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Magnetoresistance in quantum Hall metals due to Pancharatnam wavefunction transformation and degenerate Landau levels

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Abstract. We derive the trial Hall resistance formula for the quantum Hall metals to address both the integer and fractional quantum Hall effects. Within the degenerate (and crossed) Landau levels, and in the presence of changing magnetic field strength, one can invoke two physical processes responsible for the electron conduction and quantum Hall effects in Fermi metals. One of the process requires the Pancharatnam wavefunction transformation, while the second involves electron transfer between two orthogonalized wavefunctions (within the degenerate and crossed Landau levels). We discuss the relevant physical postulates with respect to these physical processes to qualitatively reproduce the measured Hall resistance's zigzag curve for both the integer and the fractional filling factors. Along the way, we give out some evidence to contradict the postulates with experiments.

Keywords. Quantum Hall metals; fractional and integer quantum Hall effects; Pancharatnam wavefunction transformation; degenerate Landau levels.

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1. Introduction

Since the discovery of electron as a negatively charged subatomic particle by Thomson [1], its spin by Uhlenbeck and Goudsmit [2], its wave nature by de Broglie [3], and its geometric and dynamic phases by Pancharatnam [4], we have always imagined (in one way or another) of their interactions and motion within a given atom or an electronic device, and how their wavefunctions transform in order to flow along the grid and between the lattice points, satisfying the principle of least action [5] (when the system is away from the critical point) and the principle of maximum interaction [6] during a phase transition. This single thought have led us to formulate proper and consistent theories to understand Fermi liquid metals, semiconductors, band insulators, conventional

superconductors, magnets, ferromagnets, ferroelectrics and the Mott–Hubbard insulators [7]. Apart from the high-temperature superconductors [8] and strange metals [9], topological insulators [10,11] and quantum Hall metals [12–21] are the other major solid-state systems that currently need proper and consistent formulations to understand their electronic properties, regardless of whether these systems have the potential for future technological marvels.

Here, we shall derive the trial Hall resistance formula for quantum Hall metals that will capture the essential physics required to understand the constant Hall resistances for certain filling factors and the magnetic field-dependent part of the Hall resistance due to the finite longitudinal resistance along the applied electric fields. The physical mechanisms invoked to formulate the Hall resistance formula involve the usual metallic Fermi liquid where the degenerate Landau levels (LLs) are crossed, and at each point of this LL crossing, there exists a finite but irrelevant energy-level spacing, which is responsible for the non-zero longitudinal resistance along the applied electric fields with increasing magnetic fields. The above energy-level spacings are termed irrelevant because they do not require wavefunction transformation beyond the phase factor. On the other hand, these spacings are known as the 'relevant' energy-level spacings, which were proved in earlier studies [22].

However, these irrelevant LL spacings do not play any role for certain LL crossings, and hence can be taken to be zero symbolically for convenience. The LLs of the system split due to Zeeman splitting, and with increasing magnetic field strength, these levels cross each other in a complicated way (within the degenerate LLs), which can be used to postulate two independent physical processes. The first physical mechanism is influenced by the irrelevant LL spacings such that the electron flow requires large changes to the phase factor and group momentum of a wavefunction, while the other mechanism for electron flow is independent of these spacings (require small changes to the phase factor, or negligible changes to the group momentum of a wavefunction). Here, we shall further elaborate on these two processes, and use them to derive and discuss the Hall resistance in quantum Hall metals for both fractional and integer filling factors.

2. Integer Hall conductance

We begin with the open sets, U_j and U_k, which are allowed to overlap such that $U_j \cap U_k \neq \emptyset$ and $\{U_j, U_k\} \in M^m_{\mathbb{C}}$, where $M^m_{\mathbb{C}}$ is a complex m-dimensional topological space and an m-dimensional differential manifold that deals with complex ($\mathbb{C}$) numbers. We also define a transition function,

$$\mathtt{t}_{jk}(\mathtt{p}) = \exp[\mathrm{i}\gamma(\mathtt{p})], \tag{1}$$

that smoothly maps U_j to U_k, following Definition 5.1 and Theorem 10.2 given by Nakahara in [23] where $\mathrm{i} = \sqrt{-1}$, $\gamma(\mathtt{p}) \in \mathbb{R}$, an arbitrary function that will be defined later as the Pancharatnam–Berry (PB) phase, and $\mathtt{p}$ refers to the same coordinate point on both U_j and U_k. Physically, we can define $U_j = \{\psi_{\mathrm{VB}}(\mathbf{k})_J\}$ and $U_k = \{\psi_{\mathrm{CB}}(\mathbf{k})_K\}$, where $\mathbf{k}$ is the wave vector in momentum space, $|\mathbf{k}|$ denotes the wave number, $j = 1$, $k = 2$, $\{J, K\} \in \mathbb{N}^*$ (excluding zero), $\psi_{\mathrm{VB}}(\mathbf{k})_J$ and $\psi_{\mathrm{CB}}(\mathbf{k})_K$ are the electron wavefunctions in the valence and conduction energy bands, respectively, and therefore J and K are the principal quantum numbers such that $J < K$. Moreover, $U_j \cap U_k \neq \emptyset$ implies the energy gap,

