PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 85, No. 4
October 2015
pp. 701–712

A computational method for the solution of one-dimensional nonlinear thermoelasticity

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MS received 20 February 2014; revised 25 June 2014; accepted 28 July 2014
DOI: 10.1007/s12043-014-0899-y; ***e*Publication:** 5 March 2015

Abstract. In this paper, one of the newest analytical methods, new homotopy perturbation method (NHPM), is considered to solve thermoelasticity equations. Results obtained by NHPM, which does not need small parameters, are compared with the numerical results and a very good agreement is found. This method provides a convenient way to control the convergence of approximation series and adjust convergence regions when necessary. The results reveal that the proposed method is explicit, effective and easy to use.

Keywords. Nonlinear coupled system of thermoelasticity; new homotopy perturbation method.

PACS No. 52.35.Mw

1. Introduction

Nonlinear equations are widely used as models to describe complex physical phenomena in various fields of sciences. A large class of nonlinear equations do not have a precise analytic solution, and so numerical methods have extensively been used to handle these equations. There are also some analytic techniques for nonlinear equations. Some of the classic analytic methods are the Lyapunov's artificial small parameter method, perturbation techniques and d-expansion method. In the last two decades, some new analytic methods have been proposed to handle functional equations, among them are Adomian decomposition method, tanh method, sinh–cosh method, homotopy analysis method (HAM), variational iteration method (VIM) and homotopy perturbation method (HPM). The homotopy perturbation method (HPM) was established by Ji-Huan He in 1999. The method has been used by many researchers to analyse a wide variety of scientific and engineering applications to solve various functional equations. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions. Using the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$, which is considered as a 'small parameter'. Considerable research work has recently been conducted in applying this method to a class of linear and nonlinear equations. This method was further developed and improved by He, and applied to nonlinear oscillators with discontinuities [1], nonlinear wave equations [2], boundary value problems [3], limit cycle and bifurcation of nonlinear problems [4] and many other subjects [5–8]. It can be said that He's HPM is a universal one, and is able to solve various kinds of nonlinear functional equations. For example, it was applied to nonlinear Schrödinger equations [9], nonlinear equations arising in heat transfer [10], the quadratic Ricatti differential equation [11] and to other equations [12–29].

Thermoelasticity is the study of the relationship between the elastic properties of a material and its temperature or between its thermal conductivity and stresses and the governing equations are

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial^2 u}{\partial x^2} + b\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial \theta}{\partial x} = f(x,t),\\
&\frac{\partial \theta}{\partial t} + b\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial^2 u}{\partial t \partial x} - c(\theta)\frac{\partial^2 \theta}{\partial x^2} = g(x,t)
\end{aligned} \tag{1}$$

with initial conditions

$$\begin{aligned}
&u(x,0) = u^0(x),\\
&u_t(x,0) = u^1(x),\\
&\theta(x,0) = \theta^0(x),
\end{aligned}$$

where $u = u(x,t)$ is the body displacement from equilibrium and $\theta = \theta(x,t)$ is the difference of the body's temperature from a reference $T_0 = 0$, and subscripts denote partial derivatives, $a, b,$ and c are given smooth functions. For more details about the physical meaning of the model, refer [30,31].

In this paper, a modified version of HPM called NHPM, is applied, which performs much better than the HPM. The two most important steps in the application of the new HPM are to construct a suitable homotopy equation and to choose a suitable initial guess. The new homotopy perturbation method (NHPM) was applied to linear and nonlinear PDEs [29].

The new modification of HPM, for solving nonlinear thermoelasticity is presented. The efficiency of NHPM is verified by the numerical results for two examples in §3. Comparisons between NHPM and HPM have been illustrated in this section. Conclusions will appear in §4.

2. NHPM applied to one-dimensional nonlinear thermoelasticity

For solving eq. (1) by the NHPM [29] the following homotopies are constructed:

$$\begin{aligned}
&(1-p)\left(\frac{\partial^2 u}{\partial t^2} - u_*\right) + p\left(\frac{\partial^2 u}{\partial t^2} - a\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial^2 u}{\partial x^2} + b\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial \theta}{\partial x} - f(x,t)\right) = 0,\\
&(1-p)\left(\frac{\partial \theta}{\partial t} - \theta_*\right) + p\left(\frac{\partial \theta}{\partial t} + b\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial^2 u}{\partial t \partial x} - c(\theta)\frac{\partial^2 \theta}{\partial x^2} - g(x,t)\right) = 0
\end{aligned} \tag{2}$$

or

$$\frac{\partial^2 u}{\partial t^2} = u_* - p\left(u_* - a\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial^2 u}{\partial x^2} + b\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial \theta}{\partial x} - f(x,t)\right),$$
$$\frac{\partial \theta}{\partial t} = \theta_* - p\left(\theta_* + b\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial^2 u}{\partial t \partial x} - c(\theta)\frac{\partial^2 \theta}{\partial x^2} - g(x,t)\right). \quad (3)$$

