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Solitary wave and periodic wave solutions for Burgers, Fisher, Huxley and combined forms of these equations by the (G'/G)-expansion method

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Abstract. An application of the (G'/G)-expansion method to search for exact solutions of nonlinear partial differential equations is analysed. This method is used for Burgers, Fisher, Huxley equations and combined forms of these equations. The (G'/G)-expansion method was used to construct periodic wave and solitary wave solutions of nonlinear evolution equations. This method is developed for searching exact travelling wave solutions of nonlinear partial differential equations. It is shown that the (G'/G)-expansion method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving nonlinear partial differential equations.

Keywords. The generalized (G'/G)-expansion method; Burgers equation; Fisher's equation; Huxley equation; Burgers–Fisher equation; Burgers–Huxley equation.

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1. Introduction

In this work, we derive travelling wave solutions for some important nonlinear equations using the generalized (G'/G)-expansion method. Nonlinear phenomena play fundamental roles in applied mathematics and physics. Recently, the study of nonlinear partial differential equations has attracted lots of attention among scientists. The investigation of the travelling wave solutions plays an important role in nonlinear sciences. Many powerful methods have been presented, including the inverse scattering transform [1], Hirota's bilinear method [2], homotopy analysis method [3–5], variational iteration method [6,7], homotopy perturbation method [8–10], Painlevé expansion [11,12], sine–cosine method [13,14], tanh-function method [15–17], Bäcklund transformation [18,19], exp-function method [20,21] and so on. Here, we use an effective method, the (G'/G)-expansion method, for constructing a range of exact solutions for the nonlinear partial differential equations, first proposed by Wang *et al* [22]. Zhang *et al* [23] have examined the generalized (G'/G)-expansion method and its applications. Zhang *et al* [24] have used the mKdV equation with variable coefficients using the expansion method. Also, Bekir [25] has used the application of the (G'/G)-expansion method for nonlinear evolution equations. In this article we explain a method called the (G'/G)-expansion method to look for travelling wave solutions of nonlinear evolution equations (NLEEs). Kheiri *et al* [26] examined the (G'/G)-expansion method for solving the Burgers, Burgers–Huxley and modified Burgers–KdV equations. Nonlinear evolution equations are solved using the (G'/G)-expansion method in [27,28]. A novel (G'/G)-expansion method and its application to the Boussinesq equation were examined in [29]. Finally, Aygün and Tarhan have discussed energy–momentum localization for Bianchi type-IV Universe in general relativity and teleparallel gravity [30]. The Burgers equation [26,31–33]

$$u_t + uu_x = u_{xx}, \tag{1.1}$$

is a nonlinear partial differential equation of second order which appears in various areas of applied mathematics, such as modelling of fluid dynamics, turbulence, boundary layer behaviour, shock wave formation, and traffic flow [34]. The Fisher's equation [35–37] is as follows:

$$u_t = u_{xx} + u(1-u). \tag{1.2}$$

Also Huxley equation [36]

$$u_t = u_{xx} + u(k-u)(u-1), \quad k \neq 0, \tag{1.3}$$

is an evolution equation that describes the nerve propagation in biology from which molecular CB properties can be calculated [36]. The two well-known combined forms of aforementioned equations are Burgers–Fisher [36,37] and Burgers–Huxley [26,36]. These equations are formulated respectively as follows:

$$u_t = u_{xx} + uu_x + u(1-u) \tag{1.4}$$

and

$$u_t = u_{xx} + uu_x + u(k-u)(u-1), \quad k \neq 0. \tag{1.5}$$

This article is organized as follows: In § 2, first we briefly give the steps of this method and apply the method to solve the nonlinear partial differential equations. In §3–7, Burgers, Fisher, Huxley equations and combined forms of these equations will be introduced briefly and exact solutions are obtained for related equations. A conclusion is given in §8.

2. Basic idea of (G'/G)-expansion method

We give a detailed description of method which was first presented by Wang [22].

Step 1. For a given NLPDE with independent variables $X = (x, t)$ and dependent variable u:

$$\mathcal{P}(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, ...) = 0, \tag{2.1}$$

can be reduced to an ODE

$$\mathcal{M}(u, -cu', u', u'', c^2u'', -cu'', ...) = 0, \tag{2.2}$$

the transformation $\xi = x - ct$ is the wave variable. Also, c is a constant to be determined later.

Step 2. We seek its solutions in the more general polynomial form as follows:

$$u(\xi) = a_0 + \sum_{k=1}^{m} a_k \left(\frac{G'(\xi)}{G(\xi)}\right)^k, \tag{2.3}$$

where $G(\xi)$ satisfies the second-order LODE of the form

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.4}$$

where $a_0, a_k(k = 1, 2, ..., m)$, λ and μ are constants to be determined later, $a_m = 0$, but the degree of which is generally equal to or less than $m - 1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in eq. (2.2).

Step 3. Substitute (2.3) and (2.4) into (2.2) with the value of n obtained in *Step* 1. Collecting the coefficients of $\left(G'(\xi)/G(\xi)\right)^k$ $(k = 0, 1, 2, ...)$, then setting each coefficient to zero, we get a set of over-determined partial differential equations for $a_0, a_i(i = 1, 2, ..., n)$, λ, c and μ with the aid of symbolic computation *Maple*.

Step 4. Solving the algebraic equations in *Step* 3, then substituting $a_i, ..., a_m, c$ and general solutions of eq. (2.4) into (2.3), we obtain a series of fundamental solutions of eq. (2.1) depending on the solution $G(\xi)$ of eq. (2.4).