$E_{\text{gap}} = 0$ and therefore, $\psi_{\text{CB}}(\mathbf{k})_K = \psi_{\text{VB}}(\mathbf{k})_J = \psi_J(\mathbf{k})$. If $U_j \cap U_k = \emptyset$, then $E_{\text{gap}} \neq 0$ and consequently $\mathtt{t}_{jk}(\mathtt{p})$ (defined in eq. (1)) does not exist because $\mathtt{t}_{jk}(\mathtt{p})$ strictly requires zero energy gap, and accordingly, we now define the respective PB connection and PB curvature [23],

$$\mathcal{A}_k(\mathtt{p}) = \mathtt{t}_{jk}(\mathtt{p})^{-1}\mathcal{A}_j(\mathtt{p})\mathtt{t}_{jk}(\mathtt{p}) + \mathtt{t}_{jk}(\mathtt{p})^{-1}\mathbf{d}\mathtt{t}_{jk}(\mathtt{p}), \tag{2}$$

$$\mathcal{F}_j(\mathtt{p}) = \mathbf{d}\mathcal{A}_j(\mathtt{p}) + \mathcal{A}_j(\mathtt{p}) \wedge \mathcal{A}_j(\mathtt{p}). \tag{3}$$

Here $\mathbf{d} = [\partial/\partial\mathtt{x}]\mathrm{d}\mathtt{x}$ ($\mathtt{x}$ can be any variable) and $\wedge$ denotes the exterior product, whereas $\mathbf{d}$ and d are the exterior derivatives. We now drop $\mathtt{p}$ for convenience, which will become clear shortly. Furthermore, the component of a PB connection is given by

$$\mathcal{A}_j = \frac{\partial \mathcal{A}_{j\mu}}{\partial x^\mu}\mathrm{d}x^\mu, \tag{4}$$

where $1 \leq \mu < \nu \leq m$. The origin of Berry's phase [24] was proved earlier by Pancharatnam using the elliptically polarized pencil beams within the Poincaré sphere [4,25,26], such that, the Berry's phase is a rediscovered special case (with a constant group momentum) within the Pancharatnam wavefunction transformation, which was proved in [26] using the generalized theory of interference put forth by Pancharatnam [4,25]. The Pancharatnam wavefunction transformation occurs whenever a wavefunction picks up or drops a phase factor due to changing phase and/or group velocities [26]. In particular, the PB phase is given by

$$\begin{aligned}\gamma(t) &= -\int_0^t \left\langle \psi_J(t') \left| \frac{\partial g(t')}{\partial t'} \right| \psi_J(t') \right\rangle \mathrm{d}t' + \mathrm{i}\int_0^t \left\langle \psi_J(t') \left| \frac{\partial}{\partial t'}\psi_J(t') \right\rangle \mathrm{d}t' \\ &= \gamma[v_{\text{p}}(t)] + \gamma[v_{\text{g}}(t)].\end{aligned} \tag{5}$$

Here $\dot{v}_{\text{p}}(t) \propto \partial^2 g(t)/\partial t^2$ and $\dot{v}_{\text{g}}(t) \propto \partial^2\psi_J(t)/\partial t^2$, where $\psi_J(t)$ is the electron wavefunction, $\psi_J(t) = \mathrm{e}^{ig(t)}\varphi_J(t)$, while $\dot{v}_{\text{p}}(t)$ and $\dot{v}_{\text{g}}(t)$ denote the phase and group accelerations or the changing phase and group momenta, respectively [26]. Moreover, the subscripts j and k now refer to $U_j \cap U_k = U_j' = U_k' = U$, where we have confined the relevant coordinate point ($\mathtt{p}$) and functions ($\mathtt{t}_{jk}$ and ψ_J) within the overlapped open set, $U \in M_{\mathbb{C}}^m$ because $\mathtt{t}_{jk}$ does not exist if $\psi_J \notin U$.

We also have used the PB phase in compact form [26,27],

$$\gamma_J(T) = \mathrm{i}\oint \langle\psi_J(\mathbf{X})|\nabla\psi_J(\mathbf{X})\rangle \mathrm{d}\mathbf{X}, \tag{6}$$

and eq. (4) to define $\mathcal{A}_{j(\mu,\nu)}$ in terms of ψ_J,

$$\mathcal{A}_{j(\mu,\nu)} = \langle\psi_J| \wedge \left(\frac{\partial|\psi_J\rangle}{\partial x^{\mu,\nu}}\mathrm{d}x^{\mu,\nu}\right), \tag{7}$$

and this definition is exactly identical to the one used by Nakahara [23]. Subsequently, we define the PB curvature in terms of ψ_J (here, we do not use eq. (3)),

$$\nabla \times \mathcal{A}_{j(\mu,\nu)} = \left[\left(\frac{\partial\langle\psi_J|}{\partial x^\mu}\right)\left(\frac{\partial|\psi_J\rangle}{\partial x^\nu}\right) - \left(\frac{\partial\langle\psi_J|}{\partial x^\nu}\right)\left(\frac{\partial|\psi_J\rangle}{\partial x^\mu}\right)\right]\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{8}$$

One needs to invoke eqs (2) and (3) to obtain the PB connection between $\mathcal{A}_{k(\mu,\nu)}$ and $\mathcal{A}_{j(\mu,\nu)}$ to arrive at eq. (10), which will be derived subsequently. Here, to derive eq. (8) we have used these equalities, $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, $\partial^2/\partial x^\mu \partial x^\nu = \partial^2/\partial x^\nu \partial x^\mu$.

