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Exceptional polynomials and SUSY quantum mechanics

K V S SHIV CHAITANYA[1,*], S SREE RANJANI[2], PRASANTA K PANIGRAHI[3], R RADHAKRISHNAN[4] and V SRINIVASAN[4]

[1]BITS Pilani, Hyderabad Campus, Jawahar Nagar, Shameerpet Mandal, Hyderabad 500 078, India
[2]Faculty of Science and Technology, ICFAI Foundation for Higher Education, Donthanapally, Hyderabad 501 203, India
[3]Indian Institute of Science Education and Research (IISER), Kolkata, Mohanpur Campus, Nadia 714 252, India
[4]Department of Theoretical Physics, University of Madras, Guindy Campus, Chennai 600 025, India
*Corresponding author. E-mail: chaitanya@hyderabad.bits-pilani.ac.in; chaitanya@imsc.res.in

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Abstract. We show that for the quantum mechanical problem which admit classical Laguerre/Jacobi polynomials as solutions for the Schrödinger equations (SE), will also admit exceptional Laguerre/Jacobi polynomials as solutions having the same eigenvalues but with the ground state missing after a modification of the potential. Then, we claim that the existence of these exceptional polynomials leads to the presence of non-trivial supersymmetry.

Keywords. Schrödinger equation; exactly solvable potentials; supersymmetry; orthogonal polynomials; exceptional orthogonal polynomials.

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1. Introduction

The well-known theorem of Bochner [1] states that if an infinite sequence of polynomials $P_n(x) = y_n$ $(n = 0, 1, 2, \ldots, \infty)$ satisfies a second-order eigenvalue equation of the form

$$p(x)y_n'' + q(x)y_n' + r(x)y_n = \lambda_n y_n, \tag{1}$$

then $p(x)$, $q(x)$ and $r(x)$ are polynomials of orders 2, 1 and 0, respectively. This theorem has been extended to the q-difference equations by Askey [2]. The polynomials y_n form a complete set with respect to a positive measure. Till recently, it was thought that only the classical orthogonal polynomial systems (OPS) such as, the Hermite, Laguerre and the Jacobi satisfy eq. (1). Note that y_0 is a constant in all these systems. It was shown [3,4] that it is possible to construct an OPS, which starts with y_n ($n = 1, 2 \ldots$) by a suitable modification of its weight function and forms a complete set. The newly constructed OPS was shown to satisfy a Sturm–Liouville equation of the form given in eq. (1), where $r(x)$ is not a constant. Only the Laguerre and the Jacobi polynomials were shown to allow such extensions, which are known as exceptional polynomials.

One can construct the exceptional X_1-Laguerre polynomials $\mathcal{L}_n^k(x)$, $k > 0$, using the Gram–Schmidt procedure from the sequence [3,4],

$$v_1 = x + k + 1; \quad v_i = (x + k)^i, \quad i \geq 2, \tag{2}$$

using the weight function

$$\hat{W}_k(x) = \frac{x^k \mathrm{e}^{-x}}{(x + k)^2}, \tag{3}$$

defined in the interval $x \in (0, \infty)$ and the scalar product

$$(f, g)_k = \int_0^\infty \mathrm{d}x\, \hat{W}_k(x) f(x) g(x). \tag{4}$$

The weight function for the normal Laguerre polynomial $W_k(x) = x^k \mathrm{e}^{-x}$, is multiplied by suitable factors such that one obtains a new $\hat{W}_k(x)$ so that one can construct the new OPS excluding the zero degree polynomial. The exceptional X_1-Laguerre differential equation is

$$T_k(y) = \lambda y, \tag{5}$$

where $\lambda = n - 1$ with $n = 1, 2, \ldots$ and

$$T_k(y) = -xy'' + \left(\frac{x - k}{x + k}\right) [(k + x + 1)y' - y]. \tag{6}$$

For more details, we refer the reader to [3,4] and the references therein.

