PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 85, No. 1
July 2015
pp. 17–29

Topological and non-topological soliton solutions to some time-fractional differential equations

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MS received 11 April 2014; revised 30 May 2014; accepted 11 June 2014
DOI: 10.1007/s12043-014-0881-8; ***e*Publication:** 29 November 2014

Abstract. This paper investigates, for the first time, the applicability and effectiveness of He's semi-inverse variational principle method and the ansatz method on systems of nonlinear fractional partial differential equations. He's semi-inverse variational principle method and the ansatz method are used to construct exact solutions of nonlinear fractional Klein–Gordon equation and generalized Hirota–Satsuma coupled KdV system. These equations have been widely applied in many branches of nonlinear sciences such as nonlinear optics, plasma physics, superconductivity and quantum mechanics. So, finding exact solutions of such equations are very helpful in the theoretical and numerical studies.

Keywords. He's semi-inverse method; ansatz method; nonlinear fractional Klein–Gordon equation; generalized Hirota–Satsuma coupled KdV system.

PACS Nos 02.30.Jr; 05.45.Yv

1. Introduction

In this paper, we consider nonlinear fractional Klein–Gordon equation [1–3]

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) + bu^3(x,t), \qquad t > 0, \qquad 0 < \alpha \leq 1, \tag{1}$$

and nonlinear fractional generalized Hirota–Satsuma coupled KdV system [1–3]

$$\begin{aligned} D_t^\alpha u &= \frac{1}{4}u_{xxx} + 3uu_x + 3\left(-v^2 + w\right)_x, \\ D_t^\alpha v &= -\frac{1}{2}v_{xxx} - 3uv_x, \\ D_t^\alpha w &= -\frac{1}{2}w_{xxx} - 3uw_x, \qquad t > 0, \qquad 0 < \alpha \leq 1, \end{aligned} \tag{2}$$

where a and b are arbitrary constants. Klein–Gordon equation is a relativistic field equation for scalar particles (spin-0) and is a relativistic generalization of the well-known Schrödinger's equation. While there are other relativistic wave equations, Klein–Gordon equation has been the most frequently studied equation for describing particle dynamics in quantum field theory [4,5]. The construction of exact and analytical travelling wave solutions of nonlinear fractional partial differential equations is one of the most important and essential tasks in nonlinear science, as these solutions will very well describe the various natural phenomena, such as vibrations, solitons, and propagation with a finite speed. In recent years, many methods have been developed to construct exact solutions of nonlinear partial differential equations [1–32]. He's semi-inverse variational principle, which is a direct and effective algebraic method for the computation of soliton solutions, was first proposed by He [19]. This method was further developed by many authors [20–29]. Biswas *et al* [21,23–27] obtained optical solitons and soliton solutions with higher-order dispersion by using the He's variational principle. Jumarie [30] has proposed a modified Riemann–Liouville derivative. With this kind of fractional derivative and some useful formulae, we can convert fractional differential equations into integer-order differential equations by variable transformation. Using the first integral method [1], exact solutions of nonlinear fractional Klein–Gordon equation, generalized Hirota–Satsuma coupled KdV system of time fractional order and nonlinear fractional Sharma–Tasso–Olever equations have been obtained. He's semi-inverse variational principle method and the ansatz method can be used to construct exact solutions for some time-fractional differential equations. The aim of this paper is to find exact solutions of nonlinear fractional Klein–Gordon equation and nonlinear fractional generalized Hirota–Satsuma coupled KdV system by using He's semi-inverse variational principle method and the ansatz method [31,32].

2. Jumarie's modified Riemann–Liouville derivative

The Jumarie's fractional derivative of order α is defined as [1]

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\xi)^{-\alpha-1} \left(f(\xi) - f(0)\right) \mathrm{d}\xi, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\xi)^{-\alpha} \left(f(\xi) - f(0)\right) \mathrm{d}\xi, & 0 < \alpha < 1, \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, & n \le \alpha \le n+1, \quad n \ge 1. \end{cases}$$

where $f: R \to R$, $t \to f(t)$ denote a continuous (but not necessarily differentiable) function. We list some important properties for Jumarie's fractional derivative as

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \qquad r > 0, \tag{3}$$

$$D_t^{\alpha} \left(f(t) g(t)\right) = g(t) D_t^{\alpha} f(t) + f(t) D_t^{\alpha} g(t), \tag{4}$$

$$D_t^{\alpha} f(g(t)) = f_g'(g(t)) D_t^{\alpha} g(t) = D_g^{\alpha} f\left(g(t)\right) \left(g'(t)\right)^{\alpha}. \tag{5}$$

Motivated by the ideas of Lu [1] and He [19,20], we now describe He's semi-inverse variational principle method and the ansatz method for finding exact solutions of nonlinear time-fractional differential equations as follows.

