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On the analytical solution of Fornberg–Whitham equation with the new fractional derivative

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Abstract. Motivated by the simplicity, natural and efficient nature of the new fractional derivative introduced by R Khalil *et al* in *J. Comput. Appl. Math.* **264**, 65 (2014), analytical solution of space-time fractional Fornberg–Whitham equation is obtained in series form using the relatively new method called q-homotopy analysis method (q-HAM). The new fractional derivative makes it possible to introduce fractional order in space to the Fornberg–Whitham equation and be able to obtain its solution. This work displays the elegant nature of the application of q-HAM to solve strongly nonlinear fractional differential equations. The presence of the auxiliary parameter h helps in an effective way to obtain better approximation comparable to exact solutions. The fraction-factor in this method gives it an edge over other existing analytical methods for nonlinear differential equations. Comparisons are made on the existence of exact solutions to these models. The analysis shows that our analytical solutions converge very rapidly to the exact solutions.

Keywords. Fornberg–Whitham equation; fractional derivative; q-homotopy analysis method; h-curve.

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1. Introduction

Calculus of non-integer order is increasingly being used to model physical systems. Caputo [1] used the modified form of the Darcy's law to incorporate the memory term in order to model transport through porous media. Other applications are in control theory of dynamical systems, electrical networks, ground water flow, astrophysics, meteorology, reactive flows, and semiconductors, see [2–5] and also [6–9] for some detailed work on fractional differential equations.

Generally, in such models one has to solve a fractional partial differential equation (PDE). Analytical methods commonly used to obtain solutions of these equations have very restricted applications and the numerical techniques give rise to the rounding of errors.

Recently, a modified homotopy analysis method called q-homotopy analysis method (q-HAM) was introduced in [10–14], which is less restricted than the previous methods.

In this paper, we apply q-HAM to initial value problems of the time-space-fractional Fornberg–Whitham equation with respect to the new fractional derivative introduced in [15]. The aim is to exploit the simple, natural and efficient nature of the so-called new fractional derivative to obtain analytical solution of the equations considered. Finally, we compare the applicability and performance of q-HAM with the exact solution for classical case and some other existing methods given in [16,17].

2. Preliminaries

This section is devoted to some definitions and some known results. The new conformable fractional derivative is adopted in this work.

DEFINITION 2.1 [15]

Let f: $[0,\infty) \longrightarrow \mathbb{R}$. Then the 'conformable fractional derivative' of f of order α is defined by

$$\mathcal{T}_t^{\alpha}(f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \tag{1}$$

for all $t > 0$, $\alpha \in (0,1)$. If f is α-differentiable in some $(0,a)$, $a > 0$, and

$$\lim_{t\to 0} f^{(\alpha)}(t) \tag{2}$$

exists, then define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t). \tag{3}$$

DEFINITION 2.2 [15]

The α-fractional integral of a function f starting from $a \geq 0$ is defined to be

$$\mathcal{J}_a^{\alpha}(f)(t) = \mathcal{J}_a^{1}(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\mathrm{d}x, \tag{4}$$

where the integral is the usual Riemann improper integral and $\alpha \in (0,1)$.

Lemma 2.1 [15]. *Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $t > 0$. Then*

(1) $\mathcal{T}^{\alpha}(af+bg) = a\mathcal{T}^{\alpha}g + b\mathcal{T}^{\alpha}g$, *for all* $a, b \in \mathbb{R}$,
(2) $\mathcal{T}^{\alpha}(t^n) = nt^{n-\alpha}$, *for all* $a, b \in \mathbb{R}$,
(3) $\mathcal{T}^{\alpha}(C) = 0$, *for all constant functions* $f(t) = C$,
(4) $\mathcal{T}^{\alpha}(\mathrm{e}^{kx}) = kx^{1-\alpha}\mathrm{e}^{kx}$, *for all* $k \in \mathbb{R}$,
(5) $\mathcal{T}^{\alpha}(fg) = f\mathcal{T}^{\alpha}(g) + g\mathcal{T}^{\alpha}(f)$,
(6) $\mathcal{T}^{\alpha}(f/g) = (g\mathcal{T}^{\alpha}(f) - f\mathcal{T}^{\alpha}(g))/g^2$, $g \neq 0$
(7) $\mathcal{T}^{\alpha}(f)(t) = t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t}(t)$, *for the differentiable function* f.