Similarly, the exceptional X_1-Jacobi polynomials $\mathcal{P}_n^{(\alpha,\beta)}(x)$ for α and β are real such that $\alpha \neq \beta$, $\alpha \geq -1$, $\beta \geq -1$, sign $[\alpha]$ = sign $[\beta]$, to form a complete set. One takes

$$u_1 = x - c, \quad u_i = (x - b)^i \quad i \geq 2, \tag{7}$$

where

$$a = \frac{1}{2}(\beta - \alpha); \quad b = \frac{\beta + \alpha}{\beta - \alpha} \quad \text{and} \quad c = b + \frac{1}{a}. \tag{8}$$

The scalar product is defined in the range $[-1, 1]$ with the weight function

$$\hat{W}_{\alpha,\beta}(x) = \frac{(1 - x)^\alpha (1 + x)^\beta}{(x - b)^2}. \tag{9}$$

As in exceptional Laguerre case, the weight function of this new OPS is a rational extension of the classical Jacobi weight function, $W_{\alpha,\beta}(x) = (1 - x)^\alpha (1 + x)^\beta$.

These exceptional X_1-Jacobi polynomials obey the eigenvalue equation

$$(x^2 - 1)y'' + 2a\left(\frac{1 - bx}{b - x}\right)[(x - c)y' - y] = \lambda y, \tag{10}$$

where $\lambda = (n-1)(\alpha+\beta+n)$ with $n = 1, 2, \ldots$.

It is well known that the solutions of the Schrödinger eigenvalue problems, with exactly solvable (ES) potentials involve classical OPS. The form of the complete solution is $\sqrt{W(x)}y_n$, where $W(x)$ is the weight function corresponding to the OPS, $y_n (n = 0, 1, 2, \ldots)$, such that $\int_a^b W(x)y_n(x)y_m(x)\mathrm{d}x = \delta_{mn}$. The weight function is determined from the boundary conditions of the given potential. By adding suitable terms to the potential, one can modify the weight function, to obtain exceptional polynomials as solutions to the Schrödinger equation (SE). We make a crucial observation that a quantum mechanical problem which also admits classical Laguerre/Jacobi polynomials as solutions of the SE, after a modification of the potential will admit exceptional Laguerre/Jacobi polynomials as solutions having the same eigenvalues but with the ground state missing in one of the Hamiltonians.

2. Exceptional polynomials in quantum mechanics

Theorem. *By adding an extra term $V_e(x,m)$ to the Laguerre/Jacobi differential equation and demanding the solutions to be*

$$g(x) = \frac{f(x)}{(x+m)} \quad \text{and} \quad g(x) = \frac{f(x)}{(x-b)}$$

for the new differential equations, where $f(x)$ is the Laguerre and Jacobi polynomials, respectively, $g(x)$ satisfies the X_1-exceptional differential equation for the Laguerre and Jacobi, respectively, $V_e(x,m)$ can be determined uniquely.

Proof.
Let $g(x) = L_\lambda^m(x)$ satisfy the Laguerre differential equation

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x) + (m+1-x)\frac{\mathrm{d}}{\mathrm{d}x}g(x) + \lambda g(x) = 0, \tag{11}$$

where λ is an integer. By adding an extra term $V_e(x,m)$ to the Laguerre differential equation:

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}h(x) + (m+1-x)\frac{\mathrm{d}}{\mathrm{d}x}h(x) + (\lambda + V_e(x,m))h(x) = 0, \tag{12}$$

and setting

$$h(x) = \frac{f(x)}{(x+m)} \quad \text{and} \quad \lambda = n-1,$$

where $f(x)$ satisfies the X_1-exceptional Laguerre differential equation

$$-xh''(x) + \left(\frac{x-m}{x+m}\right)[(m+x+1)h'(x) - h(x)] = (n-1)h(x), \tag{13}$$

determines $V_e(x,m)$ to be

$$V_e(x,m) = \frac{2m}{(x+m)^2} - \frac{1}{(x+m)}. \tag{14}$$

If $g(x) = f(x)/(x+m)^j$ with $f(x)$ satisfying the X_j-exceptional differential equation

$$-xg''(x) + \left(\frac{x-m}{x+m}\right)\left[\left((m+x+1) - \frac{2x(j-1)}{x-m}\right)g'(x) - jg(x)\right] \\ = (n-j)g(x), \tag{15}$$

one obtains $V_e(x,m)$ as

$$V_e(x,m) = \frac{j(j+1)m}{(x+m)^2} - \frac{j}{(x+m)} \tag{16}$$

for the general case. Similarly, for the Jacobi polynomials, $g(z) = P_n^{(\alpha,\beta)}(z)$, satisfying the Jacobi differential equation

$$(1-z^2)g''(z) + [\beta-\alpha-(\alpha+\beta+2)z]g'(z) + n(n+\alpha+\beta+1)g(z) = 0, \tag{17}$$

adding an extra potential $V_e(z)$ gives

$$(1-z^2)h''(z) + [\beta-\alpha-(\alpha+\beta+2)z]h'(z) \\ + n(n+\alpha+\beta+1)h(z) + V_e(z)h(z) \\ = 0. \tag{18}$$