3. The Burgers equation

In this section we employ the (G'/G)-expansion method to the Burgers equation [31–33] as follows:

$$u_t + uu_x = u_{xx}, \tag{3.1}$$

and the wave variable $\xi = x - ct$ PDE transforms to an ODE

$$-cu' + uu' = u'', \tag{3.2}$$

where by integrating eq. (3.2) we get

$$-cu + \frac{1}{2}u^2 - u' = 0. \tag{3.3}$$

In order to determine the value of m, we balance the linear term of the highest order u' with the highest order nonlinear term u^2 in eq. (3.3) and using (2.3) we get

$$u^3(\xi) = a_m^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^{2m} + \cdots ,$$

$$u_\xi(\xi) = -ma_m \left(\frac{G'(\xi)}{G(\xi)}\right)^{m+1} + \cdots. \tag{3.4}$$

Balancing u' with u^2 in eq. (3.3), we have

$$m = 1. \tag{3.5}$$

We can suppose that the solution of eq. (3.1) is of the form

$$u(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right), \quad a_1 \neq 0, \tag{3.6}$$

and therefore

$$u^2(\xi) = a_1^2 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + 2a_0a_1 \left(\frac{G'(\xi)}{G(\xi)}\right) + a_0^2 \tag{3.7}$$

and

$$u_\xi(\xi) = -a_1 \left(\frac{G'(\xi)}{G(\xi)}\right)^2 - a_1\lambda \left(\frac{G'(\xi)}{G(\xi)}\right) - a_1\mu. \tag{3.8}$$

Substituting (3.6)–(3.8), and by using the well-known *Maple* software, we obtain the following results:

$$a_0 = -\lambda \pm \sqrt{\lambda^2 - 4\mu}, \quad a_1 = -2, \quad c = \pm\sqrt{\lambda^2 - 4\mu}, \tag{3.9}$$

where λ and μ are arbitrary constants. Substituting (3.9) into expression (3.6), we get

$$u(\xi) = -\lambda \pm \sqrt{\lambda^2 - 4\mu} - 2\left(\frac{G'(\xi)}{G(\xi)}\right), \quad \xi = x \pm \sqrt{\lambda^2 - 4\mu}t. \tag{3.10}$$

Substituting the general solutions of eq. (2.4) into (3.10) we have three types of exact solutions of (3.1) as follows:

(1) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$\begin{aligned} u_1(\xi) = &-\lambda \pm \sqrt{\lambda^2 - 4\mu} - \sqrt{\lambda^2 - 4\mu} \\ &\times \left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right), \end{aligned} \tag{3.11}$$

where $\xi = x \pm \sqrt{\lambda^2 - 4\mu}t$.

(2) When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution

$$u_2(\xi) = -\lambda \pm \sqrt{\lambda^2 - 4\mu} - \sqrt{4\mu - \lambda^2} \times \left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)} \right), \tag{3.12}$$

where $\xi = x \pm \sqrt{\lambda^2 - 4\mu}t$.

(3) When $\lambda^2 - 4\mu = 0$, $\xi = x - ct$ becomes $\xi = x$.

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (3.11) gives

$$u_3(x,t) = -\lambda \tanh\left[\frac{\lambda}{2}(x - \lambda t)\right], \tag{3.13}$$

$$u_4(\xi) = -\lambda\left\{2 + \tanh\left[\frac{\lambda}{2}(x + \lambda t)\right]\right\}. \tag{3.14}$$

But, if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (3.11) gives

$$u_5(x,t) = -\lambda \coth\left[\frac{\lambda}{2}(x - \lambda t)\right], \tag{3.15}$$

$$u_6(\xi) = -\lambda\left\{2 + \coth\left[\frac{\lambda}{2}(x + \lambda t)\right]\right\}. \tag{3.16}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu > 0$, then (3.12) can be written as

$$u_7(\xi) = 2\sqrt{\mu}\left(\pm i + \tan\sqrt{\mu}\xi\right), \quad \xi = x - 2\sqrt{-\mu}\,t. \tag{3.17}$$

Also, if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu > 0$, then (3.12) can be written as

$$u_8(\xi) = 2\sqrt{\mu}\left(\pm i - \cot\sqrt{\mu}\xi\right), \quad \xi = x + 2\sqrt{-\mu}\,t. \tag{3.18}$$

In particular, if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu < 0$, then (3.11) gives

$$u_9(\xi) = 2\sqrt{-\mu}\left(1 - \tanh\sqrt{-\mu}\xi\right), \quad \xi = x - 2\sqrt{-\mu}\,t, \tag{3.19}$$

$$u_{10}(\xi) = -2\sqrt{-\mu}\left(1 + \tanh\sqrt{-\mu}\xi\right), \quad \xi = x + 2\sqrt{-\mu}\,t. \tag{3.20}$$

And also, if $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu < 0$, then eq. (3.11) gives

$$u_{11}(\xi) = 2\sqrt{-\mu}\left(1 - \coth\sqrt{-\mu}\xi\right), \quad \xi = x - 2\sqrt{-\mu}\,t, \tag{3.21}$$

$$u_{12}(\xi) = -2\sqrt{-\mu}\left(1 + \coth\sqrt{-\mu}\xi\right), \quad \xi = x + 2\sqrt{-\mu}\,t, \tag{3.22}$$

which are the exact solutions of the Burgers equation. It can be seen that some results are similar to the results in [26,34].