In eq. (6), T is the time taken for the Hamiltonian to return to its original form. Obviously, eq. (6) is non-integrable and cannot be zero if $\psi_J(\mathbf{k}) \rightarrow \psi_J[X_1(\mathbf{k}), X_2(\mathbf{k}), \ldots, X_N(\mathbf{k})] = \psi_J(\mathbf{X})$, $N > 1$ and at least $X_1(\mathbf{k}) \neq X_2(\mathbf{k})$. Using eq. (7) and the Stokes theorem,

$$\oint_{\mathtt{C}} \mathcal{A}_{j(\mu,\nu)} = \int_{\mathtt{S}} \left[\nabla \times \mathcal{A}_{j(\mu,\nu)}\right],$$

we can now rewrite eq. (6),

$$\gamma_J(nT) = \mathrm{i}\int_{\mathtt{S}} \left[\nabla \times \mathcal{A}_{j(\mu,\nu)}\right] = n2\pi, \tag{9}$$

where $\mathtt{C}$ and $\mathtt{S}$ refer to the curve (that forms a closed loop) and the surface integrals, respectively, in the momentum ($\mathbf{k}$) space, $n \in \mathbb{Z}^*$ is related to the Chern winding number, and $\mathbb{Z}^*$ is the set of positive integers, including zero. Obviously, n got to be an integer due to eq. (1) if $\exp(\mathrm{i}\gamma(\mathtt{p})) = 1$, where $\gamma(\mathtt{p}) = 0$ or $n2\pi$.

We now prove $\mathcal{F}_{j(\mu\nu)} - \mathcal{F}_{k(\mu\nu)} = \mathbf{d}_\mu \mathcal{A}_{j\nu} - \mathbf{d}_\nu \mathcal{A}_{k\mu} = \nabla \times \mathcal{A}_{j(\mu,\nu)}$ by using the fact that $\mathcal{F}_{j(\mu\nu)} = \mathbf{d}\mathcal{A}_{j(\mu,\nu)}$, which is a special case of eq. (3) that was proved to be exact by Nakahara using the Poincaré's lemma [23].

Proof. Recall eq. (2), and using eq. (1), one obtains

$$\mathcal{A}_{k(\mu,\nu)} = \mathcal{A}_{j(\mu,\nu)} + \mathrm{i}\, d\gamma. \tag{10}$$

Consequently,

$$\gamma = \mathrm{i}\oint_{\mathtt{C}} \left(\mathcal{A}_{j\nu} - \mathcal{A}_{k\mu}\right) \tag{11}$$

$$= \mathrm{i}\int_{\mathtt{S}} \left[\left(\frac{\partial\langle\psi_J|}{\partial x^\mu}\right)\left(\frac{\partial|\psi_J\rangle}{\partial x^\nu}\right) - \left(\frac{\partial\langle\psi_J|}{\partial x^\nu}\right)\left(\frac{\partial|\psi_J\rangle}{\partial x^\mu}\right)\right]\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{12}$$

$$= n2\pi. \tag{13}$$

To arrive at eq. (12), we also have used the equalities introduced after eq. (8), the Stokes theorem (eq. (9)) and the fact that

$$\mathcal{A}_{k(\mu,\nu)} = \langle\psi_J|\exp(-\mathrm{i}\gamma) \wedge \left(\frac{\partial|\psi_J\rangle}{\partial x^{\mu,\nu}}\mathrm{d}x^{\mu,\nu}\right)\exp(\mathrm{i}\gamma), \tag{14}$$

due to eq. (1). The integrand in eq. (12) is nothing but what is given in eq. (8). □

For a metallic free electron or Fermi liquid system, the definition for the Hall conductance is given by $G_{\mathrm{Hall}} = (e^2/h)n_{\mathrm{Chern}}$, where e denotes the electron charge, h is the

Planck's constant and the integer n_{Chern} is the Chern winding number that can be obtained from eq. (9) or (12) such that the Chern winding number is given by

$$n = n_{\text{Chern}} = \frac{\mathrm{i}}{2\pi}\int_{\mathrm{S}}\left[\nabla \times \mathcal{A}_{j(\mu,\nu)}\right] \tag{15}$$

$$= \frac{\mathrm{i}}{2\pi}\int_{\mathrm{S}}\left[\mathcal{F}_{j(\mu\nu)} - \mathcal{F}_{k(\mu\nu)}\right], \tag{16}$$

following the definition, $n_{\text{Chern}} = c_1(\mathcal{F}) = \mathrm{i}\mathcal{F}/2\pi$ given in [23] within the first Chern class ($c_1(\mathcal{F})$), where $\mathcal{F}$ is arbitrarily known as the field strength, which can be related to Yang–Mills, magnetic or electric fields [23] or vorticity ($\nabla \times \mathcal{A}_{\mu,\nu}$) in fluid dynamics. If we now define the wavefunction as a plane wave [33], $\psi_J = u_{J(|\mathbf{k}|_1,|\mathbf{k}|_2)} = \varphi_{J(|\mathbf{k}|_1,|\mathbf{k}|_2)}\exp(-\mathrm{i}|\mathbf{k}|_1 x - \mathrm{i}|\mathbf{k}|_2 y) = u_{J(|\mathbf{k}|^\mu,|\mathbf{k}|^\nu)} = \varphi_{J(|\mathbf{k}|^\mu,|\mathbf{k}|^\nu)}\exp(-\mathrm{i}|\mathbf{k}|^\mu x^\mu - \mathrm{i}|\mathbf{k}|^\nu x^\nu)$, and after using eqs (8), (9) and (15), one obtains

$$G_{\text{Hall}}^{\text{TKNN}} = \frac{\mathrm{i}e^2}{2\pi h}\sum\nolimits_J^L \int_{\mathrm{S}(\mathrm{d}|\mathbf{k}|^\mu\wedge\mathrm{d}|\mathbf{k}|^\nu)}\int_{\mathrm{S}(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu)}\left[\nabla \times \mathcal{A}^J_{j(\mu,\nu)}\right], \tag{17}$$

