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The functional variable method for solving the fractional Korteweg–de Vries equations and the coupled Korteweg–de Vries equations

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Abstract. This paper presents the exact solutions for the fractional Korteweg–de Vries equations and the coupled Korteweg–de Vries equations with time-fractional derivatives using the functional variable method. The fractional derivatives are described in the modified Riemann–Liouville derivative sense. It is demonstrated that the calculations involved in the functional variable method are extremely simple and straightforward and this method is very effective for handling nonlinear fractional equations.

Keywords. Korteweg–de Vries equation; coupled Korteweg–de Vries equation; functional variable method.

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1. Introduction

The physical and engineering processes have been modelled by means of fractional calculus, which are found to be best described by fractional differential equations. Unfortunately, in many cases, the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately. Fractional calculus has played a very important role in various fields such as economics, chemistry, notably control theory, electricity, mechanics, ground water problems, biology and signal image processing. Earlier, the investigation of travelling-wave solutions for nonlinear equations had a very effective role in the study of nonlinear physical phenomena. A wide range of physics phenomena has been described by means of the Korteweg–de Vries (KdV) equation which has been used to model the evolution and interaction of nonlinear waves.

The evolution of the Korteweg–de Vries equation, which has been a long process, started about 80 years ago, began with the experiments of Scott Russell in 1834 [1], the investigations of Boussinesq and Rayleigh around 1870 [2–5], and finally culminated with the paper by Korteweg and De Vries in 1895 [6]. It was derived as an evolution equation governing a one-dimensional, small-amplitude, long-surface gravity wave propagating in a shallow channel of water.

Subsequently, the KdV equation has thrived in other physical contexts as ion-acoustic waves, plasma physics, collision-free hydromagnetic waves, lattice dynamics, stratified internal waves, etc. [7]. The KdV model has been used to explain certain theoretical physics phenomena in the quantum mechanics domain. It is used as a model for shock wave formation, solitons, turbulence, boundary layer behaviour, and mass transport in fluid dynamics, aerodynamics, and continuum mechanics.

Now consider the general Korteweg–de Vries equation of the form

$$u_t + (p+1)(p+2)u^p u_x + u_{xxx} = g(x,t), \tag{1}$$

where $g(x,t)$ is a given function and $p = 1, 2, \ldots$ with $u, u_x, u_{xx} \to 0$ as $|x| \to \infty$. If $p = 0$, $p = 1$, and $p = 2$, eq. (1) becomes linearized KdV, nonlinear KdV, and modified KdV equations, respectively. The nonlinear KdV equation has been the focus of recent studies for finding exact solutions in [8–10] as well as numerical solutions in [11–13].

Recently, there has been much interest in fractional diffusion equations. These equations arise in continuous-time random walks, modelling of anomalous diffusive and sub-diffusive systems, unification of diffusion and wave propagation phenomenon, and simplification of the results [14]. The nature of the diffusion is characterized by a mean-square displacement of the form

$$\langle r^2(t) \rangle \sim t^\alpha. \tag{2}$$

For anomalous subdiffusion $\alpha < 1$ and for anomalous superdiffusion $\alpha > 1$, whereas in standard diffusion $\alpha = 1$. For applications on both types of anomalous diffusions, one can refer to [15,16].

In this work, the exact solutions of the nonlinear KdV equation with time- and space-fractional derivatives are considered which is of the following form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon u \frac{\partial^\beta u}{\partial x^\beta} + v \frac{\partial^3 u}{\partial x^3} = 0, \qquad t > 0, \tag{3}$$

where ε and v are parameters and α and β are parameters describing the order of the fractional time- and space-derivatives, respectively. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t < 0$ and $x < 0$. When $\alpha = 1$ and $\beta = 1$, the fractional equation reduces to the classical nonlinear KdV equation. Hirota and Satsuma proposed a coupled KdV equation, which describes the interactions of two long waves with different dispersion relations [17]. If one of the long waves never affects the other, the latter obeys the ordinary KdV equation. However, the behaviour of the KdV solitons accepts the influence of the existence of the former. Solutions show that the former determines the velocity of the KdV soliton. The coupled KdV equations with time-fractional derivatives are given as