Lemma 2.2 [15]. *Given any continuous function f in the domain of J^α, we have*

$$\mathcal{T}^\alpha \mathcal{J}^\alpha(f)(t) = f(t), \qquad t \geqslant 0. \tag{5}$$

3. q-homotopy analysis method (q-HAM)

We give a simple idea of the q-homotopy analysis method (q-HAM) in this section.

Consider the differential equation of the form

$$N[\mathcal{T}_t^\alpha u(x,t)] - f(x,t) = 0, \tag{6}$$

where N is a nonlinear operator, $\mathcal{T}_t^\alpha$ denote the conformable fractional derivative of order α with respect to t as defined in (1), (x,t) are independent variables, f is a known function, and u an unknown function. To generalize the original homotopy method, the zeroth-order deformation equation is constructed.

$$(1-nq)L(\phi(x,t;q)-u_0(x,t)) = qhH(x,t)\big(N[\mathcal{T}_t^\alpha \phi(x,t;q)]-f(x,t)\big), \tag{7}$$

where $n \geq 1$, $q \in [0, 1/n]$ denotes the so-called embedded parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, and $H(x,t)$ is a non-zero auxiliary function.

It is clearly seen that when $q = 0$ and $q = 1/n$, eq. (7) becomes

$$\phi(x,t;0) = u_0(x,t) \quad \text{and} \quad \phi\left(x,t;\frac{1}{n}\right) = u(x,t), \tag{8}$$

respectively. So, as q increases from 0 to $1/n$, the solution $\phi(x,t;q)$ varies from the initial value $u_0(x,t)$ to the solution $u(x,t)$.

If $u_0(x,t)$, L, h, $H(x,t)$ are chosen appropriately, solution $\phi(x,t;q)$ of eq. (7) exists for $q \in [0, 1/n]$.

Expansion of $\phi(x,t;q)$ in Taylor series gives

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{9}$$

where

$$u_m(x,t) = \left.\frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\right|_{q=0}. \tag{10}$$

Assume that the auxiliary linear operator L, the initial value u_0, the auxiliary parameter h and $H(x,t)$ are properly chosen such that the series in (9) converges at $q = 1/n$. Then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{11}$$

Let the vector u_n be defined as follows:

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \tag{12}$$

Differentiating eq. (7) m times with respect to the (embedding) parameter q, then evaluating at $q = 0$ and finally dividing them by $m!$, we have the mth-order deformation equation (Liao [18]) as

$$L[u_m(x,t) - \chi_m^* u_{m-1}(x,t)] = hH(x,t)\mathcal{R}_m(\vec{u}_{m-1}) \tag{13}$$

with initial conditions

$$u_m^{(k)}(x,0) = 0, \quad k = 0, 1, 2, ..., m-1, \tag{14}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\left(N[\mathcal{T}_t^{\alpha}\phi(x,t;q)] - f(x,t)\right)}{\partial q^{m-1}}\bigg|_{q=0} \tag{15}$$

and

$$\chi_m^* = \begin{cases} 0, & m \le 1 \\ n, & \text{otherwise.} \end{cases} \tag{16}$$

Remark 1. It should be emphasized that $u_m(x,t)$ for $m \ge 1$ is governed by the linear operator (13) with the linear boundary conditions that come from the original problem. The existence of the factor $(1/n)^m$ gives more chances for better convergence, faster than the solution obtained by the standard homotopy method. When $n = 1$, the method is called the standard homotopy method.

