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Blow-up of solutions for the sixth-order thin film equation with positive initial energy

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Abstract. In this paper, a sixth-order parabolic thin film equation with the initial boundary condition is considered. By using the improved energy estimate method and by constructing second-order elliptic problem, a blow-up result for certain solution with positive initial energy is established, which is an improve over the previous result of Li and Liu.

Keywords. Blow-up; sixth-order thin film equation; positive initial energy.

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1. Introduction and main result

In the last 20 years, higher-order nonlinear parabolic partial differential equations (PDEs), as models for applications in mechanics and physics, have become more common in the study on pure and applied PDEs. For instance, the pure sixth-order parabolic thin film equation (TFE) was first introduced in [1,2] to describe the spreading of a thin viscous fluid (with possible slip at the solid interface) under the driving force of an elastica (or light plate). There are many related works on blow-up solutions to these kinds of parabolic equations and systems (see [3–9] and the references therein).

In this paper, we consider the following initial boundary problem of the sixth-order TFE:

$$\begin{cases} u_t - \Delta(\Delta^2 u - |u|^{p-1}u) = 0, & x \in \Omega,\ \ t \in (0,T), \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega,\ \ t \in [0,T), \\ u = u_0(x), & x \in \Omega,\ \ t = 0, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded smooth domain, $p > 1$. Recently, Li and Liu [10] defined the energy Lyapunov functional

$$E(t) = \frac{1}{2}\int_\Omega |\Delta u(x,t)|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega |u|^{p+1}\mathrm{d}x, \quad t \geq 0, \tag{2}$$

and proved that if the initial datum $u_0 \in C^{6+\alpha}(\bar{\Omega})$ with the initial energy

$$E(0) = \frac{1}{2}\int_{\Omega} |\Delta u_0|^2 \mathrm{d}x - \frac{1}{p+1}\int_{\Omega} |u_0|^{p+1} \mathrm{d}x \le 0, \tag{3}$$

then the solution to problem (1) should blow up in finite time. In this paper, the above result is improved to show that certain solutions with positive initial energy can also blow up in finite time. For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|_2$.

For our purpose, it is assumed that B is the optimal constant of the embedding inequality

$$\|u\|_{p+1} \le B\|\Delta u\|_2, \quad u \in H_0^2(\Omega) \tag{4}$$

i.e.,

$$B^{-1} = \inf_{\substack{u\in H_0^2(\Omega)\\ u\neq 0}} \frac{\|\Delta u\|_2}{\|u\|_{p+1}},$$

and set

$$\alpha_1 = B^{-(p+1)/(p-1)}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{p+1}\right) B^{-(2(p+1))/(p-1)}. \tag{5}$$

Then, from [10] we get (with some minor corrections)

$$E'(t) = -\int_{\Omega} |\nabla(\Delta^2 u - |u|^{p-1}u)|^2 \mathrm{d}x \le 0, \quad \text{for } t \ge 0, \tag{6}$$

which implies that the equilibrium is stable.

Our main result reads as follows.

Theorem 1. *Assume that the initial datum $u_0 \in C^{6+\alpha}(\bar{\Omega})$ satisfy*

$$E(0) < E_1 \tag{7}$$

and

$$\|\Delta u_0\|_2 > \alpha_1. \tag{8}$$

Then the solution $u(x,t)$ of problem (1) *blows up in a finite time.*

2. Proof of Theorem 1

We shall use the improved energy estimate method, which has been successfully applied in [11] to deal with the second-order p-Laplacian equation (see also [12–14] for further applications of this method). To extend the method to the sixth-order thin film equation herein, we should combine it with the construction of a second-order elliptic problem (see (18)).

We first prove the following lemmas by applying the idea of Vitillaro in [15] where a different type of equation was discussed. The first lemma gives a lower bound estimate of $\|\Delta u\|_2$.

Lemma 2. *Suppose u is a solution of the system* (1). *Assume that $E(0) < E_1$ and $\|\Delta u_0\|_2 > \alpha_1$. Then there exists a positive constant $\alpha_2 > \alpha_1$, such that*

$$\|\Delta u\|_2 \geq \alpha_2, \quad \forall\, t \geq 0 \tag{9}$$

and

$$\|u\|_{p+1} \geq B\alpha_2, \quad \forall\, t \geq 0. \tag{10}$$

Proof. We first note that, by (2) and (4),

$$\begin{aligned} E(t) &\geq \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{p+1}B^{p+1}\|\Delta u\|_2^{p+1} \\ &= \frac{1}{2}\alpha^2 - \frac{1}{p+1}B^{p+1}\alpha^{p+1} =: g(\alpha), \end{aligned} \tag{11}$$

where $\alpha = \|\Delta u\|_2$. The method used here is also related to the so-called Fibering method introduced by Pohozev in the 1970s [16–19]. It is easy to verify that g is increasing for $0 < \alpha < \alpha_1$, decreasing for $\alpha > \alpha_1$; $g(\alpha) \to -\infty$ as $\alpha \to +\infty$ and $g(\alpha_1) = E_1$, where α_1 is given in (5). As $E(0) < E_1$, there exists $\alpha_2 > \alpha_1$ such that $g(\alpha_2) = E(0)$. Let $\alpha_0 = \|\Delta u_0\|_2$, then by (11) we have $g(\alpha_0) \leq E(0) = g(\alpha_2)$, which implies that $\alpha_0 \geq \alpha_2$.