By integrating eq. (3), we obtain

$$\begin{cases} u(x,t) = u(x,0) + t\dfrac{\partial u(x,0)}{\partial t} + \displaystyle\int_0^t\int_0^t u_* \mathrm{d}t\mathrm{d}t \\ \quad -p\displaystyle\int_0^t\int_0^t \left(u_* - a\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial^2 u}{\partial x^2} + b\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial \theta}{\partial x} - f(x,t)\right)\mathrm{d}t\,\mathrm{d}t, \\ \theta(x,t) = \theta(x,0) + \displaystyle\int_0^t \theta_* \mathrm{d}t \\ \quad -p\displaystyle\int_0^t \left(\theta_* + b\left(\frac{\partial u}{\partial x}, \theta\right)\frac{\partial^2 u}{\partial t \partial x} - c(\theta)\frac{\partial^2 \theta}{\partial x^2} - g(x,t)\right)\mathrm{d}t. \end{cases} \quad (4)$$

Let us present the solution of system (4) as

$$u = u_0 + pu_1 + p^2 u_2 + \cdots,$$
$$\theta = \theta_0 + p\theta_1 + p^2\theta_2 + \cdots, \quad (5)$$

where u_j and θ_j, $j = 0, \ldots, n$ are functions which need to be determined. Suppose that the initial approximation of eqs (1) is in the following form:

$$u_*(x,t) = \sum_{j=0}^{\infty} a_j(x) P_j(t),$$
$$\theta_*(x,t) = \sum_{j=0}^{\infty} b_j(x) P_j(t), \quad (6)$$

where $a_j(x)$ and $b_j(x)$, $j = 0, \ldots, n$, are unknown coefficients and $P_0(t)$, $P_1(t)$, $P_2(t)$, ... are specific functions. Substituting (5) and (6) into (4) and equating the coefficients of p with the same powers leads to

$$p^0: \begin{cases} u_0(x,t) = u(x,0) + t\dfrac{\partial u(x,0)}{\partial t} + \displaystyle\int_0^t\int_0^t u_* \mathrm{d}t\,\mathrm{d}t \\ \theta_0(x,t) = \theta(x,0) + \displaystyle\int_0^t \theta_* \mathrm{d}t \end{cases}$$

$$p^1: \begin{cases} u_1(x,t) = -\displaystyle\int_0^t\int_0^t \left(u_0 - a\left(\frac{\partial u_0}{\partial x}, \theta_0\right)\frac{\partial^2 u_0}{\partial x^2}\right. \\ \qquad \left. + b\left(\frac{\partial u_0}{\partial x}, \theta_0\right)\frac{\partial \theta_0}{\partial x} - f(x,t)\right)\mathrm{d}t\,\mathrm{d}t, \\ \theta_1(x,t) = -\displaystyle\int_0^t \left(\theta_0 + b\left(\frac{\partial u_0}{\partial x}, \theta_0\right)\frac{\partial^2 u_0}{\partial t \partial x} - c(\theta_0)\frac{\partial^2 \theta_0}{\partial x^2} - g(x,t)\right)\mathrm{d}t, \end{cases}$$
$$\vdots$$
$$(7)$$

Now if these equations are solved in such a way that $\begin{cases} u_1(x,t)=0, \\ \theta_1(x,t)=0, \end{cases}$ then eqs (7) result in

$$\begin{cases} u_1(x,t)=u_2(x,t)=\cdots=0, \\ \theta_1(x,t)=\theta_2(x,t)=\cdots=0. \end{cases}$$

The exact solution may be obtained as follows:

$$\begin{cases} u(x,t)=u_0(x,t)=u^0(x)+tu^1(x)+\sum_{j=0}^{\infty} a_j \int_0^t \int_0^t P_j(t)\mathrm{d}t\,\mathrm{d}t, \\ \theta(x,t)=\theta_0(x,t)=\theta^0(x)+\sum_{j=0}^{\infty} b_j \int_0^t P_j(t)\mathrm{d}t. \end{cases} \tag{8}$$

