = \pm 2\sqrt{\frac{a}{\omega}}, \quad \beta_0 = \pm\frac{\lambda\sqrt{a\omega}}{\omega}, \quad a = -\frac{4\nu_2\omega}{\lambda^4 + 4\mu_2\lambda^2 - 16\mu^2}, \tag{23}$$

$$\lambda = \pm\frac{\sqrt{-(c^4\mu_1 - c^2\mu_1 a + a^2\mu_2 - a\mu_2 c^2)nbc}}{c^4\mu_1 - c^2\mu_1 a + a^2\mu_2 - a\mu_2 c^2}, \tag{24}$$

$$\mu = \pm\frac{1}{4}\frac{\sqrt{(\nu_1^2 - \nu_2^2)(-4\mu_1\nu_2^2 + 4\nu_1^2\mu_2 + \lambda^2\nu_1^2 - \nu_2^2)\lambda}}{\nu_1^2 - \nu_2^2}. \tag{25}$$

By solving eq. (8), we deduce after some reduction that [18]

$$\frac{G'}{G} = \frac{1}{2}\sqrt{\lambda^2 - 4\mu} \times \left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) - \frac{\lambda}{2}, \tag{26}$$

where A and B are arbitrary constants.

On substituting (22)–(26) into (18)–(19), along with (11) we deduce the following traveling wave solutions:

Case 1. When $\lambda^2 - 4\mu \succ 0$, then we have

$$U = \pm\frac{\sqrt{\omega}\left(\lambda^2 - 4\mu\right)}{\omega} \times \left(\frac{A_1 \sinh\frac{1}{2}\sqrt{\left(\lambda^2 - 4\mu\right)}\xi + A_2 \cosh\frac{1}{2}\sqrt{\left(\lambda^2 - 4\mu\right)}\xi}{A_1 \cosh\frac{1}{2}\sqrt{\left(\lambda^2 - 4\mu\right)}\xi + A_2 \sinh\frac{1}{2}\sqrt{\left(\lambda^2 - 4\mu\right)}\xi}\right),$$

$$V = \pm\left(\lambda^2 - 4\mu\right)\sqrt{\frac{a}{\omega}} \times \left(\frac{A_1 \sinh\frac{1}{2}\sqrt{\left(\lambda^2 - 4\mu\right)}\eta + A_2 \cosh\frac{1}{2}\sqrt{\left(\lambda^2 - 4\mu\right)}\eta}{A_1 \cosh\frac{1}{2}\sqrt{\left(\lambda^2 - 4\mu\right)}\eta + A_2 \sinh\frac{1}{2}\sqrt{\left(\lambda^2 - 4\mu\right)}\eta}\right) \mp\lambda\sqrt{\frac{a}{\omega}} \pm \frac{\lambda\sqrt{a\omega}}{\omega}.$$

So we have

$$u_1 = \frac{4}{n} \times \tanh^{-1}\left(\frac{\pm\frac{\sqrt{\omega}(\lambda^2-4\mu)}{\omega}\left(\frac{A_1 \sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + A_2 \cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}{A_1 \cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi + A_2 \sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\xi}\right)}{\pm\left(\lambda^2-4\mu\right)\sqrt{\frac{a}{\omega}}\left(\frac{A_1 \sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\eta + A_2 \cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\eta}{A_1 \cosh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\eta + A_2 \sinh\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\eta}\right) \mp\lambda\sqrt{\frac{a}{\omega}} \pm \frac{\lambda\sqrt{a\omega}}{\omega}}\right),$$

where

$$\xi = \lambda\left(x \pm \sqrt{\frac{4\omega\nu_1}{16\mu^2 - 4\mu_1\lambda^2 - \lambda^2}t}\right), \quad \eta = \lambda\left(x - \frac{\frac{4\nu_2\omega}{\lambda^4 + 4\mu_2\lambda^2 - 16\mu^2}}{\sqrt{\frac{4\omega\nu_1}{16\mu^2 - 4\mu_1\lambda^2 - \lambda^2}}}t\right).$$