which is nothing but the Hall conductance derived by Thouless *et al* [33], which is also known as the Thouless–Kohmoto–Nightingale–den Nijs (TKNN) equation. Here, $\int_{\mathrm{S}(\mathrm{d}|\mathbf{k}|^\mu\wedge\mathrm{d}|\mathbf{k}|^\nu)}$ and $\int_{\mathrm{S}(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu)}$ integrate the Fermi surface in momentum and real spaces, respectively, while $\sum_J^L$ sums all the occupied bands $(u_J, \ldots, u_L)$ that contribute to the conductance. On the other hand, if we were to use eqs (11) and (16), then

$$G_{\text{Hall}} = -\frac{\mathrm{i}e^2}{2\pi h}\int_{\mathrm{S}}\left[\mathcal{F}^J_{k(\mu\nu)} - \mathcal{F}^J_{j(\mu\nu)}\right], \tag{18}$$

$$= -\frac{e^2}{h}n_{\text{Chern}} = G_{\text{Hall}}^{\text{Kohmoto}}. \tag{19}$$

Here, eq. (19) is exactly the conductance derived by Kohmoto in [34] where J is the Jth conduction band. However, the origin of the negative sign that appears in eq. (18) is not associated with the negative sign in the Kohmoto conductance formula $G_{\text{Hall}}^{\text{Kohmoto}}$ derived in [34], in which, the source of the negative sign in $G_{\text{Hall}}^{\text{Kohmoto}}$ is i^2. In fact, we can reversibly switch the sign ($- \longleftrightarrow +$) in eq. (18), such that the positive sign is for $\mathtt{t}_{jk}$, while $\mathtt{t}_{kj} = \mathtt{t}_{jk}^{-1}$ implies a negative conductance, or vice versa, which then, can be defined to be related to a hole or an electron conduction. This reversible sign switch is also compatible with $G_{\text{Hall}}^{\text{Kohmoto}}$ given in [34] if one were to use the Chern winding number in the form of eq. (12).

In summary, we have provided an alternative derivation to obtain the integer Hall conductance by explicitly invoking the definition for Hall conductance ($e^2 n_{\text{Chern}}/h$), the PB phase (γ) and the first Chern winding number ($c_1(\mathcal{F}) = n_{\text{Chern}}$). We should be aware here that eq. (19) is valid for a two-dimensional (2D) Fermi gas or Fermi liquid metals such that $E_{\text{gap}} = 0$, and this means that the Fermi–Dirac probability always equals one such that an electron can readily occupy an empty energy level at the Fermi surface without any energy penalty.

3. Integer and fractional Hall resistance

The Chern winding number introduced above actually refers to a winding fibre [23], a mathematical notion that is neither concerned with any physical entity nor refers to an electron's energy level and this fibre does not represent the electronic wavefunction. For example, in the presence of external magnetic fields, n_{Chern} does not count the number of complete circles (multiple of 2π) made by an electron (or a hole) during the Hall transport. This means that, a particular wavefunction picks up a phase factor (given in eq. (1)) on the right-hand side of a wavefunction (eq. (14)), not because of the right-hand action operator (defined in §10.1 in [23]), but due to a physical notion, known as the Pancharatnam phase retardation [4]. This phase retardation originates from the interaction between an electron and the vector potential giving rise to the changing phase and/or group momenta [4] as explained earlier.

Therefore, we need to redefine n (eq. (9)) properly with respect to the rediscovery of Pancharatnam wavefunction transformation (eq. (5)) such that $n \to \nu_{\text{P}}$,

$$\nu_{\text{P}} = \frac{\text{i}}{2\pi}\int_{\text{S}} \left[\left(\frac{\partial\langle\psi_J(t)|}{\partial x^{\mu}(t)}\right)\left(\frac{\partial|\psi_J(t)\rangle}{\partial x^{\nu}(t)}\right) - \left(\frac{\partial\langle\psi_J(t)|}{\partial x^{\nu}(t)}\right)\left(\frac{\partial|\psi_J(t)\rangle}{\partial x^{\mu}(t)}\right)\right] \text{d}x^{\mu}(t)\wedge\text{d}x^{\nu}(t). \tag{20}$$

Unlike $n \in \mathbb{Z}$ (from eq. (9)), $\nu_{\text{P}} \in \mathbb{R}$ where $\exp(\text{i}\gamma(\mathfrak{p})) \to \exp(\text{i}\gamma(t))$ and $\gamma(t) \in \mathbb{R}$. Apparently, $\gamma(t)$ records the changes to $g(t)$ and $\psi_J(t)$ as it should because $\psi_J(t) = \text{e}^{ig(t)}\varphi_J(t)$ (eq. (5)). This means that, ν_{P} can take any value restricted only by the physical processes that will be exposed below. Physically, there is no intrinsic connection between ν_{P} and the Laughlin–Jain fractional filling factor, $\nu_{\text{LJ}} = p/(2ps+1)$ where p and s are integers excluding ($\mathbb{Z}$) and including ($\mathbb{Z}^*$) zero, respectively [14,28–30] – because the physics are completely different, for example, in our formalism we strictly require degenerate and crossed Landau levels such that we do not need the notion of quantized fluxes per electron. There is also an alternative interpretation due to Wilczek [31], where the fractional ν_{W} is proposed to arise from the exotic particles called anyons [32]. Even from the outset, it is obvious that ν_{W} is also physically independent of ν_{P} as ν_{P} requires only the usual electron that can accelerate and get scattered due to electric and Lorentz forces in the presence of electric and magnetic fields. As such, technically we are not allowed to derive any valid relations (see below for details) between ν_{P} and ν_{LJ}, or between ν_{P} and ν_{W}. Here, ν_{P} is also known as the 'filling factor' because it captures the physical processes involving the occupation of degenerate and crossed Landau levels.