It is worth mentioning that if $f(x,t)$, $g(x,t)$, $u_*(x,t)$ and $\theta_*(x,t)$ are analytic around $t=t_0$, then their Taylor series can be defined as

$$\begin{aligned} u_*(x,t) &= \sum_{j=0}^{\infty} a_j(x)(t-t_0)^n, \\ \theta_*(x,t) &= \sum_{j=0}^{\infty} b_j(x)(t-t_0)^n, \\ f(x,t) &= \sum_{j=0}^{\infty} a_j^*(x)(t-t_0)^n, \\ g(x,t) &= \sum_{j=0}^{\infty} b_j^*(x)(t-t_0)^n, \end{aligned} \tag{9}$$

which can be used in eqs (7), where $a_j(x)$ and $b_j(x)$, $j=0,\ldots,n$, are unknown coefficients which need to be computed, and $a_j^*(x)$ and $b_j^*(x)$, $j=0,\ldots,n$. are known coefficients. To demonstrate the capability of the NHPM, it has been applied to some examples in §3.

3. Numerical results

In order to manifest the advantages and the accuracy of the NHPM, we have applied the NHPM and the HPM to the system of nonlinear thermoelasticity.

Example 1. Consider the following system of thermoelasticity equations [22] (figure 1):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}+\left(\dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial \theta}{\partial x}=f(x,t), \\ \dfrac{\partial \theta}{\partial t}-\dfrac{\partial^2 \theta}{\partial x^2}+\left(\dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial t\partial x}=g(x,t) \end{cases} \tag{10}$$

with initial conditions

$$\begin{aligned} &u(x,0)=\sin x, \\ &\frac{\partial u}{\partial t}(x,0)=-\sin x. \\ &\theta(x,0)=\cos x \end{aligned}$$

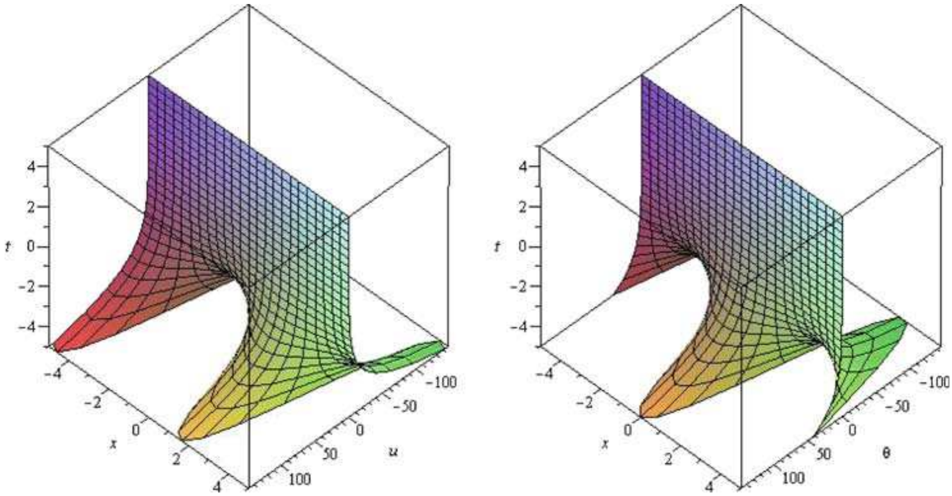

Figure 1. Numerical result of Example 1.

and defined the right-hand side of above equations by

$$f(x,t) = 2\mathrm{e}^{-t}\sin x - \mathrm{e}^{-3t}\cos^2 x \sin x,$$
$$g(x,t) = -\mathrm{e}^{-3t}\cos^3 x.$$

The exact solutions are

$$u(x,t) = \sin x\mathrm{e}^{-t},$$
$$\theta(x,t) = \cos x\mathrm{e}^{-t}.$$

The HPM and NHPM methods are used to approximate the solutions.