Setting $h(z) = f(z)/(z-b)$ and demanding that $f(z)$ satisfies the X_1-exceptional Jacobi differential equation

$$(z^2-1)h''(z) + 2a\left(\frac{1-bz}{b-z}\right)[(z-c)h'(z) - h(z)] = \lambda h(z), \tag{19}$$

with $\lambda = (n-1)(\alpha+\beta+n)$, determines $V_e(z)$ to be

$$V_e(z) = \frac{2}{(z-b)} - \frac{2-2b^2}{(z-b)^2}. \tag{20}$$

This ends the proof of our theorem.

We would like to point out here that our theorem is in accordance with the Darboux transformation [5,6] (see Appendix).

Example 1. For the 3D oscillator case in natural units the radial equation is solved by

$$R(\xi) = \xi^{l/2}\mathrm{e}^{-\xi/2}L_n^k(\xi), \tag{21}$$

where $L_n^k(\xi)$ satisfies the Laguerre differential equation

$$\xi\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}L_n^k(\xi) + \left(l+\frac{3}{2}-\xi\right)\frac{\mathrm{d}}{\mathrm{d}\xi}L_n^k(\xi) + \frac{1}{4}(\lambda-3-2l)L_n^k(\xi) = 0, \tag{22}$$

when $\xi = r^2$. Here $\lambda = 2n+3$, $n = l+2n'$ and $n' = 0, 1, 2, \ldots$ and $\lambda = 2E$, which gives $E = n + \frac{3}{2}$. Identifying with eq. (11) one has $m = l + \frac{1}{2}$ and $\lambda = n' + l + \frac{1}{2}$. Then by adding an extra potential V_e^{osc} one has

$$\xi\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}H(\xi) + \left(l+\frac{3}{2}-\xi\right)\frac{\mathrm{d}}{\mathrm{d}\xi}H(\xi) + \frac{1}{4}(\lambda-3-2l+V_e^{\mathrm{osc}})H(\xi) = 0. \tag{23}$$

Comparing the above equation with eq. (12) and demanding the solution to be

$$H(\xi) = \frac{\xi^{l/2} \mathrm{e}^{-\xi/2}}{(\xi + [(2l+1)/2])} \mathcal{L}_n^k(\xi) \tag{24}$$

one gets

$$V_e^{\mathrm{osc}}(\xi, l) = \frac{2((2l+1)/2)}{(\xi + [(2l+1)/2])^2} - \frac{1}{(\xi + (2l+1)/2)}. \tag{25}$$

Thus, one obtains the modified 3D oscillator potential as

$$V_l^+(x) = \frac{1}{2} r^2 + \frac{l(l+1)}{r^2} - E + \frac{2k}{(r^2+k)^2} - \frac{1}{(r^2+k)}, \tag{26}$$

where $k = 2l + 1$ and $\xi = \frac{1}{2} r^2$.

A quantum Hamilton–Jacobi analysis of the modified oscillator potential has been recently considered [7].

Example 2. The well-known radial equation for the Coulomb potential in natural units is

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} R(r) + \frac{2}{r} \frac{\mathrm{d}}{\mathrm{d}r} R(r) + \left[\frac{\lambda}{r} - \frac{1}{4} - \frac{l(l+1)}{r^2} \right] R(r) = 0. \tag{27}$$

The solution to $R(\rho)$, where $\rho = r/n$, is of the form

$$R(\rho) = \rho^l \mathrm{e}^{-\rho/2} K(\rho). \tag{28}$$

Here $K(\rho)$ satisfies the Laguerre differential equation

$$\rho \frac{\mathrm{d}^2}{\mathrm{d}\rho^2} K(\rho) + (2(l+1) - \rho) \frac{\mathrm{d}}{\mathrm{d}\rho} K(\rho) + (\lambda - l - 1) K(\rho) = 0. \tag{29}$$