Our proposed filling factor ν_{P} relies on the Pancharatnam wavefunction transformation (due to changes in the phase and group momenta of the electrons), which is determined by (i) the degenerate and crossed Landau levels in the presence or in the absence of **B** and (ii) irrelevant and finite Landau-level spacings. We shall see the reason for this subsequently. We first show why the irrelevant energy-level spacing, ξ_{irr}, exists even in the presence of crossed energy levels such that the band and Mott–Hubbard energy gaps are zero. In this case, the electrons still satisfy the metallic Fermi liquid theory in the usual sense. For the free-electron metals, $\xi = 0$, and on the other hand, one obtains a strange

metal if ξ is a relevant parameter [35]. When ξ is zero or irrelevant, one can define the wavefunction as a plane wave because $\mathbf{k}$ is 'energetically' continuous ($E(\mathbf{k})$ or the Fermi surface is continuous) throughout the momentum space, even within the so-called conduction band (with $E_{\rm gap} \neq 0$) or the overlapped band (with $E_{\rm gap} = 0$). This means that, even when $\xi = \xi_{\rm irr} \neq 0$, there is no energy gap for an electron to occupy the Landau level $E_b(\mathbf{k}_1)$ from its initial level $E_a(\mathbf{k}_1)$, hence the label, irrelevant in $\xi_{\rm irr}$ (see the discussion below).

Having said that, $\xi_{\rm irr}$ can be formally shown to exist from [22,35]

$$\begin{aligned} H(\mathbf{k})\varphi_a(\mathbf{k}) &= [H_0(\mathbf{k}) + \mathcal{V}(\mathbf{k})]\varphi_a(\mathbf{k}) \\ &= [h_a(\mathbf{k}) + \mathrm{v}_a(\mathbf{k})]\varphi_a(\mathbf{k}) = E_a(\mathbf{k})\varphi_a(\mathbf{k}), \qquad (21) \\ H(\mathbf{k})\varphi_b(\mathbf{k}) &= [H_0(\mathbf{k}) + \mathcal{V}(\mathbf{k})]\varphi_b(\mathbf{k}) \\ &= [h_b(\mathbf{k}) + \mathrm{v}_b(\mathbf{k})]\varphi_b(\mathbf{k}) = E_b(\mathbf{k})\varphi_b(\mathbf{k}), \qquad (22) \end{aligned}$$

where $H(\mathbf{k})$ is a solved Landau two-level ($\varphi_a(\mathbf{k})$ and $\varphi_b(\mathbf{k})$) Hamiltonian. In particular, $H_0(\mathbf{k})$ denotes the non-interacting Hamiltonian and $\mathcal{V}(\mathbf{k})$ is the interaction operator. The Landau energy levels $E_a(\mathbf{k}_1) = E_b(\mathbf{k}_1)$ at a certain $\mathbf{k}$ point ($\mathbf{k}_1$) does not imply $\xi(\mathbf{k}_1) = 0$ even though $E_{\rm gap}(\mathbf{k}_1) = E_a(\mathbf{k}_1) - E_b(\mathbf{k}_1) = 0$ because $h_a(\mathbf{k}_1) \neq h_b(\mathbf{k}_1)$ and $\mathrm{v}_a(\mathbf{k}_1) \neq \mathrm{v}_b(\mathbf{k}_1)$. These 'not-equal' signs and $E_{\rm gap}(\mathbf{k}_1) = 0$ mean that the Landau levels are degenerate and

$$\xi(\mathbf{k}) = \sum\nolimits_i E_a(\mathbf{k}_i) - \sum\nolimits_i E_b(\mathbf{k}_i) \neq 0, \qquad (23)$$

$$\sum\nolimits_i E_a(\mathbf{k}_i) < \sum\nolimits_i E_b(\mathbf{k}_i), \qquad (24)$$

where $\xi_{\rm irr}$ is the averaged irrelevant LL spacing. If $\xi(\mathbf{k}) = \xi_{\rm irr} \neq 0$, then the only required wavefunction transformation refers to picking up or dropping the PB phase factor (eq. (9)) such that, the phase and group momenta, as well as the degenerate LLs (and the filling factor, $\nu_{\rm P}$) change with increasing or decreasing magnetic fields. Here, the changes to the PB phase factor is large (because the electron flow involves many LLs) such that there is a large change in the group momentum of a wavefunction. On the other hand, $\xi(\mathbf{k}) = \xi_{\rm irr} = 0$ restricts the wavefunction to pick up or drop the same PB phase factor within the same LL where the new filling factor $\nu_{\rm P}$ is a constant, even when B changes. In this case however, the changes to the PB phase factor require a small change in the group momentum of a wavefunction (because the electron flow involves only one LL).