To establish (9), we suppose by contradiction that $\|\Delta u(\cdot, t_0)\|_2 < \alpha_2$ for some $t_0 > 0$. By the continuity of $\|\Delta u(\cdot, t)\|_2$ we can choose t_0 such that $\|\Delta u(\cdot, t_0)\|_2 > \alpha_1$. It follows from (11) that

$$E(t_0) \geq g(\|\Delta u(\cdot, t_0)\|_2) > g(\alpha_2) = E(0).$$

This is impossible because $E(t) \leq E(0)$ for all $t \geq 0$. Hence (9) is established.

To prove (10), we exploit (2) and (6) to obtain that

$$\frac{1}{p+1}\|u\|_{p+1}^{p+1} \geq \frac{1}{2}\|\Delta u\|_2^2 - E(0). \tag{12}$$

Consequently, by (9), we have

$$\frac{1}{p+1}\|u\|_{p+1}^{p+1} \geq \frac{1}{2}\alpha_2^2 - E(0) \geq \frac{1}{2}\alpha_2^2 - g(\alpha_2) = \frac{1}{p+1}B^{p+1}\alpha_2^{p+1}. \tag{13}$$

□

Therefore (10) is concluded.

In the remainder of this section we consider the case that $E(0) < E_1$ and $\|\Delta u_0\|_2 > \alpha_1$. We set

$$H(t) = E_1 - E(t), \quad t \geq 0. \tag{14}$$

Then we have the following lemma.

Lemma 3. *For all $t \geq 0$,*

$$0 < H(0) \leq H(t) \leq \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \tag{15}$$

Proof. By (6) we see that $H'(t) \geq 0$. Thus

$$H(t) \geq H(0) = E_1 - E(0) > 0, \quad t \geq 0. \tag{16}$$

From (2), we obtain

$$H(t) = E_1 - \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

and exploiting (9) and (5) we get

$$E_1 - \frac{1}{2}\|\Delta u\|_2^2 \le E_1 - \frac{1}{2}\alpha_1^2 = -\frac{1}{p+1}B^{p+1}\alpha_1^{p+1} < 0, \quad \forall\, t \ge 0.$$

Hence

$$H(t) \le \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \quad \forall\, t \ge 0. \tag{17}$$

Then (15) follows from (16) and (17). □

Let ϕ be the unique solution to

$$\begin{cases} -\Delta\phi = u, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{18}$$

Due to the elliptic L^2-theory, we have

$$\|\nabla\phi\|_2^2 \le C\|u\|_2^2. \tag{19}$$

Completion of the proof of Theorem 1. We define

$$G(t) := \frac{1}{2}\int_\Omega |\nabla\phi(x,t)|^2 \mathrm{d}x. \tag{20}$$

Differentiating $G(t)$ and exploiting (18), (1), Green's first formula

$$\int_\Omega u\Delta v \mathrm{d}x = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\mathrm{d}s - \int_\Omega \nabla u\cdot\nabla v \mathrm{d}x$$

and Green's second formula

$$\int_\Omega (u\Delta v - v\Delta u)\mathrm{d}x = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}s,$$

we get

$$\begin{aligned} G'(t) &= \int_\Omega \nabla\phi\cdot\nabla\phi_t \mathrm{d}x = \int_{\partial\Omega}\phi\frac{\partial\phi_t}{\partial n}\mathrm{d}s - \int_\Omega \phi\Delta\phi_t\mathrm{d}x \\ &= \int_\Omega \phi u_t \mathrm{d}x = \int_\Omega \phi\Delta(\Delta^2 u - |u|^{p-1}u)\mathrm{d}x \\ &= \int_\Omega \Delta\phi(\Delta^2 u - |u|^{p-1}u)\mathrm{d}x - \int_{\partial\Omega}\Bigg[(\Delta^2 u - |u|^{p-1}u)\frac{\partial\phi}{\partial n} \\ &\qquad\qquad - \phi\frac{\partial(\Delta^2 u - |u|^{p-1}u)}{\partial n}\Bigg]\mathrm{d}s \\ &= -\int_\Omega u(\Delta^2 u - |u|^{p-1}u)\mathrm{d}x = -\int_\Omega |\Delta u|^2\mathrm{d}x + \int_\Omega |u|^{p+1}\mathrm{d}x \\ &\quad + \int_{\partial\Omega}\left(\Delta u\frac{\partial u}{\partial n} - u\frac{\partial(\Delta u)}{\partial n}\right)\mathrm{d}s \\ &= -\int_\Omega |\Delta u|^2\mathrm{d}x + \int_\Omega |u|^{p+1}\mathrm{d}x. \end{aligned} \tag{21}$$