Thus, $K(\rho) = L_{n'+l}^{2l+1}(\rho)$. Comparing this with eq. (11), one gets $\lambda = n' + l$, $m = 2l + 1$ and $\lambda = 1/\sqrt{-2E}$. Now, by replacing λ by $\lambda + V_e(\rho, l)$ in eq. (29) one gets

$$\rho \frac{\mathrm{d}^2}{\mathrm{d}\rho^2} H(r) + (2(l+1) - \rho) \frac{\mathrm{d}}{\mathrm{d}\rho} H(\rho) + (\lambda - l - 1 + V_e(\rho, l)) H(\rho) = 0. \tag{30}$$

Comparing the above equation with eq. (12) and demanding the solution to be

$$H(r) = \frac{\rho^l \mathrm{e}^{-\rho/2}}{(\rho + 2l + 1)} \mathcal{L}_n^k(r) \tag{31}$$

the extra term in eqs (27) and (29) takes the form

$$V_{\mathrm{e}}(\rho, l) = \frac{2(2l+1)}{(\rho + 2l + 1)^2} - \frac{1}{(\rho + 2l + 1)}. \tag{32}$$

Thus, the new potential is given by

$$V_{\mathrm{e}}(r, l) = \frac{l(l+1)}{r^2} - \frac{1}{r} + E + \frac{2kn^3}{r(r+nk)^2} - \frac{n^2}{r(r+nk)}, \tag{33}$$

where $k = (2l + 1)$. We see that the potential depends on the quantum number n and thus the Coulomb problem is a conditionally exactly solvable model. These models were discovered in [8] and one of us has solved a class of Calogero–Sutherland-type one-dimensional N quantum mechanical systems as a conditionally exactly solvable model [9].

Example 3. For the Morse potential [10], the radial equation is solved by

$$R(y) = y^{s-n}\mathrm{e}^{-\frac{1}{2}y}K(y), \tag{34}$$

where $K(y)$ satisfies the Laguerre differential equation

$$y\frac{\mathrm{d}^2}{\mathrm{d}y^2}K(y) + (2(s-n)+1-y)\frac{\mathrm{d}}{\mathrm{d}y}K(y) + (\lambda - (s-n))K(y) = 0. \tag{35}$$

Here $K(y) = L_n^{2s-2n}(y)$ with $y = (2B/\alpha)\mathrm{e}^{-\alpha x}$, and $s = A/\alpha$ and the eigenvalues, $\lambda = A^2 - (A - n\alpha)^2$. By adding an extra potential to eq. (35) one has

$$y\frac{\mathrm{d}^2}{\mathrm{d}y^2}H(y)+(2(s-n)+1-y)\frac{\mathrm{d}}{\mathrm{d}y}H(y)+(\lambda-(s-n)+V_e^{\mathrm{Mor}}(y,s))H(y) = 0. \tag{36}$$

Comparing the above equation with eq. (12) and demanding the solution to be

$$H(y) = \frac{y^{s-n}\mathrm{e}^{-\frac{1}{2}y}}{(y+s-n)}\mathcal{L}_n^k(y) \tag{37}$$

the additional part is given by

$$V_e^{\mathrm{Mor}}(y,s) = \frac{2(s-n)}{(y+s-n)^2} - \frac{1}{(y+s-n)}. \tag{38}$$

The solution of the Schrödinger equation with extra potential $V_e(x)$ added for bound-state problems forms a complete set. Note that solutions are of the form $\sqrt{W(x)}\mathcal{L}_\lambda^m$ which gives a proof for the completeness of the exceptional polynomials.

A similar analysis is used for the Jacobi polynomials. Consider the Scarf potential [5,11]:

$$V(x) = -A^2 + (A^2 + B^2 - A\alpha)\sec^2(\alpha x) - B(2A-\alpha)\tan(\alpha x), \tag{39}$$

the solution of which is

$$\psi(z) = (1-z)^{(s-\lambda)/2}(1+z)^{(s+\lambda)/2}P_n^{(s-\lambda-1/2,s+\lambda-1/2)}(z), \tag{40}$$

where $z = \sin(\alpha x)$, $s = A/\alpha$ and $\lambda = B/\alpha$. Here, $P_n^{(s-\lambda-1/2,s+\lambda-1/2)}(z) \equiv y$ are the classical Jacobi polynomials which satisfy the Jacobi differential equation

$$(1-z^2)y'' + [\beta - \alpha - (\alpha+\beta+2)z]y' + n(n+\alpha+\beta+1)y = 0. \tag{41}$$