The HPM method:

According to the HPM method, we have

$$\begin{cases} u_0 = (1-t)\sin x, \\ \theta_0 = \cos x, \end{cases}$$
$$\begin{cases} u_1 = \left(\frac{1}{2}t^2 + t^3\left(\frac{\cos 2x}{6}\right) + t^4\left(-\frac{5}{48} - \frac{3\cos 2x}{16}\right) + t^5\left(\frac{23}{440} + 9\frac{\cos 2x}{80}\right)\right)\sin x + \cdots, \\ \theta_1 = -t\cos x + \left(t^2 - \frac{3}{2}t^3 + \frac{9}{8}t^4 + \cdots\right)\cos^3 x, \end{cases}$$
$$\begin{cases} u_2 = \frac{1}{80640}(t^3(-42(160 + t(-100 + t(-100 + t(120 + t(-89 + 29t))))) \sin x \\ \quad +(-6720 + t(9240 + t - 10668 + 83(91 - 31t)t))) \sin 3x \\ \quad +3t(560 - 3t(252 + t(-175 + 51t))) \sin 5x))), \\ \theta_2 = \frac{1}{1920}(t^2 \cos x(20(-48 + t(-96 + t(154 + (-96 + t)t))) \cos 2x \\ \quad +(-160 + 9t(40 + t(-34 + 13t))) \cos 4x)), \end{cases}$$
$$\vdots$$

Therefore, the solution will be as follows:

$$\begin{cases} u = u_0 + u_1 + u_2 + \cdots , \\ \theta = \theta_0 + \theta_1 + \theta_2 + \cdots . \end{cases}$$

To solve eq. (1), by the NHPM, the following homotopies are constructed:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = u_0 - p\left(u_0 - \dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial \theta}{\partial x} - f(x,t)\right), \\ \dfrac{\partial \theta}{\partial t} = \theta_0 - p\left(\theta_0 - \dfrac{\partial^2 \theta}{\partial x^2} + \left(\dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial t\partial x} - g(x,t)\right). \end{cases} \tag{11}$$

By integrating both sides of the above equations, we obtain

$$\begin{cases} u(x,t) = u(x,0) + u_t(x,0)t + \displaystyle\int_0^t\int_0^t u_*(x,t)\mathrm{d}t\mathrm{d}t \\ \quad -p\displaystyle\int_0^t\int_0^t \left(u_0 - \dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial \theta}{\partial x} - f(x,t)\right)\mathrm{d}t\mathrm{d}t, \\ \theta(x,t) = \theta(x,0) + \displaystyle\int_0^t \theta_*(x,t)\mathrm{d}t\mathrm{d}t - p\int_0^t\left(\theta_0 - \dfrac{\partial^2 \theta}{\partial x^2} + \left(\dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial t\partial x} - g(x,t)\right)\mathrm{d}t. \end{cases} \tag{12}$$

Suppose the solutions of system (12) are, as assumed in (5), substituting eqs (5) into eqs (12), collecting the same powers of p, and equating each coefficient of p to zero, results in

$$p^0 : \begin{cases} u_0(x,t) = u(x,0) + u_t(x,0)t + \displaystyle\int_0^t\int_0^t u_*(x,t)\mathrm{d}t\,\mathrm{d}t, \\ \theta_0(x,t) = \theta(x,0) + \displaystyle\int_0^t \theta_*(x,t)\mathrm{d}t, \end{cases}$$

$$p^1 : \begin{cases} u_1(x,t) = -\displaystyle\int_0^t\int_0^t \left(u_* - \dfrac{\partial^2 u_0}{\partial x^2} + \left(\dfrac{\partial u_0}{\partial x}\theta_0\right)\dfrac{\partial \theta_0}{\partial x} - f(x,t)\right)\mathrm{d}t\,\mathrm{d}t, \\ \theta_1(x,t) = -\displaystyle\int_0^t \left(\theta_* - \dfrac{\partial^2 \theta_0}{\partial x^2} + \left(\dfrac{\partial u_0}{\partial x}\theta_0\right)\dfrac{\partial^2 u_0}{\partial t\partial x} - g(x,t)\right)\mathrm{d}t, \end{cases}$$

$$p^2 : \begin{cases} u_2(x,t) = \displaystyle\int_0^t\int_0^t \left(\dfrac{\partial^2 u_1}{\partial x^2} - \left(\dfrac{\partial u_1}{\partial x}\theta_0\right)\dfrac{\partial \theta_0}{\partial x} - \left(\dfrac{\partial u_0}{\partial x}\theta_1\right)\dfrac{\partial \theta_0}{\partial x} - \left(\dfrac{\partial u_0}{\partial x}\theta_0\right)\dfrac{\partial \theta_1}{\partial x}\right)\mathrm{d}t\,\mathrm{d}t, \\ \theta_2(x,t) = \displaystyle\int_0^t \left(\dfrac{\partial^2 \theta_1}{\partial x^2} - \left(\dfrac{\partial u_1}{\partial x}\theta_0\right)\dfrac{\partial^2 u_0}{\partial t\partial x} - \left(\dfrac{\partial u_0}{\partial x}\theta_1\right)\dfrac{\partial^2 u_0}{\partial t\partial x} - \left(\dfrac{\partial u_0}{\partial x}\theta_0\right)\dfrac{\partial^2 u_1}{\partial t\partial x}\right)\mathrm{d}t, \end{cases}$$