4. The Fisher's equation

We next consider the Fisher's equation as follows:

$$u_t = u_{xx} + u(1-u). \tag{4.1}$$

The wave variable $\xi = x - ct$ PDE transforms to an ODE

$$-cu' = u'' + u(1-u). \tag{4.2}$$

By the same procedure as illustrated in §3, we can determine value of m by balancing u'' and u^2 in eq. (4.2). We find $m = 2$. We can suppose that the solutions of eq. (4.1) is of the form

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \quad a_2 \neq 0, \tag{4.3}$$

$$u_\xi(\xi) = -ma_m\left(\frac{G'(\xi)}{G(\xi)}\right)^{m+1} + \cdots \tag{4.4}$$

$$u_{\xi\xi}(\xi) = m(m+1)a_m\left(\frac{G'(\xi)}{G(\xi)}\right)^{m+2} + \cdots. \tag{4.5}$$

As stated before, substituting (4.3)–(4.5) and by using the well-known *Maple* software, we obtain the following results:

$$a_0 = 5\lambda^2 + \frac{3}{2}\lambda\sqrt{49\lambda^2 - 16\mu} + 6\mu + \frac{1}{2}, \quad a_1 = 21\lambda + 3\sqrt{49\lambda^2 - 16\mu},$$

$$a_2 = 6, \quad c = -\frac{25}{2}\lambda + \frac{5}{2}\sqrt{49\lambda^2 - 16\mu} \tag{4.6}$$

or

$$a_0 = 5\lambda^2 - \frac{3}{2}\lambda\sqrt{49\lambda^2 - 16\mu} + 6\mu + \frac{1}{2}, \quad a_1 = 21\lambda - 3\sqrt{49\lambda^2 - 16\mu},$$

$$a_2 = 6, \quad c = -\frac{25}{2}\lambda - \frac{5}{2}\sqrt{49\lambda^2 - 16\mu}, \tag{4.7}$$

where λ and μ are arbitrary constants. Substituting (4.6)–(4.7) into expression (4.3), we get

$$\begin{aligned} u(\xi) &= 5\lambda^2 + \frac{3}{2}\lambda\sqrt{49\lambda^2 - 16\mu} + 6\mu + \frac{1}{2} + 21\lambda \\ &\quad + 3\sqrt{49\lambda^2 - 16\mu}\left(\frac{G'(\xi)}{G(\xi)}\right) + 6\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \\ \xi &= x + \frac{25}{2}\lambda + \frac{5}{2}\sqrt{49\lambda^2 - 16\mu}\; t \end{aligned} \tag{4.8}$$

or

$$u(\xi) = 5\lambda^2 - \frac{3}{2}\lambda\sqrt{49\lambda^2 - 16\mu} + 6\mu + \frac{1}{2} + 21\lambda$$
$$-3\sqrt{49\lambda^2 - 16\mu}\left(\frac{G'(\xi)}{G(\xi)}\right) + 6\left(\frac{G'(\xi)}{G(\xi)}\right)^2,$$
$$\xi = x + \frac{25}{2}\lambda - \frac{5}{2}\sqrt{49\lambda^2 - 16\mu}\, t. \quad (4.9)$$

Substituting the general solutions of eq. (2.4) into (4.8) and (4.9) we get three types of exact solutions of eq. (4.1) as follows:

(1) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions

$$u_1(\xi) = \frac{3}{2}(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2 \quad (4.10)$$
$$+\frac{24\mu + 8 - \lambda^2 + 5\lambda\sqrt{49\lambda^2 - 16\mu}}{4},$$
$$\xi = x + \left(\frac{25}{2}\lambda + \frac{5}{2}\sqrt{49\lambda^2 - 16\mu}\right)t$$

and

$$u_2(\xi) = \frac{3}{2}(\lambda^2 - 4\mu)\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)^2$$
$$+\frac{24\mu + 8 - \lambda^2 - 5\lambda\sqrt{49\lambda^2 - 16\mu}}{4},$$
$$\xi = x + \left(\frac{25}{2}\lambda - \frac{5}{2}\sqrt{49\lambda^2 - 16\mu}\right)t. \quad (4.11)$$

(2) When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solutions

$$u_3(\xi) = \frac{3}{2}(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2$$
$$+\frac{24\mu + 8 - \lambda^2 + 5\lambda\sqrt{49\lambda^2 - 16\mu}}{4},$$
$$\xi = x + \left(\frac{25}{2}\lambda + \frac{5}{2}\sqrt{49\lambda^2 - 16\mu}\right)t \quad (4.12)$$

and

$$u_4(\xi) = \frac{3}{2}(4\mu - \lambda^2)\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)^2$$

$$+\frac{24\mu + 8 - \lambda^2 - 5\lambda\sqrt{49\lambda^2 - 16\mu}}{4},$$

$$\xi = x + \left(\frac{25}{2}\lambda - \frac{5}{2}\sqrt{49\lambda^2 - 16\mu}\right)t. \tag{4.13}$$

(3) When $\lambda^2 - 4\mu = 0$, we get the rational solution

$$u_5(\xi) = \frac{6C_2^2}{(C_1 + C_2\xi)^2} + \frac{24\mu + 8 - \lambda^2 + 5\lambda\sqrt{49\lambda^2 - 16\mu}}{4}, \tag{4.14}$$

where

$$\xi = x + \left(\frac{25}{2}\lambda + \frac{5}{2}\sqrt{49\lambda^2 - 16\mu}\right)t$$

and

$$u_6(\xi) = \frac{6C_2^2}{(C_1 + C_2\xi)^2} + \frac{24\mu + 8 - \lambda^2 + 5\lambda\sqrt{49\lambda^2 - 16\mu}}{4}, \tag{4.15}$$

where

$$\xi = x + \left(\frac{25}{2}\lambda - \frac{5}{2}\sqrt{49\lambda^2 - 16\mu}\right)t.$$

Case 1

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then eqs (4.10) and (4.11) give respectively

$$u_7(\xi) = 2 + \frac{17}{2}\lambda^2 + \frac{3}{2}\lambda^2 \tanh^2\left(\frac{\lambda\xi}{2}\right), \quad \xi = x + 30\lambda t, \tag{4.16}$$

$$u_8(\xi) = 2 - 9\lambda^2 + \frac{3}{2}\lambda^2 \tanh^2\left(\frac{\lambda\xi}{2}\right), \quad \xi = x - 5\lambda t. \tag{4.17}$$