Adding an extra potential $V_e(z)$ and demanding the solution to be of the form

$$\psi(z) = \frac{(1-z)^{(s-\lambda)/2}(1+z)^{(s+\lambda)/2}}{z-b}\mathcal{P}_n^{(s-\lambda-1/2,s+\lambda-1/2)}(z), \tag{42}$$

with $\mathcal{P}_n^{(s-\lambda-1/2,s+\lambda-1/2)}(z)$ satisfying the X_1-exceptional Jacobi polynomials eq. (10), $V_e(z)$ becomes

$$V_e(z) = \frac{A(2A-1)}{2A-1-2Bz} - \frac{2(A(2A-1)^2 - 4B^2)}{(2A-1-2Bz)^2}. \tag{43}$$

The Scarf potential in eq. (39) has the Jacobi polynomials as solutions and all the potentials having Jacobi polynomials as solutions can be mapped to the Scarf potential by a suitable change of variable.

3. Supersymmetric quantum mechanics

The discovery of exceptional polynomials led Quesne to guess the superpotential $\mathcal{W}(x)$, demanding isospectrality [11]. In conventional supersymmetric quantum mechanics the wave functions of the two sets of Hamiltonians, which are isospectral, have the same set of OPS. In the previous section, we have constructed two sets of isospectral Hamiltonians, of which, one Hamiltonian has the Laguerre/Jacobi solution and the other has the exceptional Laguerre/Jacobi solution. The new potentials are isospectral to the old potentials except for the ground state. Therefore, it is natural to ask, "Can the two potentials thus constructed be partner potentials? Can one construct a superpotential, $\mathcal{W}(x)$, from which these two potentials can be constructed?" Here V^- cannot be obtained from V^+ through shape invariant arguments. Below, we answer these questions in the affirmative.

First, we briefly review the conventional SUSY and refer the reader to [10,12] for more details. In supersymmetry, the superpotential $\mathcal{W}(x)$ is defined in terms of the intertwining operators $\hat{A}$ and $\hat{A}^\dagger$ as

$$\hat{A} = \frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{W}(x), \quad \hat{A}^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + \mathcal{W}(x). \tag{44}$$

This allows one to define a pair of factorized Hamiltonians $H^\pm$ as

$$H^+ = \hat{A}^\dagger \hat{A} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^+(x) - E, \tag{45}$$

$$H^- = \hat{A}\hat{A}^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^-(x) - E, \tag{46}$$

where E is the factorization energy.

The partner potentials $V^\pm(x)$ are related to $\mathcal{W}(x)$ by

$$V^\pm(x) = \mathcal{W}^2(x) \mp \mathcal{W}'(x) + E, \tag{47}$$

where $'$ denotes differentiation with respect to x. Equations (45) and (46) imply

$$H^+\hat{A}^\dagger = \hat{A}^\dagger H^-, \quad \hat{A}H^+ = H^-\hat{A}. \tag{48}$$

From the above, one can see that the operators $\hat{A}$ and $\hat{A}^\dagger$ act as intertwining operators. These operators allow one to go from wave functions $|\psi_\nu^+\rangle$ to $|\psi_\nu^-\rangle$ and vice versa. Our aim is to construct operator $\hat{A}$ for the isospectral Hamiltonians given in the previous section. We would also like to add that conventional methods for obtaining the superpotential from the ground-state eigenfunction [13,14] do not work here. In our case, the wave functions for both the Hamiltonians are known and one can construct the operator $\hat{A}$ from the following relation:

$$\hat{A}H^+|\psi_\nu^+\rangle = E_\nu \hat{A}|\psi_\nu^+\rangle = E_{\nu+1}|\psi_{\nu+1}^-\rangle = H^-|\psi_{\nu+1}^-\rangle \tag{49}$$

using the operator $\hat{\mathcal{O}}$ [3,4,11], which connects the ordinary Laguerre polynomials to the exceptional Laguerre polynomials

$$\hat{\mathcal{O}}L_\nu^{k-1}(x) = \mathcal{L}_{\nu+1}^k(x), \tag{50}$$

where $\hat{\mathcal{O}} = (x+k)(\mathrm{d}/\mathrm{d}x - 1) - 1$. The superpotential, $\mathcal{W}(x)$, can be obtained by replacing $\mathrm{d}/\mathrm{d}x$ in $\hat{A}$ with $\hat{\mathcal{O}}$ and the superpotential $\mathcal{W}(x)$ is determined.