$$\vdots$$

$$p^j : \begin{cases} u_j(x,t) = \displaystyle\int_0^t\int_0^t \left(\dfrac{\partial^2 u_{j-1}}{\partial x^2} - \sum_{k=0}^{j}\left(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} u_i\theta_k \dfrac{\partial \theta_{j-k-i-1}}{\partial x}\right)\right)\mathrm{d}t\,\mathrm{d}t, \\ \theta_j(x,t) = \displaystyle\int_0^t \left(\dfrac{\partial^2 \theta_{j-1}}{\partial x^2} - \sum_{k=0}^{j}\left(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} u_i\theta_k \dfrac{\partial \theta_{j-k-i-1}}{\partial x}\right)\right)\mathrm{d}t, \end{cases}$$

$$\vdots$$

Assume

$$\begin{cases} u_0(x,t) = \sum_{n=0}^{\infty} a_n(x) P_n(t), \quad P_k(t) = t^k, \\ \theta_0(x,t) = \sum_{n=0}^{\infty} b_n(x) P_n(t), \quad P_k(t) = t^k. \end{cases}$$

Solving the above equations for $u_1(x,t), \theta_1(x,t)$ leads to

$$\begin{aligned} u_1(x,t) \;=\; & \left(-\frac{1}{2}a_0(x) + \frac{1}{2}\sin x\right) t + \left(-\frac{1}{6}a_1(x) - \frac{1}{6}\sin x + \frac{1}{3}\cos^2 x \sin x \right. \\ & \left. + \frac{1}{6}b_0(x)\cos x \sin x - \frac{1}{6}b_{0x}(x)\cos^2 x\right) t^2 + \cdots, \end{aligned}$$

$$\begin{aligned} \theta_1(x,t) \;=\; & (-b_0(x) - \cos x)t + \left(-\frac{1}{2}b_1(x) + \frac{1}{2}b_{0xx}(x) + \frac{1}{2}b_0(x)\cos^2(x) \right. \\ & \left. - a_{0x}(x)\cos^2 x + \frac{3}{2}\cos^3 x\right) t^2 + \cdots. \end{aligned}$$

By disintegrating $u_1(x,t), \theta_1(x,t)$ coefficients $a_n(x), b_n(x) \quad (n = 1, 2, 3, \ldots)$ are determined as

$$\begin{aligned} & a_0(x) = \sin x, \quad a_1(x) = -\sin x, \\ & a_2(x) = \frac{1}{2}\sin x, \quad a_3(x) = -\frac{1}{20}\sin x, \ldots \\ & b_0(x) = -\cos x, \quad b_1(x) = \cos x, \\ & b_2(x) = -\frac{1}{2}\cos x, \quad b_3(x) = \frac{1}{6}\cos x, \ldots. \end{aligned}$$

Therefore, an exact solution of eq. (10) is obtained as

$$\begin{aligned} u(x,t) \;=\; & u_0(x,t) = \sin x - t\sin x + \frac{1}{2}a_0(x)t + \frac{1}{6}a_1(x)t^2 \\ & + \frac{1}{12}a_2(x)t^3 + \frac{1}{20}a_3(x)t^4 + \cdots = \sin x \mathrm{e}^{-t}, \end{aligned}$$

$$\theta(x,t) = \cos x + b_0(x)t + \frac{1}{2}b_1(x)t^2 + \frac{1}{3}b_2(x)t^3 + \frac{1}{4}b_3(x)t^4 + \cdots = \cos x \mathrm{e}^{-t}.$$

Example 2. Consider the following nonlinear coupled system in one-dimensional thermoelasticity [23] (figure 2):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \left(2 - \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial x^2} + \left(2 + \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial \theta}{\partial x} = f(x,t), \\ \dfrac{\partial \theta}{\partial t} + \left(2 + \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial t \partial x} - \theta\dfrac{\partial^2 \theta}{\partial x^2} = g(x,t) \end{cases} \tag{13}$$

with initial conditions

$$\begin{aligned} & u(x,0) = \frac{1}{1+x^2}, \\ & \frac{\partial u}{\partial t}(x,0) = 0, \\ & \theta(x,0) = \frac{1}{1+x^2}, \end{aligned}$$