But, if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then eqs (4.10) and (4.11) give respectively

$$u_9(\xi) = 2 + \frac{17}{2}\lambda^2 + \frac{3}{2}\lambda^2 \coth^2\left(\frac{\lambda\xi}{2}\right), \quad \xi = x + 30\lambda t, \tag{4.18}$$

$$u_{10}(\xi) = 2 - 9\lambda^2 + \frac{3}{2}\lambda^2 \coth^2\left(\frac{\lambda\xi}{2}\right), \quad \xi = x - 5\lambda t. \tag{4.19}$$

Case 2

If $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu > 0$, then eqs (4.12) and (4.13)) give

$$u_{11}(\xi) = 2 + 6\mu \sec^2\left(\sqrt{\mu}\xi\right), \quad \xi = x + 10\sqrt{-\mu}\, t, \tag{4.20}$$

$$u_{12}(\xi) = 2 + 6\mu \sec^2\left(\sqrt{\mu}\xi\right), \quad \xi = x - 10\sqrt{-\mu}\, t. \tag{4.21}$$

Also, if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu > 0$, then eqs (4.12) and (4.13) give

$$u_{13}(\xi) = 2 + 6\mu \csc^2\left(\sqrt{\mu}\xi\right), \quad \xi = x + 10\sqrt{-\mu}\, t, \tag{4.22}$$

$$u_{14}(\xi) = 2 + 6\mu \csc^2\left(\sqrt{\mu}\xi\right), \quad \xi = x - 10\sqrt{-\mu}\, t, \tag{4.23}$$

which are the exact solutions of the Fisher's equation. It can be seen that the results are similar to the comparing results in [34].

5. The Huxley equation

Consider the Huxley equation of the form

$$u_t = u_{xx} + u(k - u)(u - 1), \quad k \neq 0. \tag{5.1}$$

The wave variable $\xi = x - ct$ PDE transforms to an ODE

$$cu' + u'' + u(k - u)(u - 1) = 0. \tag{5.2}$$

Applying the procedure given in the previous sections and balancing u'' and u^3 in eq. (5.2) we obtain

$$m = 1. \tag{5.3}$$

Proceeding as before we get

$$a_0 = \pm\frac{\sqrt{2}}{2}\lambda + \frac{k}{2}, \quad a_1 = \pm\sqrt{2}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{1}{8}k^2, \quad c = \mp\frac{k-2}{\sqrt{2}} \tag{5.4}$$

$$a_0 = \pm\frac{\sqrt{2}}{2}\lambda + \frac{1}{2}, \quad a_1 = \pm\sqrt{2}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{1}{8}, \quad c = \pm\frac{2k-1}{\sqrt{2}} \tag{5.5}$$

$$a_0 = \pm\frac{\sqrt{2}}{2}\lambda + \frac{1}{2} + \frac{k}{2}, \quad a_1 = \pm\sqrt{2}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{1}{8} + \frac{k}{4} - \frac{k^2}{8}, \quad c = \mp\frac{k+1}{\sqrt{2}} \tag{5.6}$$

or

$$a_0 = \pm\frac{\sqrt{2}}{2}\lambda - \frac{k}{3} \pm \frac{\sqrt{2}}{6}c, \quad a_1 = \pm\sqrt{2}, \quad c = \pm\sqrt{12\mu - 3\lambda^2 - 6k + 2k^2}, \tag{5.7}$$

where λ and μ are arbitrary constants. Substituting (5.4)–(5.7) into expression (3.6), we get

$$u(\xi) = \pm\frac{\sqrt{2}}{2}\lambda + \frac{k}{2} \pm \sqrt{2}\left(\frac{G'(\xi)}{G(\xi)}\right), \quad \xi = x - ct \tag{5.8}$$

$$u(\xi) = \pm\frac{\sqrt{2}}{2}\lambda + \frac{1}{2} \pm \sqrt{2}\left(\frac{G'(\xi)}{G(\xi)}\right), \quad \xi = x - ct \tag{5.9}$$

$$u(\xi) = \pm\frac{\sqrt{2}}{2}\lambda + \frac{1}{2} + \frac{k}{2} \pm \sqrt{2}\left(\frac{G'(\xi)}{G(\xi)}\right), \quad \xi = x - ct \tag{5.10}$$

or

$$u(\xi) = \pm\frac{\sqrt{2}}{2}\lambda - \frac{k}{3} \pm \frac{\sqrt{2}}{6}c \pm \sqrt{2}\left(\frac{G'(\xi)}{G(\xi)}\right), \quad \xi = x - ct. \tag{5.11}$$

Case 1. When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solution from eqs (5.8)–(5.10) as

$$u_1(\xi) = \pm\sqrt{\frac{\lambda^2 - 4\mu}{2}}\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) \pm\frac{\sqrt{2}}{2}\lambda + \frac{k}{2},$$
$$\xi = x \pm \frac{k-2}{\sqrt{2}}t, \tag{5.12}$$

$$u_2(\xi) = \pm\sqrt{\frac{\lambda^2 - 4\mu}{2}}\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) \pm\frac{\sqrt{2}}{2}\lambda + \frac{1}{2},$$
$$\xi = x \mp \frac{2k-1}{\sqrt{2}}t, \tag{5.13}$$

$$u_3(\xi) = \pm\sqrt{\frac{\lambda^2 - 4\mu}{2}}\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) \pm\frac{\sqrt{2}}{2}\lambda + \frac{1}{2} + \frac{k}{2},$$
$$\xi = x \pm \frac{k+1}{\sqrt{2}}t. \tag{5.14}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (5.12)–(5.14) can be written respectively as

$$u_4(\xi) = \frac{k}{2} \pm \frac{\lambda}{\sqrt{2}}\left(1 + \tanh\left(\frac{\lambda}{2}\xi\right)\right), \quad \xi = x \pm \frac{k-2}{\sqrt{2}}t, \tag{5.15}$$

$$u_5(\xi) = \frac{1}{2} \pm \frac{\lambda}{\sqrt{2}}\left(1 + \tanh\left(\frac{\lambda}{2}\xi\right)\right), \quad \xi = x \mp \frac{2k-1}{\sqrt{2}}t, \tag{5.16}$$

$$u_6(\xi) = \frac{1}{2} + \frac{k}{2} \pm \frac{\lambda}{\sqrt{2}}\left(1 + \tanh\left(\frac{\lambda}{2}\xi\right)\right), \quad \xi = x \pm \frac{k+1}{\sqrt{2}}t. \tag{5.17}$$