Case 2. When $\lambda^2 - 4\mu \succ 0$, then we have

$$u_2 = \frac{4}{n} \times \tanh^{-1}\left(\frac{\pm\frac{\sqrt{\omega}(4\mu-\lambda^2)}{\omega}\left(\frac{A_1 \sin\frac{1}{2}\sqrt{(4\mu-\lambda^2)}\xi + A_2\cos\frac{1}{2}\sqrt{(4\mu-\lambda^2)}\xi}{A_1\cos\frac{1}{2}\sqrt{(4\mu-\lambda^2)}\xi + A_2\sin\frac{1}{2}\sqrt{(4\mu-\lambda^2)}\xi}\right)}{\pm\left(4\mu-\lambda^2\right)\sqrt{\frac{a}{\omega}}\left(\frac{A_1\sin\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\eta + A_2\cos\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\eta}{A_1\cos\frac{1}{2}\sqrt{(4\mu-\lambda^2)}\eta + A_2\sin\frac{1}{2}\sqrt{(4\mu-\lambda^2)}\eta}\right)\mp\lambda\sqrt{\frac{a}{\omega}}\pm\frac{\lambda\sqrt{a\omega}}{\omega}}\right).$$

Case 3. When $\lambda^2 - 4\mu = 0$, then we have

$$u_3 = \frac{4}{n}\tanh^{-1}\left(\frac{\pm\frac{2c\sqrt{\omega}}{\omega}\left(\frac{A_2}{A_1+A_2\xi}\right)\pm\frac{\lambda c\sqrt{\omega}}{\omega}}{\pm2\sqrt{\frac{a}{\omega}}\left(\frac{A_2}{A_1+A_2\eta}\right)\pm\frac{\lambda\sqrt{a\omega}}{\omega}}\right).$$

When the parameters are given special values, the solitary waves are derived from the travelling waves. In particular, if $A_1 \neq 0$, $A_2 = 0$, $\lambda \succ 0$, $\mu = 0$, u_1 will be

$$u_1 = \frac{4}{n}\tanh^{-1}\left(\frac{\pm\frac{\lambda\sqrt{\omega}}{\omega}\left(\tanh\frac{1}{2}\lambda\xi\right)}{\pm\lambda\sqrt{\frac{a}{\omega}}\left(\tanh\frac{1}{2}\lambda\eta\right)\mp\sqrt{\frac{a}{\omega}}\pm\frac{\sqrt{a\omega}}{\omega}}\right),$$

$$\xi = \lambda\left(x \pm \sqrt{\frac{4\omega\nu_1}{16\mu^2 - 4\mu_1\lambda^2 - \lambda^2}t}\right), \quad \eta = \lambda\left(x - \frac{\frac{4\nu_2\omega}{\lambda^4+4\mu_2\lambda^2-16\mu^2}}{\sqrt{\frac{4\omega\nu_1}{16\mu^2-4\mu_1\lambda^2-\lambda^2}}}t\right),$$

where

$$\lambda = \pm\frac{\sqrt{-(c^4\mu_1 - c^2\mu_1 a + a^2\mu_2 - a\mu_2c^2)nbc}}{c^4\mu_1 - c^2\mu_1 a + a^2\mu_2 - a\mu_2 c^2},$$

$$\mu = \pm\frac{1}{4}\frac{\sqrt{(\nu_1^2-\nu_2^2)(-4\mu_1\nu_2^2 + 4\nu_1^2\mu_2 + \lambda^2\nu_1^2 - \nu_2^2)\lambda}}{\nu_1^2 - \nu_2^2}$$

λ and μ satisfy in all the above cases.

4. Conclusion

In this paper, we successfully constructed the new exact travelling wave solutions by introducing the binary (G'/G)-expansion method and obtained new exact solutions of a generalized sinh-Gordon equation. This method is reliable and effective and gives more solutions. This method has more advantages. It is direct and concise. The proposed method in this paper can also be extended to solve some NLPDEs in mathematical physics.

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