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} + 6au\frac{\partial u}{\partial x} - 2bv\frac{\partial v}{\partial x} + a\frac{\partial^3 u}{\partial x^3} = 0, \qquad 0 < \alpha \le 1, \\ &\frac{\partial^\beta v}{\partial t^\beta} + 3bu\frac{\partial v}{\partial x} + b\frac{\partial^3 v}{\partial x^3} = 0, \qquad 0 < \beta \le 1, \end{aligned} \tag{4}$$

where a and b are constants, α and β are parameters describing the order of the time-fractional derivatives of $u(x;t)$ and $v(x;t)$, respectively. The functions $u(x;t)$ and $v(x;t)$ are assumed to be causal functions of time and space, i.e., vanishing for $t<0$ and $x<0$. When $\alpha=\beta=1$, the above system reduces to the classical coupled KdV equations.

Many effective analytic methods such as the Adomian decomposition method [18,19], homotopy perturbation method [20–22], variational iteration method [23], fractional subequation method [24], Lagrange characteristic method [25], first integral method [26–29], and so on [30–32], have been developed to derive approximate or exact solutions of fractional ordinary differential equations, integral equations, and fractional partial differential equations.

A direct and effective method to solve nonlinear partial differential equations was first proposed by Zerarka [33]. Soon after, this method became popular among the other researchers and was developed by many in [34–36].

The aim of this paper is to show the ability of this powerful method for finding exact solutions of the fractional KdV equations and coupled KdV equations.

The rest of this paper is organized as follows. In §2, the functional variable method is briefly described. In §3, the functional variable method is applied for finding exact solutions of the fractional KdV equations and coupled KdV equations.

2. The modified Riemann–Liouville derivative and the functional variable method

The modified Riemann–Liouville derivative was defined by Jumarie [37,38] as

$$D_x^{a} f(x)=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^x (x-\xi)^{-\alpha-1}\left[f(\xi)-f(0)\right]\mathrm{d}\xi, & \alpha<0,\\ \dfrac{1}{\Gamma(-\alpha)}\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_0^x (x-\xi)^{-\alpha}\left[f(\xi)-f(0)\right]\mathrm{d}\xi, & \alpha<0<1,\\ \left(f^{(n)}(x)\right)^{(\alpha-n)},\ n\le\alpha\le n+1, & n\ge 1,\end{cases} \tag{5}$$

Here $\Gamma(\cdot)$ is the gamma function and $f\colon R\to R, x\to f(x)$ denote a continuous (but not necessarily differentiable) function.

One of the properties of the fractional-modified Riemann–Liouville derivative can be stated as follows:

$$D_x^{\alpha}x^{\gamma}=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha},\quad \gamma>0, \tag{6}$$

$$D_x^{\alpha}\left(u(x)v(x)\right)=v(x)D_x^{\alpha}u(x)+u(x)D_x^{\alpha}v(x), \tag{7}$$

$$D_x^{\alpha}\left[f(u(x))\right]=f_u'(u)D_x^{\alpha}u(x)=D_x^{\alpha}f(u)\left(u_x'\right)^{\alpha}. \tag{8}$$

For other properties, refer to [26,39].

Let us present the features of the functional variable method. For a given time-fractional differential equation that it is written in several independent variables $\{t, x, y, z, ...\}$ and a dependent variable u as

$$D\left(u, D_t^{\alpha}u, u_x, u_y, u_z, D_t^{2\alpha}u, u_{xy}, u_{yz}, u_{xz}, ...\right) = 0, \tag{9}$$

where the subscripts denote partial derivatives.

First, the variable transformation is introduced

$$\xi = l_1 x + l_2 y + l_3 z + \cdots - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}, \tag{10}$$

where l_i and λ are constants to be determined later to find the travelling wave solution of eq. (9), so that

$$u(t, x, y, z, ...) = U(\xi). \tag{11}$$

On using this transformation eq. (9) can be reduced to an ordinary differential equation (ODE)

$$Q\big(U, U_{\xi}, U_{\xi\xi}, U_{\xi\xi\xi}, \ldots\big) = 0, \tag{12}$$

where Q is a polynomial in u and its total derivatives.

Then we make a transformation in which the unknown function $U(\xi)$ is considered as a functional variable in the form

$$U_{\xi} = F(u)\,, \tag{13}$$

and some successive derivatives of U are

$$\begin{aligned} U_{\xi\xi} &= \frac{1}{2}\left(F^2\right)', \\ U_{\xi\xi\xi} &= \frac{1}{2}\left(F^2\right)''\sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \frac{1}{2}\left[\left(F^2\right)'''' F^2 + \left(F^2\right)''\left(F^2\right)'\right], \end{aligned} \tag{14}$$

where $'$ stands for $\mathrm{d}/\mathrm{d}U$.