But, if $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (5.12)–(5.14) can be written respectively as

$$u_7(\xi) = \frac{k}{2} \pm \frac{\lambda}{\sqrt{2}}\left(1 + \coth\left(\frac{\lambda}{2}\xi\right)\right), \quad \xi = x \pm \frac{k-2}{\sqrt{2}}t, \tag{5.18}$$

$$u_8(\xi) = \frac{1}{2} \pm \frac{\lambda}{\sqrt{2}}\left(1 + \coth\left(\frac{\lambda}{2}\xi\right)\right), \quad \xi = x \mp \frac{2k-1}{\sqrt{2}}t, \tag{5.19}$$

$$u_9(\xi) = \frac{1}{2} + \frac{k}{2} \pm \frac{\lambda}{\sqrt{2}}\left(1 + \coth\left(\frac{\lambda}{2}\xi\right)\right), \quad \xi = x \pm \frac{k+1}{\sqrt{2}}t. \tag{5.20}$$

Also, if $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu < 0$, then (5.12)–(5.14) can be written respectively as

$$u_{10}(\xi) = \frac{k}{2} \pm \sqrt{-2\mu}\tanh\left(\sqrt{-\mu}\xi\right), \quad \xi = x \pm \frac{k-2}{\sqrt{2}}t, \tag{5.21}$$

$$u_{11}(\xi) = \frac{1}{2} \pm \sqrt{-2\mu}\tanh\left(\sqrt{-\mu}\xi\right), \quad \xi = x \mp \frac{2k-1}{\sqrt{2}}t, \tag{5.22}$$

$$u_{12}(\xi) = \frac{1}{2} + \frac{k}{2} \pm \sqrt{-2\mu}\tanh\left(\sqrt{-\mu}\xi\right), \quad \xi = x \pm \frac{k+1}{\sqrt{2}}t. \tag{5.23}$$

And, if $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu < 0$, then (5.12)–(5.14) can be written respectively as

$$u_{13}(\xi) = \frac{k}{2} \pm \sqrt{-2\mu}\coth\left(\sqrt{-\mu}\xi\right), \quad \xi = x \pm \frac{k-2}{\sqrt{2}}t, \tag{5.24}$$

$$u_{14}(\xi) = \frac{1}{2} \pm \sqrt{-2\mu}\coth\left(\sqrt{-\mu}\xi\right), \quad \xi = x \mp \frac{2k-1}{\sqrt{2}}t, \tag{5.25}$$

$$u_{15}(\xi) = \frac{1}{2} + \frac{k}{2} \pm \sqrt{-2\mu}\coth\left(\sqrt{-\mu}\xi\right), \quad \xi = x \pm \frac{k+1}{\sqrt{2}}t. \tag{5.26}$$

Case 2. We have three types of exact solutions of eq. (5.1) for eq. (5.11) as follows:

(1) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$\begin{aligned} u_{16}(\xi) = &\pm\sqrt{\frac{\lambda^2-4\mu}{2}}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) \\ &\pm\frac{\sqrt{2}}{2}\lambda - \frac{k}{3} \pm \frac{\sqrt{2}}{6}c, \\ \xi = x &\mp \sqrt{12\mu - 3\lambda^2 - 6k + 2k^2}t. \end{aligned} \tag{5.27}$$

(2) When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution

$$u_{17}(\xi) = \pm\sqrt{\frac{4\mu - \lambda^2}{2}} \left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)} \right)$$

$$\pm \frac{\sqrt{2}}{2}\lambda - \frac{k}{3} \pm \frac{\sqrt{2}}{6}c,$$

$$\xi = x \mp \sqrt{12\mu - 3\lambda^2 - 6k + 2k^2}t. \tag{5.28}$$

(3) When $\lambda^2 - 4\mu = 0$, we get the rational solution

$$u_{18}(\xi) = \pm\frac{\sqrt{2}C_2}{(C_1 + C_2\xi)} \pm \frac{\sqrt{2}}{2}\lambda - \frac{k}{3} \pm \frac{\sqrt{2}}{6}c,$$

$$\xi = x \mp \sqrt{-6k + 2k^2}\, t. \tag{5.29}$$

(I) The first set:

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (5.27) can be written as

$$u_{19}(\xi) = \pm\frac{\lambda}{\sqrt{2}}\left(1 + \tanh\left(\frac{\lambda}{2}\xi\right)\right) - \frac{k}{3} \pm \frac{\sqrt{2}}{6}c,$$

$$\xi = x \mp \sqrt{-3\lambda^2 - 6k + 2k^2}t, \tag{5.30}$$

but, if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (5.27) can be written as

$$u_{20}(\xi) = \pm\frac{\lambda}{\sqrt{2}}\left(1 + \coth\left(\frac{\lambda}{2}\xi\right)\right) - \frac{k}{3} \pm \frac{\sqrt{2}}{6}c,$$

$$\xi = x \mp \sqrt{-3\lambda^2 - 6k + 2k^2}t. \tag{5.31}$$

(II) The second set:

If $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu > 0$, then (5.28) can be written as

$$u_{21}(\xi) = \mp\sqrt{2\mu}\tan(\sqrt{\mu}\xi) - \frac{k}{3} \pm \frac{\sqrt{2}}{6}c,$$
$$\xi = x \mp \sqrt{12\mu - 6k + 2k^2}t. \tag{5.32}$$

Also, if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu > 0$, then eq. (5.28) can be written as

$$u_{22}(\xi) = \mp\sqrt{2\mu}\cot(\sqrt{\mu}\xi) - \frac{k}{3} \pm \frac{\sqrt{2}}{6}c,$$
$$\xi = x \mp \sqrt{12\mu - 6k + 2k^2}t, \tag{5.33}$$

which are the exact solutions of the Huxley equation. It can be seen that the results are similar to the results in [34].