We obtain the intertwining operators defined in eq. (44), which takes one from $|\psi_\nu^+\rangle$ to $|\psi_\nu^-\rangle$ and vice versa. Thus, the form of these intertwining operators is universal. Hence, we prove that the existence of exceptional polynomials leads to the presence of non-trivial supersymmetry.

It should be noted that to find the isospectral potentials, it is not necessary that the Hamiltonians should be of the form $\hat{A}^\dagger\hat{A}$ and $\hat{A}\hat{A}^\dagger$. It turns out that in case of the 3D oscillator, as the Hamiltonian is positive definite, the supersymmetry machinery goes through and the partner Hamiltonians are of the form $\hat{A}^\dagger\hat{A}$ and $\hat{A}\hat{A}^\dagger$. To demonstrate the supersymmetry for the 3D oscillator we substitute the wave function for the 3D oscillator

$$|\psi_\nu^+\rangle = \frac{\xi^{l/2}\mathrm{e}^{-\xi/2}}{(\xi + (2l+1)/2)}\mathcal{L}_n^k(\xi) \tag{51}$$

and

$$|\psi_\nu^-\rangle = \xi^{l/2}\mathrm{e}^{-\xi/2}L_n^k(\xi) \tag{52}$$

using (50) we get $\mathcal{W}(x)$ as

$$\mathcal{W}(x) = -\frac{l}{2\xi} - \frac{1}{2} - \frac{2}{2\xi + k}. \tag{53}$$

Here we take $k = 2l + 1$. We recover the results of Quesne [11]

$$\hat{A} = \frac{\mathrm{d}}{\mathrm{d}x} - \frac{l}{2\xi} - \frac{1}{2} - \frac{2}{2\xi + k}. \tag{54}$$

By taking $\xi = \frac{1}{2}x^2$, we get $\mathcal{W}(x)$ as

$$\mathcal{W}(x) = -\frac{l}{x} - \frac{x}{2} - \frac{2}{x^2 + k}. \tag{55}$$

Then one gets

$$\mathcal{W}^2(x) + \mathcal{W}'(x) = V_l^+(x) = \frac{1}{2}x^2 + \frac{l(l+1)}{x^2} - E$$

and

$$\mathcal{W}^2(x) - \mathcal{W}'(x) = V^-(x) = V_{l-1}^+(x) + V_e(x).$$

However, in the Coulomb problem one has negative eigenvalues. This makes the Hamiltonian negative definite. This can be overcome by taking $-\hat{A}^\dagger\hat{A}$ to be $\hat{B}^\dagger\hat{B}$, which is positive semidefinite. One can always add a zero point energy as in conventional SUSY, i.e., $H_0 + \epsilon = \epsilon + E'$, where E' is the conventional energy of the bound-state hydrogen atom which can be written as $\lambda = 1/\sqrt{-2E'}$, that makes $H_0 + \epsilon$ positive semidefinite without changing the wave function. In this case, we are unable to show

$$\mathcal{W}^2(r) + \mathcal{W}'(r) = V_l^+(r) = \frac{l(l+1)}{r^2} - \frac{1}{r} + E$$

and

$$\mathcal{W}^2(r) - \mathcal{W}'(r) = V^-(r) = V_{l-1}^+(r) + V_e(r).$$

We have shown that Coulomb model is a conditionally exactly solvable model. We argue that for $n = 1$, there exists a non-trival supersymmetry for the Coulomb potential by using the following arguments.

It should be noted that all these potentials admitting the Laguerre differential eq. (11) can be brought to the same form by a point canonical transformation. They only differ in the constant values of λ and m. For example, in natural units $V^{\rm osc} = \frac{1}{2}x^2 - E$, by making a change of variable $y = x^2$, one gets the Coulomb potential $V^{\rm Coul} = (V^{\rm osc}/yE) = (1/2E) - (1/y)$, without altering the centrifugal term. Similarly, starting from the Morse potential, $V = A^2 + B^2\exp(-2\alpha x) - 2B(A + \alpha/2)\exp(-\alpha x)$, by changing the variable $y = \exp(\alpha x)$, we obtain the Coulomb potential.