Substituting (14) in (12), the ODE (12) can be reduced in terms of u, f, and its derivatives as

$$R\left(U, F, F', F'', F''', \ldots\right) = 0. \tag{15}$$

Equation (15) is particularly important because it admits analytical solutions for a large class of nonlinear wave-type equations. After integration, eq. (15) provides the expression for F, and this together with eq. (13) give relevant solutions to the original problem. In order to illustrate how the method works, we examine some examples treated by other approaches. This is discussed in the following section.

3. Applications

In this section, two cases of fractional KdV equations and coupled-KdV equations are studied using of the functional variable method.

Example 3.1. The KdV equation has an important role to play in nonlinear physics and it has been used in a number of other physical contexts. Now, consider the time-fractional KdV equation

$$D_t^{\alpha} u + 6uu_x + u_{xxx} = 0, \quad 0 < \alpha \le 1, \quad t > 0, \tag{16}$$

where α is a parameter describing the order of the fractional time derivative, t is the time, and x is the space coordinate in the direction of propagation. Now, on applying the transformation

$$u(x,t) = U(\xi), \quad \xi = lx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}, \tag{17}$$

to eq. (16) and integrating the resulting equation once, we get the ODE

$$l^3 U'' - \lambda U + 3lU^2 = 0$$

or

$$U_{\xi\xi} = \frac{\lambda}{l^3} U - \frac{3}{l^2} U^2. \tag{18}$$

Then we use the transformation

$$U_{\xi} = F(U). \tag{19}$$

Using the above transformation in eq. (16) leads to

$$\frac{1}{2}\left(F^2(U)\right)' = \frac{\lambda}{l^3} U - \frac{3}{l^2} U^2 \tag{20}$$

or

$$F^2(U) = \frac{\lambda}{l^3} U^2 - \frac{2}{l^2} U^3. \tag{21}$$

According to eq. (14), we get from eq. (21), the expression for the function $F(U)$ which reads as

$$F(U) = \pm\sqrt{\frac{\lambda}{l^3}} U \sqrt{1 - \frac{2l}{\lambda} U}. \tag{22}$$

After applying the change of variables

$$Z = \frac{2l}{\lambda} U, \tag{23}$$

and using the relation (11), eq. (22) gives the following solution:

$$U(\xi) = \frac{\lambda}{2l}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{l^3}}\xi\right). \tag{24}$$

Obviously, by setting eq. (24) in eq. (17) the hyperbolic solutions of the time-fractional KdV equation is obtained as

$$u_1(x,t) = \frac{\lambda}{2l}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{l^3}}\left(lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\right), \tag{25}$$

$$u_2(x,t) = -\frac{\lambda}{2l}\,\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{l^3}}\left(lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\right). \tag{26}$$

For $\lambda/l^3 < 0$, the periodic solutions can be obtained as follows:

$$u_3(x,t) = \frac{\lambda}{2l}\,\sec^2\left(\frac{1}{2}\sqrt{-\frac{\lambda}{l^3}}\left(lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\right), \tag{27}$$

$$u_4(x,t) = \frac{\lambda}{2l}\,\csc^2\left(\frac{1}{2}\sqrt{-\frac{\lambda}{l^3}}\left(lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\right). \tag{28}$$

Example 3.2. Consider the following space and time-fractional KdV equation:

$$D_t^\alpha u + u D_x^\beta + u_{xxx} = 0. \tag{29}$$

For solving this equation first, we introduce the following transformation:

$$u(x,t) = U(\xi), \quad \xi = \frac{kx^\beta}{\Gamma(1+\beta)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}. \tag{30}$$

Substituting (30) in eq. (29) we get the ODE

$$-\lambda U' + kUU' + k^3 U''' = 0. \tag{31}$$

After one integration, we get

$$-\lambda U + \frac{kU^2}{2} + k^3 U'' = 0. \tag{32}$$

We use the transformation

$$U_\xi = F(U). \tag{33}$$

So eq. (32) can be converted to

$$\frac{1}{2}\left(F^2(U)\right)' = \frac{\lambda}{k^3}U - \frac{1}{2k^2}U^2. \tag{34}$$