As a consequence, the Hall resistance consists of two independent physical processes, which are activated when $\mathbf{B}$ increases, where changing $\mathbf{B}$ initiates changes to the LL crossings and Zeeman splittings, in such a way that there are two possibilities: (i) for a certain set of crossed LLs (within the degenerate LLs), one requires the electrons to flow within the same LLs (where $\nu_{\rm P}$ remains unchanged) governed by eq. (19) and (ii) the second process is also activated for a different set of LL crossings where they now require the electrons to flow from one LL to another degenerate LL that gives $\mathrm{d}R_\perp/\mathrm{d}B \neq 0$ and/or $\mathrm{d}R_\perp/\mathrm{d}B$ is a positive slope. In the latter case, $\nu_{\rm P}$ is not a constant. Therefore, the

total Hall resistance for quantum Hall metals can be constructed to read (we can suppress the negative sign for an obvious reason),

$$R_{\rm Hall}^{\rm Pancharatnam} = \sum_q \left[\frac{h}{e^2}(\nu_{\rm P})_q\right]_{\rm LL,\ \xi_{irr}=0}^{\rm Same} + \alpha_B \sum_r \left[\frac{{\rm d}R_\perp}{{\rm d}B}\right]_r B\Big|_{\rm LLs,\ \xi_{irr}\neq 0}^{\rm Many}. \quad (25)$$

Recall that the standard Hall resistance should read as

$$R_{\rm Hall}^{\rm std} = \frac{h}{e^2}(n_{\rm Chern}, \nu_{\rm LJ})^{-1}, \quad (26)$$

which differs from the first term on the right-hand side of eq. (25) due to different physics (see §4 for details), involving degenerate and crossed LLs (eq. (27)). Here, the magnitude of applied magnetic fields is denoted by B, α_B is a numerical constant of proportionality due to applied magnetic fields, and recall that $\xi_{\rm irr} = 0$ is not literally true, but a symbolic way of saying that only one particular LL contributes to $R_{\rm Hall}^{\rm Pancharatnam}$ (by means of the first term in eq. (25)), which eventually means, $\xi_{\rm irr} \neq 0$ does not play any role. In particular, the first term in eq. (25) is independent of $\mathbf{B}$ (because ${\rm d}R_\perp/{\rm d}B \approx 0$ and/or ${\rm d}R_\perp/{\rm d}B$ reaches a minimum), and comes from the resistance due to electron flow in the same LL, labelled q, whereas, the second term arises from the electron flow involving many LLs within the degenerate LLs, labelled r, which is proportional to the change in the longitudinal resistance ($R_\perp$) along the applied electric fields, perpendicular to $R_{\rm Hall}^{\rm Pancharatnam}$. For example, $R_{\rm Hall}^{\rm Pancharatnam} \propto {\rm d}R_\perp/{\rm d}B$ because $R_\perp$ is not entirely due to the applied electric fields if $\mathbf{B} \neq 0$, where ${\rm d}R_\perp/{\rm d}B$ can be nonlinear. Besides that, whenever $q = r$, ${\rm d}R_\perp/{\rm d}B \approx 0$ (and/or ${\rm d}R_\perp/{\rm d}B$ reaches a minimum) where $r \neq q$ represents the electron's path with many $\nu_{\rm P}$ (due to many LLs or many quantitatively different wavefunctions are involved), while q requires a constant $\nu_{\rm P}$ (due to a single LL or only an identical wavefunction is involved). Apparently, 'quantitatively different wavefunctions' here means that they are orthonormalized, satisfying eqs (21)–(23).

In figure 1, we use eq. (25) to plot the zigzag Hall resistance curve by approximating $\alpha_B {\rm d}R_\perp/{\rm d}B = \alpha_B \Delta R_\perp/(1/2)\Delta B = \alpha_B \times$ slope and $h/e^2 = 1$ where the extra factor, the one-half that appears in the stated equality implies that we can approximately and correctly consider only the part where $R_\perp$ increases with increasing B, which then contributes to the total $R_{\rm Hall}^{\rm Pancharatnam}$. However, the factor $1/2$ is for a symmetric $R_\perp(B)$ peak, where this factor of course differs for asymmetric peaks ($<\frac{1}{2}$ or $>\frac{1}{2}$) or for peaks with plateaus ($<\frac{1}{2}$).

In summary, the quantum Hall metals and its trial Hall resistance formula (eq. (25)) addressed and constructed herein are strictly valid for non-interacting ($\xi_{\rm irr} = 0$) and weakly interacting ($\xi_{\rm irr} \neq 0$) degenerate Fermi liquid (with LL crossings). Here, the electron flow in a non- or weakly-interacting system needs a continuous momentum transfer, $\hbar(\mathbf{k} - \mathbf{k}')$ within the degenerate and partially filled LL, without requiring electron excitation or wavefunction transformation beyond the phase factor. This means that, we cannot apply the ionization energy theory to study quantum Hall metals. Rightly so, we did not apply the ionization energy theory nor the energy-level spacing renormalization group method to construct the postulates and the trial Hall resistance formula. In fact, the quantum Hall effects in a (strongly) interacting and gapped system ($\xi \neq 0$: ξ is relevant and trivial) or in a (strongly) interacting and gapless system ($\xi \neq 0$: ξ is relevant and non-trivial) cannot obey the two postulates presented above, and therefore, eq. (25) is not applicable to interacting systems by definition.

4. Additional notes

We have stated earlier (see the text after eq. (25)) that $R_{\rm Hall}^{\rm std} = (h/e^2)(n_{\rm Chern}, \nu_{\rm LJ})^{-1}$ differs from $R_{\rm Hall}^{\rm Pancharatnam} = (h/e^2)\nu_{\rm P}$ due to different physics. What we mean by this is that the accepted version of the standard integer Hall resistance [37], $R_{\rm Hall}^{\rm integer} = (h/e^2)n_{\rm Chern}^{-1}$ exploits the Kubo formula [36] and the perturbed eigenstate [37],

$$|N_E\rangle = |N\rangle + \sum_{M\neq N} \frac{\langle M|(-eEy)|N\rangle}{E_M - E_N}|M\rangle + \cdots \tag{27}$$