4. Fornberg–Whitham equation with the new fractional derivative

The Fornberg–Whitham equation with the new fractional derivative as defined in (1) in time and space is considered. This is given as

$$\begin{aligned} &\mathcal{T}_t^{\alpha}u - u_{xxt} + \mathcal{T}_x^{\beta}u = uu_{xxx} - uu_x + 3u_x u_{xx}, \\ &0 < x \le 1, \quad t > 0, \quad 0 < \alpha, \quad \beta \le 1, \quad \alpha \ne 0.5, \end{aligned} \tag{17}$$

subjected to the initial condition

$$u(x,0) = \mathrm{e}^{(x/2)}. \tag{18}$$

The exact solution to this problem, when $\alpha = \beta = 1$, is

$$u(x,t) = \mathrm{e}^{(x/2)-(2t/3)}. \tag{19}$$

4.1 *Application of q-HAM*

In order to use q-HAM to solve the problem considered in (17), we choose the linear operator

$$L[\phi(x,t;q)] = \mathcal{T}_t^{\alpha}\phi(x,t;q) \tag{20}$$

with the property that $L[c_1] = 0$, c_1 is constant.

We use initial approximation $u_0(x,t) = \mathrm{e}^{x/2}$. We can then define the nonlinear operator as

$$\begin{aligned} N[\phi(x,t;q)] = {} & \mathcal{T}_t^{\alpha}\phi(x,t;q) - \phi_{xxt}(x,t;q) + \mathcal{T}_x^{\beta}\phi(x,t;q) \\ & -\phi(x,t;q)\phi_{xxx}(x,t;q) + \phi(x,t;q)\phi_x(x,t;q) \\ & -3\phi_x(x,t;q)\phi_{xx}(x,t;q). \end{aligned} \tag{21}$$

We construct the zeroth-order deformation equation as

$$(1-nq)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\mathcal{T}_t^{\alpha}\phi(x,t;q)]. \tag{22}$$

We choose $H(x,t) = 1$ to obtain the mth-order deformation equation as

$$L[u_m(x,t) - \chi_m^* u_{m-1}(x,t)] = h\mathcal{R}_m(\vec{u}_{m-1}), \tag{23}$$

with initial condition for $m \geq 1$, $u_m(x,0) = 0$, χ_m^* is as defined in (16) and

$$\begin{aligned} \mathcal{R}_m(\vec{u}_{m-1}) = {} & \mathcal{T}_t^{\alpha}u_{m-1} - u_{(m-1)xxt} + \mathcal{T}_x^{\beta}u_{m-1} - \sum_{k=0}^{m-1} u_k u_{(m-1-k)xxx} \\ & + \sum_{k=0}^{m-1} u_k u_{(m-1-k)x} - 3\sum_{k=0}^{m-1} u_{kx} u_{(m-1-k)xx}. \end{aligned} \tag{24}$$

So, the solution to eq. (17) for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m^* u_{m-1} + h\mathcal{J}_t^{\alpha}[\mathcal{R}_m(\vec{u}_{m-1})]. \tag{25}$$

We therefore obtain components of the solution using q-HAM successively as follows:

$$\begin{aligned} u_1(x,t) = {} & \chi_1^* u_0 + h\mathcal{J}_t^{\alpha}[\mathcal{T}_t^{\alpha}u_0 - (u_0)_{xxt} + \mathcal{T}_x^{\beta}u_0 - u_0(u_0)_{xxx} \\ & + u_0(u_0)_x - 3(u_0)_x(u_0)_{xx}] \\ = {} & \frac{h}{2\alpha}x^{1-\beta}\mathrm{e}^{x/2}t^{\alpha} \end{aligned} \tag{26}$$

$$\begin{aligned} u_2(x,t) = {} & \chi_2^* u_1 + h\mathcal{J}_t^{\alpha}[\mathcal{T}_t^{\alpha}u_1 - (u_1)_{xxt} + \mathcal{T}_x^{\beta}u_1 - u_0(u_1)_{xxx} \\ & - u_1(u_0)_{xxx} + u_0(u_1)_x +] + h\mathcal{J}_t^{\alpha}\,[u_1(u_0)_x - 3(u_0)_x(u_1)_{xx} \\ & - 3(u_1)_x(u_0)_{xx}] \end{aligned} \tag{27}$$