From (2) and (14), we get

$$-\int_\Omega |\Delta u|^2 \mathrm{d}x = -2E_1 - \frac{2}{p+1}\int_\Omega |u|^{p+1}\mathrm{d}x + 2H(t). \tag{22}$$

It follows from (21) and (22) that

$$G'(t) = -2E_1 + \left(1 - \frac{2}{p+1}\right)\|u\|_{p+1}^{p+1} + 2H(t). \tag{23}$$

By using (5) and (10), we have

$$\begin{aligned} 2E_1 &= \frac{2(p-1)}{p+1}\alpha_1^2 = \frac{2(p-1)}{p+1}B^{p+1}\alpha_1^{p+1} \\ &= \frac{\alpha_1^{p+1}}{\alpha_2^{p+1}}\frac{2(p-1)}{p+1}B^{p+1}\alpha_2^{p+1} \\ &\le \frac{\alpha_1^{p+1}}{\alpha_2^{p+1}}\frac{2(p-1)}{p+1}\|u\|_{p+1}^{p+1}. \end{aligned} \tag{24}$$

It follows from (23), (24) and (15) that

$$\begin{aligned} G'(t) &\ge \frac{2(p-1)}{p+1}\left(1 - \frac{\alpha_1^{p+1}}{\alpha_2^{p+1}}\right)\|u\|_{p+1}^{p+1} + 2H(t) \\ &= C_0\|u\|_{p+1}^{p+1} + 2H(t) \ge C_0\|u\|_{p+1}^{p+1} \ge 0, \end{aligned} \tag{25}$$

where

$$C_0 = \frac{2(p-1)}{p+1}\left(1 - \frac{\alpha_1^{p+1}}{\alpha_2^{p+1}}\right) > 0$$

due to the fact that $p > 1$ and $\alpha_2 > \alpha_1$.

Next, we use (19) and Hölder's inequality to estimate $G^{(p+1)/2}(t)$ as

$$G^{(p+1)/2}(t) \le C\|u\|_2^{p+1} \le C|\Omega|^{(p-1)/2}\|u\|_{p+1}^{p+1}. \tag{26}$$

By combining (25) and (26) we have

$$G'(t) \ge \gamma G^{(p+1)/2}(t), \tag{27}$$

where $\gamma = C_0/\left[C|\Omega|^{(p-1)/2}\right]$. A direct integration of (27) then yields

$$G^{(p-1)/2}(t) \ge \frac{1}{G^{(1-p)/2}(0) - (p-1)\gamma t/2}.$$

Therefore $G(t)$ blows up in a time $t^* \le (2G^{(1-p)/2}(0))/((p-1)\gamma)$. So does $\|u\|_2^2$, according to (19).

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References

[1] J R King, *Mathematical aspects of semiconductor process modelling*, Ph.D. Thesis (University of Oxford, Oxford, 1986)
[2] J R King, *SIAM J. Appl. Math*. **49**, 1064 (1989)
[3] A L Bertozzi and M C Pugh, *Comm. Pure Appl. Math.* **51**, 625 (1998)
[4] J D Evans, V A Galaktionov and J R King, *Eur. J. Appl. Math.* **18**, 195 (2007)
[5] J D Evans, V A Galaktionov and J R King, *Nonlinearity* **20**, 1799 (2007)
[6] P Souplet, *Math. Methods Appl. Sci.* **19**, 1317 (1996)
[7] F Q Sun, *Electron. J. Differ. Eq.* **2010**, 9 (2010)
[8] S Q Yu, *Appl. Anal.* **91**, 1321 (2012)
[9] J P Zhao, *Z. Naturforsch.* **67a**, 479 (2012)
[10] Z B Li and C C Liu, *Miskolc Math. Notes* **13**, 429 (2012)
[11] W J Liu and M X Wang, *Acta Appl. Math.* **103**, 141 (2008)
[12] Z B Fang, L Sun and C Li, *Bound. Value Probl.* **2013**, 181 (2013)
[13] W J Gao and Y Z Han, *Appl. Math. Lett.* **24**, 784 (2011)
[14] X L Wu and W J Gao, *Acta Math. Sci. Ser. B Engl. Ed.* **33**, 1024 (2013)
[15] E Vitillaro, *Arch. Ration. Mech. Anal.* **149**, 155 (1999)
[16] P Álvarez-Caudevilla and V A Galaktionov, *Adv. Nonlinear Stud.* **12**, 315 (2012)
[17] K J Brown and T-F Wu, *Differential Integral Equations* **22**, 1097 (2009)
[18] S I Pohožaev, An approach to nonlinear equations, *Dokl. Akad. Nauk SSSR* **247**, 1327 (1979), English trans1.: *Sov. Math. Dokl.* **20**, 912 (1979)
[19] S I Pohožaev, The fibering method in nonlinear variational problems, in *Topological and variational methods for nonlinear boundary value problems (Cholín,* 1995*)*, pp. 35–88, Pitman Res. Notes Math. Ser., 365, Longman, Harlow