Then the expression for the function $F(U)$ is obtained as

$$F(U) = \pm\sqrt{\frac{\lambda}{k^3}}U\sqrt{1 - \frac{k}{3\lambda}U}. \tag{35}$$

For solving the above equation, the following change of variable is applied:

$$Z = \frac{k}{3\lambda} U. \tag{36}$$

Using the relation (13), eq. (35) has the following solution:

$$U(\xi) = \frac{3\lambda}{k} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{\lambda}{k^3}} \xi \right). \tag{37}$$

When $\lambda / k^3 > 0$, we can easily get the hyperbolic as follows:

$$u_1(x,t) = \frac{3\lambda}{k} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{\lambda}{k^3}} \left(\frac{k x^\beta}{\Gamma(1+\beta)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \right) \right), \tag{38}$$

$$u_2(x,t) = -\frac{3\lambda}{k} \operatorname{csch}^2 \left(\frac{1}{2} \sqrt{\frac{\lambda}{k^3}} \left(\frac{k x^\beta}{\Gamma(1+\beta)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \right) \right). \tag{39}$$

If $\lambda / k^3 < 0$, it is evident that solutions (38) and (39) can reduce to periodic solutions as follows:

$$u_3(x,t) = \frac{3\lambda}{k} \sec^2 \left(\frac{1}{2} \sqrt{-\frac{\lambda}{k^3}} \left(\frac{k x^\beta}{\Gamma(1+\beta)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \right) \right), \tag{40}$$

$$u_4(x,t) = \frac{3\lambda}{k} \csc^2 \left(\frac{1}{2} \sqrt{-\frac{\lambda}{k^3}} \left(\frac{k x^\beta}{\Gamma(1+\beta)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \right) \right). \tag{41}$$

Example 3.3. In this section, the coupled KdV equations are solved with time-fractional derivatives of the form

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} + 6au \frac{\partial u}{\partial x} - 2bv \frac{\partial v}{\partial x} + a \frac{\partial^3 u}{\partial x^3} = 0, \quad 0 < \alpha \le 1, \\ &\frac{\partial^\beta v}{\partial t^\beta} + 3bu \frac{\partial v}{\partial x} + b \frac{\partial^3 v}{\partial x^3} = 0, \quad 0 < \beta \le 1. \end{aligned} \tag{42}$$

In order to obtain solutions of eq. (42), the following transformations are introduced:

$$v(x,t) = \frac{1}{l} U(\xi), \quad u(x,t) = U(\xi), \quad \xi = x - \frac{\lambda_1 t^\alpha}{\Gamma(1+\alpha)} - \frac{\lambda_2 t^\beta}{\Gamma(1+\beta)}. \tag{43}$$

Substituting (43) in eq. (42), the equation can be converted to the ODE

$$\left(-\lambda_1 - \frac{\lambda_2}{l} \right) U' + \left(6a + \frac{3b}{l} - \frac{2b}{l^2} \right) U U' + \left(a + \frac{b}{l} \right) U''' = 0. \tag{44}$$

After one integration, we get

$$\left(-\lambda_1 - \frac{\lambda_2}{l} \right) U + \left(6a + \frac{3b}{l} - \frac{2b}{l^2} \right) \frac{U^2}{2} + \left(a + \frac{b}{l} \right) U'' = 0. \tag{45}$$

Then we use the transformation

$$U_\xi = F(U). \tag{46}$$

So, eq. (45) will convert to

$$\frac{1}{2}\left(F^2(U)\right)' = \frac{l\lambda_1+\lambda_2}{al+b}U - \frac{6al^2+3bl-2b}{2l(al+b)}U^2. \tag{47}$$

Thus, we get from eq. (47) the expression for the function $F(U)$ which reads as

$$F(U) = \pm\sqrt{\frac{l\lambda_1+\lambda_2}{al+b}}U\sqrt{1-\frac{6al^2+3bl-2b}{3l(l\lambda_1+\lambda_2)}U}. \tag{48}$$

The solutions of eq. (48) can be obtained as

$$U(\xi) = \frac{3l(l\lambda_1+\lambda_2)}{6al^2+3bl-2b}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{l\lambda_1+\lambda_2}{al+b}}\xi\right). \tag{49}$$