to arrive at $R_{\rm Hall}^{\rm integer}$ where $(-eEy)$ denotes the 'degeneracy-lifting' perturbation potential. This implies that eq. (27) is only true for quantum Hall metals with non-degenerate (due to lifted degeneracy) LLs to avoid $E_M - E_N = 0$. On the contrary, $R_{\rm Hall}^{\rm Pancharatnam}$ is strictly valid for degenerate and crossed LLs (eqs (21–24)) such that a complete 'degeneracy-lifting' interaction is not allowed to operate, and therefore, the electron flow only requires the electron wavefunction to pick up or drop the phase factor (due to Pancharatnam wavefunction transformation). In our formalism, this transformation never goes beyond the phase factor (geometric and dynamic) because $\xi \to \xi_{\rm irr}$ (eq. (25)). We stress here that a completely lifted degeneracy does not allow any crossed LLs to exist, and recall here that the degenerate and crossed energy levels have been proven to exist in a different system [35] (in particular, see figure 1(E) in [35]). In this work, the crossed energy levels refer to crossed LLs. In §3, we have pointed out why $\nu_{\rm P}$ is not associated with this ratio, the number of quantized fluxes per number of electrons where $\nu_{\rm P}$ is strictly defined by eq. (20).

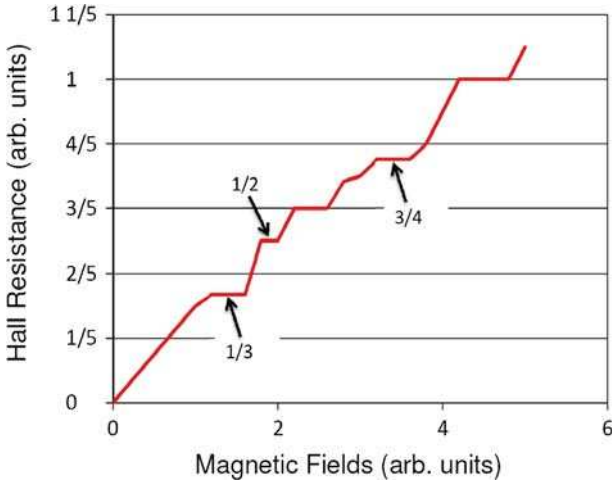

Figure 1. Calculated Hall resistance using eq. (25). The constant $R_{\rm Hall}^{\rm Pancharatnam}$ (independent of the magnetic field strength) is set to occur for $\nu_{\rm P} = \{1/3, 1/2, 3/5, 3/4, 1, ...$ contributed by the first term in eq. (25), while the second term is by definition zero because in these cases, $q = r$. For $r \neq q$, the electron conduction involves many LLs (or many $\nu_{\rm P}$), and therefore, the second term in eq. (25) also contributes to the total $R_{\rm Hall}^{\rm Pancharatnam}$ as a result of non-zero and/or large longitudinal resistance ($R_\perp$).

If we attempt to fit the experimental resistance data using the standard equation (eq. (26)), which contains n_{Chern}, then one gets integer plateaus, exclusively for certain quantum Hall metals. In contrast, if we use eq. (25) (after ignoring the second term in eq. (25) because it is irrelevant in this context) to plot the experimental data, then the integer (n_{Chern}) plateaus become fractionalized. The reason for this transformation (from integer to fractional plateaus) is that n_{Chern} (from eq. (26)) transforms into $1/\nu_{\text{P}}$ ($1/n_{\text{Chern}} \rightarrow \nu_{\text{P}}$). We have been careful not to mention $\nu_{\text{P}} = 1/n_{\text{Chern}}$ because even though this equality is a mathematically valid relation to transform n_{Chern} into ν_{P}, physically this relation is incorrect because ν_{P} has a precise physical interpretation based on the derivation of ν_{P} given in eq. (20). On the other hand, n_{Chern} comes to existence from a mathematical notion known as winding fibre (eqs (15) and (16)), which happens to represent integers.

The above transformation also applies to ν_{LJ} such that $\nu_{\text{P}} \rightarrow 1/\nu_{\text{LJ}}$. As ν_{LJ} represents fractional plateaus, the said transformation can give rise to both fractional or integer ν_{P}, depending on the fraction. For example, if $\nu_{\text{LJ}} = 1/2$, $\nu_{\text{P}} = 2$, and if $\nu_{\text{LJ}} = 2/3$, then $\nu_{\text{P}} = 3/2$.

We have handpicked the values for ν_{P} to plot the resistance curve in figure 1 using ideal and symmetric $\text{d}R_{\perp}/\text{d}B$ peaks (hence, the factor $1/2$). For constant ν_{P}, the second term in eq. (25) is made to be zero manually. If ν_{P} is not a constant, then the second term is made to be finite, again manually. For eq. (25) to be valid, we have assumed that the non-interacting and weakly interacting 2D quantum Hall metals have random LL crossings as a function of magnetic field. This assumption can be independently verified with a generic non-interacting or weakly interacting Hamiltonian. However, we did neither construct such a Hamiltonian nor the wavefunction because they are not necessary, or do not uniquely guide us to obtain eq. (25) and the conduction mechanism in 2D quantum Hall metals. In particular, the said Hamiltonian and its wavefunction, even if constructed, will be independent of our postulates presented above to obtain eq. (25). That is why our mechanism is independent of Laughlin wavefunction [14,15], Murthy–Shankar effective Hamiltonian theories [30] and Jain trial states [28,29].