$$\begin{aligned} = {} & \frac{(n+h)h}{2\alpha}x^{1-\beta}\mathrm{e}^{x/2}t^{\alpha} + \frac{h^2}{8\alpha^2}x^{2(1-\beta)}\mathrm{e}^{x/2}t^{2\alpha} + \frac{h^2(1-\beta)}{4\alpha^2}x^{-\beta}\mathrm{e}^{x/2}t^{2\alpha} \\ & + \frac{\beta(1-\beta)h}{2(2\alpha-1)}x^{-(1+\beta)}\mathrm{e}^{x/2}t^{2\alpha-1} - \frac{h}{8(2\alpha-1)}x^{1-\beta}\mathrm{e}^{x/2}t^{2\alpha-1} \\ & - \frac{\beta(\beta+1)(1-\beta)h}{4\alpha^2}x^{-(2+\beta)}\mathrm{e}^{x}t^{2\alpha} - \frac{(1-\beta)h}{2(2\alpha-1)}x^{-\beta}\mathrm{e}^{x/2}t^{2\alpha-1} \\ & - \frac{(1-\beta)h}{2\alpha^2}x^{-\beta}\mathrm{e}^{x}t^{2\alpha} + \frac{3\beta(1-\beta)h}{4\alpha^2}x^{-(1+\beta)}\mathrm{e}^{x}t^{2\alpha}. \end{aligned}$$

In the same way, $u_m(x,t)$ for $m = 3, 4, 5, \ldots$ can be obtained using *Mathematica 9*.

Then the series solution expression by q-HAM can be written in the form

$$u(x, t; n; h) = e^{x/2} + \sum_{i=1}^{\infty} u_i(x, t; n; h) \left(\frac{1}{n}\right)^i . \tag{28}$$

Equation (28) is an appropriate solution to the problem (17) in terms of convergence parameters h and n.

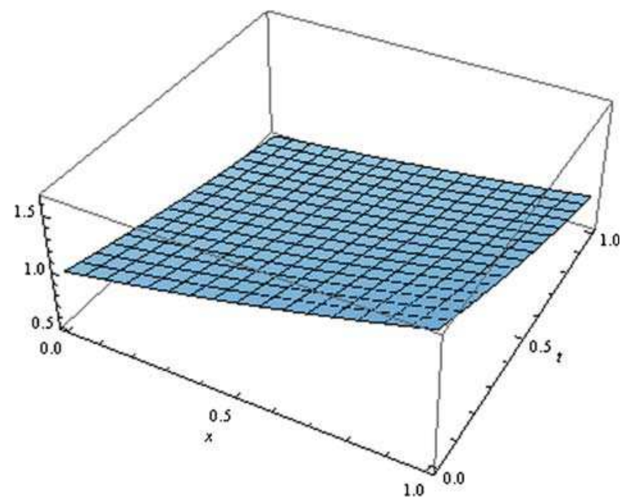

Figure 1. q-HAM solution of eq. (17): $\alpha = \beta = 1$ and $h = -1$.

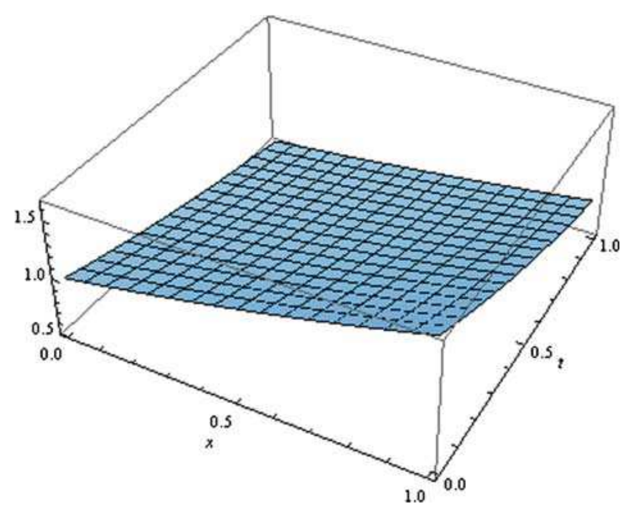

Figure 2. Exact solution of eq. (17) for $\alpha = \beta = 1$.