6. The Burgers–Fisher equation

Suppose we solve the following Burgers–Fisher equation in the form

$$u_t = u_{xx} + uu_x + u(1-u). \tag{6.1}$$

Proceeding as before, eq. (6.1) reduces to an ODE

$$-cu' - u'' - uu' - u(1-u) = 0, \tag{6.2}$$

upon using the wave variable $\xi = x - ct$. Balancing u^2 with u'' gives $m = 2$. However, balancing u'' with uu' gives $m = 1$. We found that $m = 1$ is the only value that works. Based on this, the (G'/G)-expansion method admits the use of

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right), \quad a_1 \neq 0. \tag{6.3}$$

Substituting (6.3) into eq. (6.2), we obtain an equation that contains $\left(G'(\xi)/G(\xi)\right)^n$, $0 \leq n \leq 3$. Solving the system that results from the coefficients of $\left(G'(\xi)/G(\xi)\right)^n$ gives the following sets of solutions:

$$a_0 = \frac{1}{2} + \lambda, \quad a_1 = 2, \quad c = \frac{-5}{2}, \quad \mu = -\frac{1}{16} + \frac{1}{4}\lambda^2, \tag{6.4}$$

where λ and μ are arbitrary constants. Substituting (6.4) into expression (6.3), we obtain

$$u(\xi) = \frac{1}{2} + \lambda + 2\left(\frac{G'(\xi)}{G(\xi)}\right), \quad \xi = x + \frac{5}{2}t. \tag{6.5}$$

We have the exact solutions of eq. (6.1) as follows:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u_1(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) + \frac{1}{2} + \lambda, \quad \xi = x + \frac{5}{2}t. \tag{6.6}$$

(I) The first set:

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (6.6) gives

$$u_2(x,t) = \lambda\left(1 + \tanh\left[\frac{\lambda}{2}\left(x + \frac{5}{2}t\right)\right]\right) + \frac{1}{2}, \tag{6.7}$$

and, if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (6.6) gives

$$u_3(x,t) = \lambda\left(1 + \coth\left[\frac{\lambda}{2}\left(x + \frac{5}{2}t\right)\right]\right) + \frac{1}{2}. \tag{6.8}$$

(II) The second set:

If $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu < 0$, then (6.6) gives

$$u_4(x,t) = 2\sqrt{-\mu}\tan\left[\sqrt{-\mu}\left(x + \frac{5}{2}t\right)\right] + \frac{1}{2}, \tag{6.9}$$

but, if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (6.6) gives

$$u_5(x,t) = 2\sqrt{-\mu}\cot\left[\sqrt{-\mu}\left(x + \frac{5}{2}t\right)\right] + \frac{1}{2}, \tag{6.10}$$

which are the exact solutions of the Burgers–Fisher equation. It can be seen that the results are similar to the results in [34].

7. The Burgers–Huxley equation

As the last example, we consider the following Burgers–Huxley equation:

$$u_t = u_{xx} + uu_x + u(k-u)(u-1), \quad k \neq 0, \tag{7.1}$$

and using the wave variable $\xi = x - ct$ reduces it to an ODE

$$-cu' - u'' - uu' - u(k-u)(u-1) = 0. \tag{7.2}$$

Balancing u^3 with u'' gives $m = 1$. This allows us to set

$$u(\xi) = a_0 + a_1\left(\frac{G'(\xi)}{G(\xi)}\right), \quad a_1 \neq 0. \tag{7.3}$$

Proceeding as before, we get

$$a_0 = \frac{k+1}{2} - \lambda, \quad a_1 = -2, \quad c = \frac{k+1}{2}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{(k-1)^2}{16} \tag{7.4}$$

$$a_0 = \frac{1}{2} - \lambda, \quad a_1 = -2, \quad c = -\frac{4k-1}{2}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{1}{16} \tag{7.5}$$

$$a_0 = \frac{k}{2} - \lambda, \quad a_1 = -2, \quad c = \frac{k-4}{2}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{k^2}{16} \tag{7.6}$$

$$a_0 = \frac{\lambda+k+1}{2}, \quad a_1 = 1, \quad c = -k-1, \quad \mu = \frac{1}{4}\lambda^2 - \frac{(k-1)^2}{16} \tag{7.7}$$

$$a_0 = \frac{1}{2}\lambda + \frac{1}{2}, \quad a_1 = 1, \quad c = k-1, \quad \mu = \frac{1}{4}\lambda^2 - \frac{1}{4} \tag{7.8}$$

or

$$a_0 = \frac{1}{2}\lambda + \frac{k}{2}, \quad a_1 = 1, \quad c = -k+1, \quad \mu = \frac{1}{4}\lambda^2 - \frac{k^2}{4}, \tag{7.9}$$

where λ and μ are arbitrary constants. Substituting (7.4)–(7.9) into expression (7.3), we get

$$u(\xi) = -2\left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{k+1}{2} - \lambda, \quad \xi = x - \frac{k+1}{2}t \tag{7.10}$$

$$u(\xi) = -2\left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{1}{2} - \lambda, \quad \xi = x + \frac{4k-1}{2}t \tag{7.11}$$

$$u(\xi) = -2\left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{k}{2} - \lambda, \quad \xi = x - \frac{k-4}{2}t \tag{7.12}$$

$$u(\xi) = \left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{\lambda+k+1}{2}, \quad \xi = x + (k+1)t \tag{7.13}$$

$$u(\xi) = \left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{\lambda+1}{2}, \quad \xi = x - (k-1)t \tag{7.14}$$

or

$$u(\xi) = \left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{\lambda+k}{2}, \quad \xi = x + (k-1)t. \tag{7.15}$$