5. Conclusions

We have used the connection between the Pancharatnam phase retardation (changing phase and/or group momenta) and $R_{\text{Hall}}^{\text{Pancharatnam}}$ via ν_{P}, $\xi_{\text{irr}} = 0$ and $\xi_{\text{irr}} \neq 0$ to construct eq. (25). In addition, the physics that has led us to write the trial $R_{\text{Hall}}^{\text{Pancharatnam}}$ given in eq. (25) does not require any exotic particles, namely, anyons or composite fermions where $R_{\text{Hall}}^{\text{std}} = (h/e^2)(n_{\text{Chern}}, \nu_{\text{LJ}})^{-1} \neq R_{\text{Hall}}^{\text{Pancharatnam}} = (h/e^2)\nu_{\text{P}}$ such that ν_{P} can acquire any integer or fractions with odd or even denominators because with increasing $\mathbf{B}$ (or other valid external disturbances, namely, the gate potential ($\text{d}R_{\perp}/\text{d}V_{\text{gate}}$) or external pressure ($\text{d}R_{\perp}/\text{d}P$)) one can initiate complicated changes to the LL crossings and Zeeman splittings (this is only true for $\mathbf{B} \neq 0$), which in turn give rise to effectively random LL crossings and ν_{P}. These postulates on electron flow, including the trial Hall resistance formula, are strictly valid only for non-interacting and weakly interacting fermions. The non-equal sign in $R_{\text{Hall}}^{\text{std}} \neq R_{\text{Hall}}^{\text{Pancharatnam}}$ means that the physics developed for ν_{P} differs significantly from that of ν_{LJ} and n_{Chern}. This also means that, the trial $R_{\text{Hall}}^{\text{Pancharatnam}}$ and

its physics can be proven to be false with proper Hall measurements for different types of samples. For example, if the integer and fractional values for ν_{LJ} can be shown to be fundamental such that the same values of ν_{LJ} (but for different sets of $R_{\mathrm{Hall}}^{\mathrm{Std}}$ and **B**) can also be systematically observed for other quantum Hall metals, then our physical postulates are definitely incorrect.

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References

[1] J J Thomson, *Phil. Mag.* **44**, 293 (1897)
[2] G E Uhlenbeck and S Goudsmit, *Naturwissenschaften* **13**, 953 (1925)
[3] L V de Broglie, *Ann. de Phys.* **10**, 22 (1925)
[4] S Pancharatnam, *Proc. Indian Acad. Sci. A* **44**, 427 (1956)
[5] A E H Emery, *J. Med. Genet.* **25**, 561 (1988)
[6] A D Arulsamy, *J. Chem. Sci.* **125**, 1223 (2013)
[7] N W Ashcroft and N D Mermin, *Solid state physics* (Holt, Rinehart and Winston, New York, 1976)
[8] J G Bednorz and K A Mueller, *Z. Phys. B* **64**, 189 (1986)
[9] P W Anderson, *The theory of superconductivity in the high-T_c cuprates* (Princeton University Press, Princeton, 1995)
[10] Y Hatsugai, *Phys. Rev. Lett.* **71**, 3697 (1993)
[11] C L Kane and E J Mele, *Phys. Rev. Lett.* **95**, 146802 (2005)
[12] K von Klitzing, G Dorda and M Pepper, *Phys. Rev. Lett.* **45**, 494 (1980)
[13] D C Tsui, H L Stormer and A C Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982)
[14] R B Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983)
[15] R B Laughlin, *Phys. Rev. B* **23**, 5632 (1981)
[16] V L Pokrovsky and A L Talapov, *Pramana – J. Phys.* **28**, 596 (1987)
[17] V E Arkhincheev, *Pramana – J. Phys.* **70**, 271 (2008)
[18] A Pinczuk, B S Dennis, L N Pfeiffer and K W Wes, *Phil. Mag. B* **70**, 429 (1994)
[19] T Chakraborty and P Pietilainen, *The fractional quantum Hall effect* (Springer, New York, 1988)
[20] B Karmakar and B M Arora, *Pramana – J. Phys.* **67**, 191 (2006)
[21] R N Bhatt and W Xin, *Pramana – J. Phys.* **58**, 271 (2002)
[22] A D Arulsamy, *Prog. Theor. Phys.* **126**, 577 (2011)
[23] M Nakahara, *Geometry, topology and physics* (IOP Publishing, Bristol, 2003)
[24] M V Berry, *Proc. R. Soc. A* **392**, 45 (1984)
[25] S Pancharatnam, *Collected works of Shivaramakrishnan Pancharatnam* (Oxford University Press, Oxford, 1975)
[26] A D Arulsamy, http://arxiv.org/abs/1302.5050 (2013)
[27] D J Griffiths, *Introduction to quantum mechanics* (Prentice-Hall, New Jersey, 2005)

[28] J K Jain, *Phys. Rev. Lett.* **63**, 199 (1989)
[29] J K Jain, *Adv. Phys.* **41**, 105 (1992)
[30] G Murthy and R Shankar, *Rev. Mod. Phys.* **75**, 1101 (2003)
[31] F Wilczek, *Phys. Rev. Lett.* **49**, 957 (1982)
[32] X M Chen and J J Quinn, *Phil. Mag. B* **70**, 617 (1994)
[33] D J Thouless, M Kohmoto, M P Nightingale and M den Nijs, *Phys. Rev. Lett.* **49**, 405 (1982)
[34] M Kohmoto, *Ann. Phys.* **160**, 355 (1985)
[35] A D Arulsamy, *J. Supercond. Nov. Magn.* **27**, 309 (2014)
[36] R Kubo, *J. Phys. Soc. Jpn* **12**, 570 (1957)
[37] Y Ando, *J. Phys. Soc. Jpn* **82**, 102001 (2013)