4.2 *Numerical results and discussion*

We present the numerical results of the series solution of the space-time fractional Fornberg–Whithem equation obtained by q-HAM. The graphs give clear pictures of the simple but elegant nature of the method used to solve strongly nonlinear problems of the fractional type.

The plots of the 3-term series solution of (17) obtained by q-HAM and that of the exact solution are presented in figures 1 and 2, respectively with an appropriate $h = -1, n = 1$, and $\alpha = \beta = 1$. The appropriate choices of h are displayed in the so-called h-curve of figure 3. Also, the plots of the solution for different α and different β are displayed in figures 4 and 5, respectively.

Remark 2. It can be observed that the 3-term solution given by q-HAM matched excellently with the exact solution, despite the simple and less computational nature of the q-HAM compared with other analytical methods (see [16,17]).

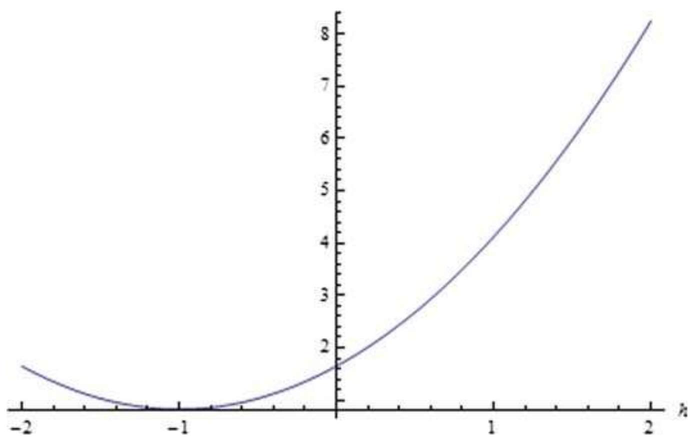

Figure 3. The h-curve of $u(1, 1)$ given by the third-order q-HAM approximate solution.

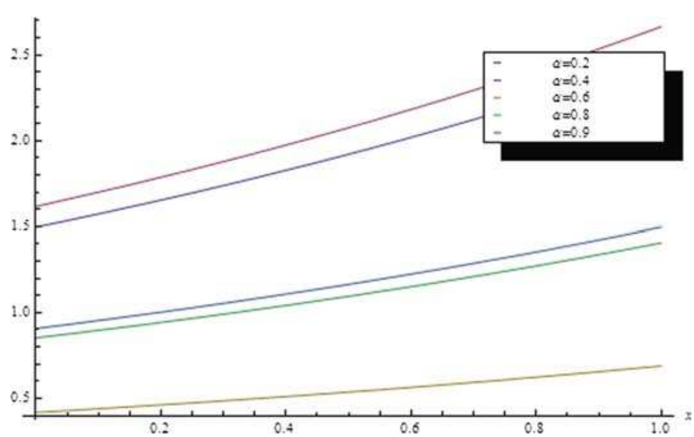

Figure 4. q-HAM solution of eq. (17) for different values of α when $\beta = 1$.