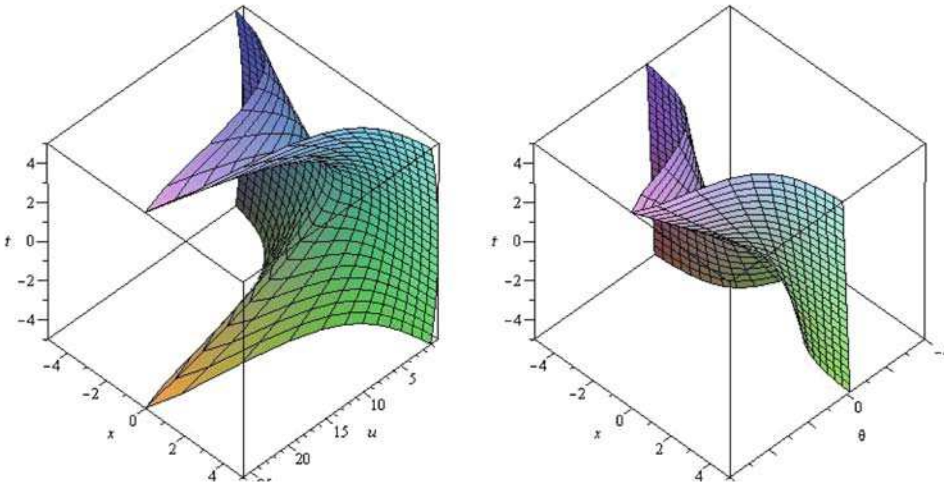

Figure 2. Numerical result of Example 2.

and defining the right-hand side of the above equations by

$$
\begin{aligned}
f(x,t) &= \frac{2}{1+x^2} - \frac{2(1+t^2)(3x^2-1)}{(1+x^2)^3}\left(2-\left(\frac{-2x(1+t^2)}{(1+x^2)^2}\right)\left(\frac{1+t}{1+x^2}\right)\right) \\
&\quad - \frac{2x(1+t)}{(1+x^2)^2}\left(2+\left(\frac{-2x(1+t^2)}{(1+x^2)^2}\right)\left(\frac{1+t}{1+x^2}\right)\right), \\
g(x,t) &= \frac{1}{1+x^2} - \frac{4xt}{(1+x^2)^2}\left(2+\frac{-2x(1+t^2)}{(1+x^2)^2}\left(\frac{1+t}{1+x^2}\right)\right) \\
&\quad - \frac{2(1+t)(3x^2-1)}{(1+x^2)^3}\left(\frac{1+t}{1+x^2}\right).
\end{aligned}
$$

This system is solved by HPM and NHPM.

HPM Method

According to the HPM, we have

$$
\begin{cases}
u_0 = \dfrac{1}{1+x^2}, \\
\theta_0 = \dfrac{1}{1+x^2}, \\
u_1 = \dfrac{1}{105(1+x^2)^6}\left(\begin{array}{l} 105t^2(1+5x^2+10x^4+10x^6+5x^8+x^{10} \\ +70t^3(2x^2-7x^3+2x^4-6x^5-4x^7-x^9) \\ +35t^4(1+x+2x^2-6x^3-2x^4-8x^6-3x^8) \\ +42t^5(x+x^2-3x^3+x^4)+10t^7(x-3x^3) \\ +14t^6(x+x^2-3x^3+x^4) \end{array}\right), \\
\theta_1 = \dfrac{1}{15(1+x^2)^5}\left(\begin{array}{l} 15t(1+4x^2+6x^4+4x^6+x^8) \\ +30t^2(1-2x-62x^3-3x^4-6x^5-2x^7) \\ +10t^3(1+2x^2-3x^4)+30t^4x^2+24t^5x^2 \end{array}\right),
\end{cases}
$$

$\vdots$

Therefore, the solution will be as follows:

$$\begin{cases} u = u_0 + u_1 + u_2 + \cdots, \\ \theta = \theta_0 + \theta_1 + \theta_2 + \cdots, \end{cases}$$