3. The semi-inverse variational principle (SVP) method

Let us consider a general form of the time-fractional differential equation

$$P\left(u, D_t^{\alpha} u, u_x, D_t^{2\alpha} u, u_{xx}, ...\right) = 0, \tag{6}$$

where P is a polynomial in its arguments. We now summarize He's semi-inverse method, established by Jabbari *et al* [29], the details of which can be found in [19–28] among many others.

Step 1: To find the exact solution of eq. (6) we introduce the variable transformation [1]

$$u(x,t) = U(\xi), \qquad \xi = lx - \frac{\lambda}{\Gamma(1+\alpha)} t^{\alpha}, \tag{7}$$

where l and λ are constants to be determined later.

Using eq. (7) changes eq. (6) to an ODE

$$Q\left(U, \frac{\mathrm{d}U}{\mathrm{d}\xi}, \frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}, ...\right) = 0, \tag{8}$$

where $U = U(\xi)$ is an unknown function, Q is a polynomial in variable U and its derivatives.

Step 2: If possible, integrate eq. (8) term by term, one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3: According to He's semi-inverse method, we construct the following trial-functional

$$J(U) = \int L \,\mathrm{d}\xi, \tag{9}$$

where L is an unknown function of U and its derivatives.

Step 4: By Ritz method, we can obtain different forms of solitary wave solutions, such as

$$\begin{aligned} U(\xi) &= A\,\mathrm{sech}(B\xi), \quad U(\xi) = A\,\mathrm{csch}(B\xi), \quad U(\xi) = A\tanh(B\xi), \\ U(\xi) &= A\coth(B\xi) \end{aligned} \tag{10}$$

and so on. For example, in this paper, we search a solitary wave solution in the form

$$U(\xi) = A\,\mathrm{sech}(B\xi), \tag{11}$$

where A and B are constants to be determined later. Substituting eq. (11) into eq. (9) and making J stationary with respect to A and B results in

$$\frac{\partial J}{\partial A} = 0, \tag{12}$$

$$\frac{\partial J}{\partial B} = 0. \tag{13}$$

Solving eqs (12) and (13), we obtain A and B. Hence the solitary wave solution (11) is well determined.

3.1 *Application of SVP method to nonlinear fractional Klein–Gordon equation*

In order to solve eq. (1) by He's semi-inverse method, we use the following wave transformation [1]:

$$u(x,t) = U(\xi), \qquad \xi = lx - \frac{\lambda}{\Gamma(1+\alpha)} t^{\alpha}. \tag{14}$$

By replacing eq. (14) into eq. (1), we have

$$\left(\lambda^2 - l^2\right) U'' - aU - bU^3 = 0. \tag{15}$$

By He's semi-inverse principle [19,20], we can obtain the following variational formulation:

$$J = \int_0^{\infty} \left[\frac{\left(l^2 - \lambda^2\right)}{2} \left(U'\right)^2 - \frac{a}{2} U^2 - \frac{b}{4} U^4 \right] \mathrm{d}\xi. \tag{16}$$

By a Ritz-like method, we search a solitary wave solution in the form

$$U(\xi) = A \operatorname{sech}(B\xi), \tag{17}$$

where A and B are unknown constants to be determined later. Substituting eq. (17) into eq. (16), we have

$$\begin{aligned} J &= \int_0^{\infty} \left[\frac{A^2 B^2 \left(l^2 - \lambda^2\right)}{2} \operatorname{sech}^2(B\xi) \tanh^2(B\xi) - \frac{aA^2}{2} \operatorname{sech}^2(B\xi) \right. \\ &\qquad \left. - \frac{bA^4}{4} \operatorname{sech}^4(B\xi) \right] \mathrm{d}\xi \\ &= \frac{A^2 B \left(l^2 - \lambda^2\right)}{6} - \frac{aA^2}{2B} - \frac{bA^4}{6B}. \end{aligned} \tag{18}$$

Making J stationary with A and B yields

$$\frac{\partial J}{\partial A} = \frac{AB\left(l^2 - \lambda^2\right)}{3} - \frac{aA}{B} - \frac{2bA^3}{3B} = 0, \tag{19}$$

$$\frac{\partial J}{\partial B} = \frac{A^2\left(l^2 - \lambda^2\right)}{6} + \frac{aA^2}{2B^2} + \frac{bA^4}{6B^2} = 0. \tag{20}$$

From eqs (19) and (20), we have

$$A = \pm\sqrt{-\frac{2a}{b}}, \quad B = \pm\sqrt{\frac{a}{\left(\lambda^2 - l^2\right)}}. \tag{21}$$

Using the travelling wave transformation (14), we have the following bright (bell-shaped) soliton solutions of eq. (1):

$$u(x,t) = \pm\sqrt{-\frac{2a}{b}}\,\operatorname{sech}\left[\pm\sqrt{\frac{a}{(\lambda^2-l^2)}}\left(lx-\frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}\right)\right]. \tag{22}$$