Substituting the general solutions of (2.4) into (7.10)–(7.15) we have three types of exact solutions of eq. (7.1) as follows:

(I) The first set

(1) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u_1(\xi) = -\sqrt{\lambda^2-4\mu}\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) + \frac{k+1}{2} - \lambda,$$

$$\xi = x - \frac{k+1}{2}t. \tag{7.16}$$

(2) When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution

$$u_2(\xi) = -\sqrt{4\mu-\lambda^2}\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right) + \frac{k+1}{2} - \lambda,$$

$$\xi = x - \frac{k+1}{2}t. \tag{7.17}$$

(3) When $\lambda^2 - 4\mu = 0$, we get the rational solution

$$u_3(\xi) = -\frac{2C_2}{(C_1 + C_2\xi)} + \frac{k+1}{2} - \lambda, \quad \xi = x - \frac{k+1}{2}t. \tag{7.18}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (7.16) can be written as

$$u_4(\xi) = -\lambda\left(1+\tanh\frac{\lambda}{2}\xi\right)+\frac{k+1}{2}, \quad \xi = x-\frac{k+1}{2}t \tag{7.19}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (7.16) can be written as

$$u_5(\xi) = -\lambda\left(1+\coth\frac{\lambda}{2}\xi\right)+\frac{k+1}{2}, \quad \xi = x-\frac{k+1}{2}t. \tag{7.20}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu > 0$, then (7.17) gives

$$u_6(\xi) = 2\sqrt{\mu}\tan(\sqrt{\mu}\xi)+\frac{k+1}{2}, \quad \xi = x-\frac{k+1}{2}t \tag{7.21}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu > 0$, then (7.17) gives

$$u_7(\xi) = -2\sqrt{\mu}\cot(\sqrt{\mu}\xi)+\frac{k+1}{2}, \quad \xi = x-\frac{k+1}{2}t. \tag{7.22}$$

(II) The second set

(1) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u_8(\xi) = -\sqrt{\lambda^2-4\mu}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) +\frac{1}{2}-\lambda,$$

$$\xi = x+\frac{4k-1}{2}t. \tag{7.23}$$

(2) When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution

$$u_9(\xi) = -\sqrt{4\mu-\lambda^2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right) +\frac{1}{2}-\lambda,$$

$$\xi = x+\frac{4k-1}{2}t. \tag{7.24}$$

(3) When $\lambda^2 - 4\mu = 0$, we get the rational solution

$$u_{10}(\xi) = -\frac{2C_2}{(C_1+C_2\xi)}+\frac{1}{2}-\lambda, \quad \xi = x+\frac{4k-1}{2}t. \tag{7.25}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (7.23) can be written as

$$u_{11}(\xi) = -\lambda\left(1+\tanh\frac{\lambda}{2}\xi\right)+\frac{1}{2}, \quad \xi = x+\frac{4k-1}{2}t \tag{7.26}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (7.23) can be written as

$$u_{12}(\xi) = -\lambda\left(1 + \coth\frac{\lambda}{2}\xi\right) + \frac{1}{2}, \quad \xi = x + \frac{4k-1}{2}t. \tag{7.27}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu > 0$, then (7.24) gives

$$u_{13}(\xi) = 2\sqrt{\mu}\tan(\sqrt{\mu}\xi) + \frac{1}{2}, \quad \xi = x + \frac{4k-1}{2}t \tag{7.28}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu > 0$, then (7.24) gives

$$u_{14}(\xi) = -2\sqrt{\mu}\cot(\sqrt{\mu}\xi) + \frac{1}{2}, \quad \xi = x + \frac{4k-1}{2}t. \tag{7.29}$$

(III) The third set

(1) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u_{15}(\xi) = -\sqrt{\lambda^2 - 4\mu}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right)$$
$$+\frac{k}{2} - \lambda,$$
$$\xi = x - \frac{k-4}{2}t. \tag{7.30}$$

(2) When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution

$$u_{16}(\xi) = -\sqrt{4\mu - \lambda^2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right)$$
$$+\frac{k}{2} - \lambda,$$
$$\xi = x - \frac{k-4}{2}t. \tag{7.31}$$

(3) When $\lambda^2 - 4\mu = 0$, we get the rational solution

$$u_{17}(\xi) = -\frac{2C_2}{(C_1 + C_2\xi)} + \frac{k}{2} - \lambda, \quad \xi = x - \frac{k-4}{2}t. \tag{7.32}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (7.30) can be written as

$$u_{18}(\xi) = -\lambda\left(1 + \tanh\frac{\lambda}{2}\xi\right) + \frac{k}{2}, \quad \xi = x - \frac{k-4}{2}t \tag{7.33}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (7.30) can be written as

$$u_{19}(\xi) = -\lambda\left(1 + \coth\frac{\lambda}{2}\xi\right) + \frac{k}{2}, \quad \xi = x - \frac{k-4}{2}t. \tag{7.34}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu > 0$, then (7.31) gives

$$u_{20}(\xi) = 2\sqrt{\mu}\tan(\sqrt{\mu}\xi) + \frac{k}{2}, \quad \xi = x - \frac{k-4}{2}t \tag{7.35}$$

and, if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu > 0$, then (7.31) gives

$$u_{21}(\xi) = -2\sqrt{\mu}\cot(\sqrt{\mu}\xi) + \frac{k}{2}, \quad \xi = x - \frac{k-4}{2}t. \tag{7.36}$$

(IV) The fourth set

(1) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u_{22}(\xi) = \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) + \frac{\lambda+k+1}{2},$$
$$\xi = x + (k+1)t. \tag{7.37}$$

(2) When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution

$$u_{23}(\xi) = \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right) + \frac{\lambda+k+1}{2},$$
$$\xi = x + (k+1)t. \tag{7.38}$$