To solve eq (13), by the NHPM, the following homotopies are constructed:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = u_* - p\left(u_0 - \left(2 - \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial x^2} + \left(2 + \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial \theta}{\partial x} - f(x,t)\right), \\ \dfrac{\partial \theta}{\partial t} = \theta_* - p\left(\theta_0 + \left(2 + \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial t \partial x} - \theta\dfrac{\partial^2 \theta}{\partial x^2} - g(x,t)\right). \end{cases} \tag{14}$$

By integrating both sides of the above equations, we obtain

$$\begin{cases} u(x,t) = u(x,0) + u_t(x,0)t + \displaystyle\int_0^t\int_0^t u_*(x,t)\mathrm{d}t\mathrm{d}t \\ -p\displaystyle\int_0^t\int_0^t \left(u_* - \left(2 - \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial x^2} + \left(2 + \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial \theta}{\partial x} - f(x,t)\right)\mathrm{d}t\mathrm{d}t, \\ \theta(x,t) = \theta(x,0) + \displaystyle\int_0^t \theta_*(x,t)\mathrm{d}t\mathrm{d}t \\ -p\displaystyle\int_0^t \left(\theta_* + \left(2 + \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial t \partial x} - \theta\dfrac{\partial^2 \theta}{\partial x^2} - g(x,t)\right)\mathrm{d}t. \end{cases} \tag{15}$$

Suppose the solution of system (15) is similar to (5), substituting eqs (5) into eqs (15), collecting the same powers of p, and equating each coefficient of p to zero, the results will be

$$\begin{cases} u(x,t) = u(x,0) + u_t(x,0)t + \displaystyle\int_0^t\int_0^t u_*(x,t)\mathrm{d}t\,\mathrm{d}t \\ \qquad -p\displaystyle\int_0^t\int_0^t \left(u_* - \left(2 - \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial x^2} + \left(2 + \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial \theta}{\partial x} - f(x,t)\right)\mathrm{d}t\,\mathrm{d}t, \\ \theta(x,t) = \theta(x,0) + \displaystyle\int_0^t \theta_*(x,t)\mathrm{d}t\,\mathrm{d}t \\ -p\displaystyle\int_0^t \left(\theta_* + \left(2 + \dfrac{\partial u}{\partial x}\theta\right)\dfrac{\partial^2 u}{\partial t \partial x} - \theta\dfrac{\partial^2 \theta}{\partial x^2} - g(x,t)\right)\mathrm{d}t, \end{cases}$$

$$p^0: \begin{cases} u_0(x,t) = u(x,0) + u_t(x,0)t + \displaystyle\int_0^t\int_0^t u_*(x,t)\mathrm{d}t\,\mathrm{d}t, \\ \theta_0(x,t) = \theta(x,0) + \displaystyle\int_0^t \theta_*(x,t)\mathrm{d}t, \end{cases}$$

$$p^1: \begin{cases} u_1(x,t) = \displaystyle\int_0^t\int_0^t \left(-u_* + \left(2 - \dfrac{\partial u_0}{\partial x}\theta_0\right)\dfrac{\partial^2 u_0}{\partial x^2} - \left(2 + \dfrac{\partial u_0}{\partial x}\theta_0\right)\dfrac{\partial \theta_0}{\partial x} + f(x,t)\right)\mathrm{d}t\,\mathrm{d}t, \\ \theta_1(x,t) = \displaystyle\int_0^t \left(-\theta_* - \left(2 + \dfrac{\partial u_0}{\partial x}\theta_0\right)\dfrac{\partial^2 u_0}{\partial t \partial x} + \theta_0\dfrac{\partial^2 \theta_0}{\partial x^2} + g(x,t)\right)\mathrm{d}t, \end{cases}$$

$$\vdots$$

$$p^j:\begin{cases} u_j(x,t)=\int_0^t\int_0^t\left(2\frac{\partial^2 u_{j-1}}{\partial x^2}-2\frac{\partial\theta_{j-1}}{\partial x}-\sum_{k=0}^{j}\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\frac{\partial u_i}{\partial x}\theta_k\frac{\partial^2 u_{j-k-i-1}}{\partial x^2}\right.\\ \qquad\left.-\sum_{k=0}^{j}\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\frac{\partial u_i}{\partial x}\theta_k\frac{\partial\theta_{j-k-i-1}}{\partial x}\right)\mathrm{d}t\,\mathrm{d}t,\\ \theta_j(x,t)=\int_0^t\left(-2\frac{\partial^2\theta_{j-1}}{\partial t\partial x}-\sum_{k=0}^{j}\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}\frac{\partial u_i}{\partial x}\theta_k\frac{\partial^2 u_{j-k-i-1}}{\partial t\partial x}\right.\\ \qquad\left.+\sum_{k=0}^{j-1}\theta_k\frac{\partial^2\theta_{j-1-k}}{\partial x^2}\right)\mathrm{d}t, \end{cases}$$