For $(l\lambda_1+\lambda_2)/(al+b) > 0$, we get the following hyperbolic solutions:

$$u_1(x,t) = \frac{3l(l\lambda_1+\lambda_2)}{6al^2+3bl-2b} \times \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{l\lambda_1+\lambda_2}{al+b}}\left(x-\frac{\lambda_1 t^\alpha}{\Gamma(1+\alpha)}-\frac{\lambda_2 t^\beta}{\Gamma(1+\beta)}\right)\right), \tag{50}$$

$$v_1(x,t) = \frac{1}{l}\left(\frac{3l(l\lambda_1+\lambda_2)}{6al^2+3bl-2b}\right) \times \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{l\lambda_1+\lambda_2}{al+b}}\left(x-\frac{\lambda_1 t^\alpha}{\Gamma(1+\alpha)}-\frac{\lambda_2 t^\beta}{\Gamma(1+\beta)}\right)\right), \tag{51}$$

and

$$u_2(x,t) = -\frac{3l(l\lambda_1+\lambda_2)}{6al^2+3bl-2b} \times \operatorname{csch}^2\left(\frac{1}{2}\sqrt{\frac{l\lambda_1+\lambda_2}{al+b}}\left(x-\frac{\lambda_1 t^\alpha}{\Gamma(1+\alpha)}-\frac{\lambda_2 t^\beta}{\Gamma(1+\beta)}\right)\right), \tag{52}$$

$$v_2(x,t) = -\frac{1}{l}\left(\frac{3l(l\lambda_1+\lambda_2)}{6al^2+3bl-2b}\right) \times \operatorname{csch}^2\left(\frac{1}{2}\sqrt{\frac{l\lambda_1+\lambda_2}{al+b}}\left(x-\frac{\lambda_1 t^\alpha}{\Gamma(1+\alpha)}-\frac{\lambda_2 t^\beta}{\Gamma(1+\beta)}\right)\right). \tag{53}$$

For $(l\lambda_1 + \lambda_2)/(al + b) < 0$, we get the following periodic solutions:

$$u_3(x,t) = \frac{3l\,(l\lambda_1 + \lambda_2)}{6al^2 + 3bl - 2b} \times \sec^2\left(\frac{1}{2}\sqrt{-\frac{l\lambda_1 + \lambda_2}{al + b}}\left(x - \frac{\lambda_1 t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\lambda_2 t^{\beta}}{\Gamma\,(1+\beta)}\right)\right), \tag{54}$$

$$v_3(x,t) = \frac{1}{l}\left(\frac{3l\,(l\lambda_1 + \lambda_2)}{6al^2 + 3bl - 2b}\right) \times \sec^2\left(\frac{1}{2}\sqrt{-\frac{l\lambda_1 + \lambda_2}{al + b}}\left(x - \frac{\lambda_1 t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\lambda_2 t^{\beta}}{\Gamma\,(1+\beta)}\right)\right), \tag{55}$$

and

$$u_4(x,t) = \frac{3l\,(l\lambda_1 + \lambda_2)}{6al^2 + 3bl - 2b} \times \csc^2\left(\frac{1}{2}\sqrt{-\frac{l\lambda_1 + \lambda_2}{al + b}}\left(x - \frac{\lambda_1 t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\lambda_2 t^{\beta}}{\Gamma\,(1+\beta)}\right)\right), \tag{56}$$

$$v_4(x,t) = \frac{1}{l}\left(\frac{3l\,(l\lambda_1 + \lambda_2)}{6al^2 + 3bl - 2b}\right) \times \csc^2\left(\frac{1}{2}\sqrt{-\frac{l\lambda_1 + \lambda_2}{al + b}}\left(x - \frac{\lambda_1 t^{\alpha}}{\Gamma(1+\alpha)} - \frac{\lambda_2 t^{\beta}}{\Gamma\,(1+\beta)}\right)\right). \tag{57}$$

4. Conclusions

In this paper, the functional variable method and the modified Riemann–Liouville derivative are presented for solving the fractional KdV and the coupled KdV equations. It is predicted that the obtained solutions in this paper will be useful for further investigating the complicated nonlinear physical phenomena. This method introduces a promising tool for solving many fractional partial differential equations and it is also a reliable technique to handle nonlinear fractional differential equation. The calculations of functional variable method are very simple and straightforward. Thus, we deduce that this method can be applied to solve many systems of nonlinear fractional partial differential equations.

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