(3) When $\lambda^2 - 4\mu = 0$, we get the rational solution

$$u_{24}(\xi) = \frac{C_2}{(C_1 + C_2\xi)} + \frac{\lambda+k+1}{2}, \quad \xi = x + (k+1)t. \tag{7.39}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (7.37) can be written as

$$u_{25}(\xi) = \frac{\lambda}{2}\left(1 + \tanh\frac{\lambda}{2}\xi\right) + \frac{k+1}{2}, \quad \xi = x + (k+1)t \tag{7.40}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (7.37) can be written as

$$u_{26}(\xi) = \frac{\lambda}{2}\left(1 + \coth\frac{\lambda}{2}\xi\right) + \frac{k+1}{2}, \quad \xi = x + (k+1)t. \tag{7.41}$$

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu > 0$, then (7.38) gives

$$u_{27}(\xi) = -\sqrt{\mu}\tan(\sqrt{\mu}\xi) + \frac{k+1}{2}, \quad \xi = x + (k+1)t, \tag{7.42}$$

and, if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu > 0$, then (7.38) gives

$$u_{28}(\xi) = \sqrt{\mu}\cot(\sqrt{\mu}\xi) + \frac{k+1}{2}, \quad \xi = x + (k+1)t. \tag{7.43}$$

(V) The fifth set

(1) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u_{29}(\xi) = \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) + \frac{\lambda+1}{2},$$
$$\xi = x - (k-1)t. \tag{7.44}$$

(2) When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution

$$u_{30}(\xi) = \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right) + \frac{\lambda+1}{2},$$
$$\xi = x - (k-1)t. \tag{7.45}$$

(3) When $\lambda^2 - 4\mu = 0$, we get the rational solution

$$u_{31}(\xi) = \frac{C_2}{(C_1 + C_2\xi)} + \frac{\lambda+1}{2}, \quad \xi = x - (k-1)t. \tag{7.46}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (7.44) can be written as

$$u_{32}(\xi) = \frac{\lambda}{2}\left(1 + \tanh\frac{\lambda}{2}\xi\right) + \frac{1}{2}, \quad \xi = x - (k-1)t \tag{7.47}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (7.44) can be written as

$$u_{33}(\xi) = \frac{\lambda}{2}\left(1 + \coth\frac{\lambda}{2}\xi\right) + \frac{1}{2}, \quad \xi = x - (k-1)t. \tag{7.48}$$

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu > 0$, then (7.45) gives

$$u_{34}(\xi) = -\sqrt{\mu}\tan(\sqrt{\mu}\xi) + \frac{1}{2}, \quad \xi = x - (k-1)t \tag{7.49}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu > 0$, then (7.45) gives

$$u_{35}(\xi) = \sqrt{\mu}\cot(\sqrt{\mu}\xi) + \frac{1}{2}, \qquad \xi = x - (k-1)t. \tag{7.50}$$

(VI) The sixth set

(1) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution

$$u_{36}(\xi) = \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}\right) + \frac{\lambda+1}{2},$$
$$\xi = x + (k-1)t. \tag{7.51}$$

(2) When $\lambda^2 - 4\mu < 0$, we have the trigonometric function solution

$$u_{37}(\xi) = \frac{\sqrt{4\mu - \lambda^2}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}\xi}{2}\right)}\right) + \frac{\lambda+k}{2},$$
$$\xi = x + (k-1)t. \tag{7.52}$$

(3) When $\lambda^2 - 4\mu = 0$, we get the rational solution

$$u_{38}(\xi) = \frac{C_2}{(C_1 + C_2\xi)} + \frac{\lambda+k}{2}, \quad \xi = x + (k-1)t. \tag{7.53}$$

If $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then (7.51) can be written as

$$u_{39}(\xi) = \frac{\lambda}{2}\left(1 + \tanh\frac{\lambda}{2}\xi\right) + \frac{k}{2}, \quad \xi = x + (k-1)t, \tag{7.54}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda > 0$, $\mu = 0$, then (7.51) can be written as

$$u_{40}(\xi) = \frac{\lambda}{2}\left(1 + \coth\frac{\lambda}{2}\xi\right) + \frac{k}{2}, \quad \xi = x + (k-1)t. \tag{7.55}$$

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$, $\mu > 0$, then (7.52) gives

$$u_{41}(\xi) = -\sqrt{\mu}\tan(\sqrt{\mu}\xi) + \frac{k}{2}, \qquad \xi = x + (k-1)t, \tag{7.56}$$

and if $C_2 \neq 0$, $C_1 = 0$, $\lambda = 0$, $\mu > 0$, then (7.52) gives

$$u_{42}(\xi) = \sqrt{\mu}\cot(\sqrt{\mu}\xi) + \frac{k}{2}, \qquad \xi = x + (k-1)t, \tag{7.57}$$

which are the exact solutions of the Burgers–Huxley equation. We observe that the results (7.16)–(7.57) agree with the results obtained in [26,34]. Also, new results are formally developed and obtained in this paper. All analytical solutions obtained by the generalized (G'/G)-expansion method are solutions of aforementioned nonlinear equations.

8. Conclusion

In this article we investigated Burgers, Fisher, Huxley equations and combined forms of these equations. Also we developed solitary wave and periodic wave solutions for the aforementioned equations. Generalized (G'/G)-expansion method is useful for finding travelling wave solutions of nonlinear evolution equations. This method has been successfully applied to obtain some new generalized solitonary solutions to the Burgers, Fisher, Huxley and combined forms of these equations. These exact solutions are of three types: (1) hyperbolic function solution, (2) trigonometric function solution and (3) rational solution. The generalized (G'/G)-expansion method is a powerful tool to search for exact solutions of NLPDEs. Some of these results are in agreement with the results reported specially by Wazwaz [34]. Also, new results are formally developed in this article. It can be concluded that this method is a very powerful and efficient technique for finding exact solutions for a wide class of problems.

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