Therefore, we consider $2\mathcal{W}'_{\rm osc}(x)$ of the oscillator as

$$2\mathcal{W}'_{\rm osc}(x) = V_e^{\rm osc} = V^+ - V^- = -\frac{l}{x^2} + \frac{1}{(x^2+k)^2} - \frac{2k}{(x^2+k)}. \tag{56}$$

Then making a change of variable $r = x^2$ one obtains

$$2\mathcal{W}'_{\rm Coul}(r) = \frac{2\mathcal{W}'_{\rm osc}(r)}{r} = -\frac{l}{r^2} + \frac{1}{r}\left(\frac{1}{(r+k)} - \frac{2k}{(r+k)^2}\right) = V^-(r) - V^+(r). \tag{57}$$

The superpotential for the Coulomb problem can also be obtained.

The same method will work for ES potentials which have classical Jacobi polynomials as solutions. Here the operator, which allows us to go from classical to exceptional X_1-Jacobi polynomials is given as

$$O_j P_n^{\alpha-1,\beta+1}(x) = 2(\beta-\alpha)(\beta+n)\mathcal{P}_{n+1}^{\alpha,\beta}, \tag{58}$$

where $O_j = [\alpha+\beta-(\beta-\alpha)x]\left((1+x)\frac{\rm d}{{\rm d}x} + \beta + 1\right) + (\beta-\alpha)(1+x)$. Taking the wave functions given in eqs (40) and (42) as $\psi^+(x)$ and $\psi^-(x)$, respectively, we can construct $\mathcal{W}(x)$, which allows us to construct the intertwining operators whose form is universal. We recover the Quesne's results for the constructed Scarf potential [11].

4. Conclusion

We have proved a theorem that whenever the Schrödinger equation is associated with the Laguerre/Jacobi polynomials as solutions, one can also construct the exceptional Laguerre/Jacobi polynomials as solutions for the Schrödinger equation by adding an extra term $V_e(x)$ to the original potential. The form of the potential is universal and it depends on the extra terms, $1/(x+k)$ for the Laguerre and $1/(x-b)$ for the Jacobi polynomials. We have also proved this theorem for the general X_j exceptional Laguerre/Jacobi polynomials. We have shown for the 3D oscillator the constructed new potential from our method and the old potential from two sets of Hamiltonians which satisfy a new kind of supersymmetry called the non-trivial supersymmetry. We have also shown that the Coulomb potential is a conditionally exactly solvable model when it admits exceptional Laguerre polynomials as its solutions. The most important point we would like to bring forward in this paper is that the existence of these exceptional polynomials leads to non-trivial supersymmetry.

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Appendix

The following theorm on Darboux transformation is proved in [15].

Theorem. *For any solution* ψ *of equation the Darboux transformation*

$$\tilde{\psi} = \psi_z - \frac{\psi_{1,z}}{\psi_1}\psi, \tag{59}$$

gives a solution of the following equation:

$$\sigma(z)\tilde{\psi}_{zz} + \tau_1(z)\tilde{\psi}_z + \{u_1(z) + \lambda\}\tilde{\psi} = 0, \tag{60}$$

where the functions $\tau_1(z)$, $u_1(z)$ *are given by*

$$\tau_1(z) = \tau(z) + \sigma_z(z), \tag{61}$$

$$u_1(z) = u(z) + \tau_z(z) + \sigma_z(z)\{\ln\psi_1\}_z + 2\sigma(z)\{\ln\psi_1\}_{zz}. \tag{62}$$

Here, ψ is a solution of the general second-order equation

$$\sigma(z)\psi_{zz} + \tau(z)\psi_z + \{u(z) + \lambda\}\psi = 0 \tag{63}$$

with $\sigma(z)$, $\tau(z)$, $u(z)$ being sufficiently smooth functions. Let us denote a non-trivial solution of eq. (63) with the parameter $\lambda = \lambda_1$ as ψ_1.

By comparing with our theorem, the differential eq. (63) is the Laguerre/Jacobi differential equation. Then the differential eq. (60) is the exceptional Laguerre/Jacobi differential equation. In our case, we obtained the exceptional Laguerre/Jacobi differential equation from the Laguerre/Jacobi differential equation by adding an extra potential $V_e(z)$. From this theorem the extra potential $V_e(z)$ is given by eq. (62)

$$-V_e(z) = u(z) - u_1(z) = \tau_z(z) + \sigma_z(z)\{\ln\psi_1\}_z + 2\sigma(z)\{\ln\psi_1\}_{zz}. \tag{64}$$

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