$$\vdots$$

By assuming

$$\begin{cases} u_0(x,t)=\sum_{n=0}^{\infty}a_n(x)P_n(t),\ \ P_k(t)=t^k,\\ \theta_0(x,t)=\sum_{n=0}^{\infty}b_n(x)P_n(t),\ \ P_k(t)=t^k, \end{cases}$$

and solving the above equations for $u_1(x,t),\theta_1(x,t)$ leads to the result

$$\begin{aligned} u_1(x,t) &= \left(-\frac{1}{2}a_0(x)+\frac{1}{1+x^2}\right)t^2\\ &\quad+\left(-\frac{1}{6}a_1(x)+\frac{1}{6}b_0(x)\left(\frac{-2x(-2+6x^2)}{(1+x^2)^5}\right)-\frac{1}{3}b_{0x}(x)+\frac{xb_{0x}(x)}{3(1+x^2)^3}\right.\\ &\quad\left.-\frac{-2x^2b_0(x)}{3(1+x^2)^4}-\frac{(-6x^3+2x)}{3(1+x^2)^6}+\frac{3x^2}{3(1+x^2)^5}-\frac{2x}{3(1+x^2)^2}\right)t^3\\ &\quad+\cdots,\\ \theta_1(x,t) &= \left(-b_0(x)+\frac{1}{1+x^2}\right)t\\ &\quad+\left(-\tfrac{1}{2}b_1(x)-2a_{0x}(x)-a_{0x}(x)\left(\tfrac{-2x}{(1+x^2)^3}\right)\right.\\ &\quad+\frac{1}{2}b_0(x)\left(\frac{6x^2-2}{(1+x^2)^3}\right)+\frac{1}{2(1+x^2)}b_{0xx}(x)+\frac{8x}{2(1+x^2)^2}\\ &\quad\left.-\frac{4x^2}{(1+x^2)^5}-\frac{6x^2-2}{(1+x^2)^5}\right)t^2\\ &\quad+\cdots. \end{aligned}$$

By disintegrating $u_1(x,t),\theta_1(x,t)$ coefficients $a_n(x),b_n(x)$ $(n=1,2,3,\ldots)$ are determined as

$$a_0(x)=\frac{2}{1+x^2},\ a_1(x)=0,\quad a_2(x)=0,\ a_3(x)=0,\ldots$$

$$b_0(x)=\frac{1}{1+x^2},\ b_1(x)=0,\quad b_2(x)=0,\ b_3(x)=0,\ldots.$$

The solution of eq. (13) is given as

$$\begin{aligned} u(x,t) &= u_0(x,t) = \frac{1}{1+x^2} + \frac{1}{2}a_0(x)t^2 + \frac{1}{6}a_1(x)t^3 \\ &\quad + \frac{1}{12}a_2(x)t^4 + \frac{1}{20}a_3(x)t^5 + \cdots = \frac{1+t^2}{1+x^2}, \\ \theta(x,t) &= \theta_0(x,t) = \frac{1}{1+x^2} + b_0(x)t + \frac{1}{2}b_1(x)t^2 \\ &\quad + \frac{1}{3}b_2(x)t^3 + \frac{1}{4}b_3(x)t^4 + \cdots = \frac{1+t}{1+x^2}. \end{aligned}$$

In the above example we have also derived the exact solutions.

4. Conclusions

The HPM and NHPM solutions of the problem under study is assumed to be a summation of a power series in p. The difference between the two methods starts from the initial approximation of the solution. An appropriate choice of the initial solution may lead to ideal results with simple calculations. However, an unsuitable choice of the initial solution requires infinite iterations with complex solution procedures and infinite iterations will result in undesirable outcomes. These two equations are applicable in one-dimensional nonlinear thermoelasticity problems [30,31]. The solutions introduced in this study can be used to obtain the closed form of solutions if they are required. This method will be applied to soliton equations such as KdV and NLSE which will be worked on later and will be reported elsewhere. The computations associated with the examples were performed using *Maple* 14.

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