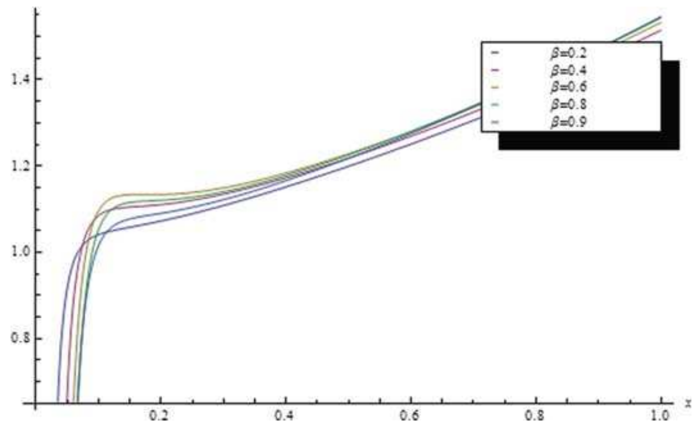

Figure 5. q-HAM solution of eq. (17) for different values of β when $\alpha = 1$.

Remark 3. Using the h-curve, it is possible to locate the valid region of h which corresponds to the line segment nearly parallel to the horizontal axis.

5. Conclusion

In this paper, the applicability of the fractional q-homotopy analysis method to the solution of the space-time fractional Fornberg–Whitham equation with appropriate initial condition has been proved. The simple, natural, and efficient nature of the new fractional derivative discussed in [15] makes it possible to introduce fractional order in space which is complicated in the case of other types of fractional derivatives. It should also be noted that the physical interpretation of this derivative coincides with the physical interpretation of classical derivative when α and β are integers. Indeed, further investigation is still open for future work regarding physical interpretations of the new fractional derivative depending on the problem considered.

Our results show that q-HAM can be applied to many complicated linear and strongly nonlinear partial differential equations. The method to choose the appropriate auxiliary parameter h for better convergence of the series solution is given in the h-curve. All the numerical analyses in this study were carried out using *Mathematica 9*.

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References

[1] M Caputo, *Geothermics* **28**, 113 (1999)
[2] E Cumberbatch and A Fitt, *Mathematical modeling: Case studies from industries* (Cambridge University Press, UK, 2001)

[3] G I Marchuk, *Mathematical models in environmental problems* (North-Holland, Elsevier Science Publishers, 1986)

[4] E S Oran and J P Boris, *Numerical simulation of reactive flow*, 2nd edn (Cambridge University Press, UK, 2001)

[5] K S Miller and B Ross, *An introduction to the fractional calculus and fractional differential equations* (John Wiley and Sons Inc., New York, 2003)

[6] K A Gepreel and A A Al-Thobaiti, *Indian J. Phys.* **88**, 293 (2014)

[7] K M Furati, O S Iyiola and M Kirane, *Appl. Math. Comput.* **249**, 24 (2014)

[8] O S Iyiola, *British J. Math. Comp. Sci.* **4**(**10**) (2014)

[9] M Eslami, B F Vajargah, M Mirzazadeh and A Biswas, *Indian J. Phys.* **88**(**2**), 177 (2014)

[10] M A El-Tawil and S N Huseen, *Int. J. Appl. Math. Mech.* **8**(**15**), 51 (2012)

[11] O S Iyola and F D Zaman, *AIP Adv.* **4**, 107121 (2014)

[12] O S Iyiola, *Adv. Math. Sci. J.* **2**(**2**), 71 (2013)

[13] O S Iyiola, M E Soh and C D Enyi, *Math. Eng. Sci. Aerospace.* **4**(**4**), 105 (2013)

[14] O S Iyiola and O G Olayinka, *Ain Shams Engng J.* **5**(**3**), 999 (2014)

[15] R Khalil, M Al Horani, A Yousef and M Sababhehb, *J. Comput. Appl. Math.* **264**, 65 (2014)

[16] M Merdan, A Gokdogan, A Yıldırım and S T Mohyud-Din, *Abstract Appl. Anal.* Vol. (2012), Article ID 965367, 8 pages, DOI:10.1155/2012/965367

[17] Y Mahmoudi and M Kazemian, *J. Basic. Appl. Sci. Res.* **2(3)**, 2985 (2012)

[18] S-J Liao, *Int. J. Non-linear Mech.* **30(3)**, 371 (1995)