3.2 *Application of ansatz method to nonlinear fractional Klein–Gordon equation*

This section will utilize the ansatz method to solve the nonlinear fractional Klein–Gordon equation. The bright, dark and singular soliton solutions to eq. (1) will be obtained with the help of ansatz method. In order to solve eq. (1) by the ansatz method, we use the following wave transformation:

$$u(x,t) = U(\tau), \qquad \tau = B\left(lx-\frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}\right). \tag{23}$$

By replacing eq. (23) in eq. (1), we have

$$B^2\left(\lambda^2-l^2\right)U'' - aU - bU^3 = 0. \tag{24}$$

3.2.1 *Bright soliton solution.* For bright soliton, the hypothesis is

$$U(\tau) = A\,\operatorname{sech}^p\tau, \tag{25}$$

where

$$\tau = B\left(lx-\frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}\right). \tag{26}$$

The value of the unknown exponent p will decrease during the course of derivation of the soliton solutions. Also A and B are free parameters, while λ is the speed of the soliton. Thus, from (25), we have

$$\frac{\mathrm{d}^2U(\tau)}{\mathrm{d}\tau^2} = Ap^2\,\operatorname{sech}^p\tau - Ap(p+1)\,\operatorname{sech}^{p+2}\tau \tag{27}$$

and

$$U^3(\tau) = A^3\,\operatorname{sech}^{3p}\tau. \tag{28}$$

Substitution of (25) in eq. (24) leads to

$$B^2\left(\lambda^2-l^2\right)\left\{Ap^2\,\operatorname{sech}^p\tau - Ap(p+1)\,\operatorname{sech}^{p+2}\tau\right\} - aA\,\operatorname{sech}^p\tau - bA^3\,\operatorname{sech}^{3p}\tau = 0. \tag{29}$$

By virtue of the balancing principle, on equating the exponents $3p$ and $p+2$, from (29), we get

$$p = 1. \tag{30}$$

Next, from (29) setting the coefficients of the linearly independent functions to zero implies

sech^1 coeff.:

$$B^2\left(\lambda^2 - l^2\right) - a = 0,$$

sech^3 coeff.:

$$2B^2\left(\lambda^2 - l^2\right) + bA^2 = 0. \tag{31}$$

Solving the above equations yields

$$A = \pm\sqrt{-\frac{2a}{b}} \tag{32}$$

and

$$B = \pm\sqrt{\frac{a}{\left(\lambda^2 - l^2\right)}}. \tag{33}$$

Equations (32) and (33) prompt the constraints

$$-ab > 0 \tag{34}$$

and

$$a\left(\lambda^2 - l^2\right) > 0, \tag{35}$$

respectively. Thus, the bright 1-soliton solution to eq. (1) is given by

$$u(x,t) = \pm\sqrt{-\frac{2a}{b}}\ \text{sech}\left[\pm\sqrt{\frac{a}{\left(\lambda^2 - l^2\right)}}\left(lx - \frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}\right)\right]. \tag{36}$$

3.2.2 *Topological (dark) soliton solution.* The initial hypothesis for dark 1-soliton solution to eq. (24) is

$$U(\tau) = A\tanh^p\tau, \tag{37}$$

where τ is the same as (26). However, for dark solitons the parameters A and B are indeed free soliton parameters, although λ still represents the velocity of the dark soliton. Thus, from (37), we have

$$\frac{\mathrm{d}^2U(\tau)}{\mathrm{d}\tau^2} = Ap(p-1)\tanh^{p-2}\tau - 2Ap^2\tanh^p\tau + Ap(p+1)\tanh^{p+2}\tau \tag{38}$$

and

$$U^3(\tau) = A^3\tanh^{3p}\tau. \tag{39}$$

In this case, substituting the hypothesis (37) into (24) leads to

$$B^2\left(\lambda^2 - l^2\right)\left\{Ap(p-1)\tanh^{p-2}\tau - 2Ap^2\tanh^p\tau + Ap(p+1)\tanh^{p+2}\tau\right\}$$
$$-aA\tanh^p\tau - bA^3\tanh^{3p}\tau = 0. \tag{40}$$

By balancing the power of $\tanh^{p+2}$ and $\tanh^{3p}$ in eq. (40) we have

$$p = 1. \tag{41}$$

Now, from eq. (40), setting the coefficients of the linearly independent functions $\tanh^{(p+j)}\tau$ to zero, where $j = 0,\ 2$, gives

$\tanh^1$ coeff.:

$$-2B^2\left(\lambda^2 - l^2\right) - a = 0,$$

$\tanh^3$ coeff.:

$$2B^2\left(\lambda^2 - l^2\right) bA^2 = 0. \tag{42}$$

Solving the above equations yields

$$A = \pm\sqrt{-\frac{a}{b}} \tag{43}$$

and

$$B = \pm\sqrt{\frac{a}{2\left(l^2 - \lambda^2\right)}}. \tag{44}$$

Equations (43) and (44) prompt the constraints

$$-ab > 0 \tag{45}$$

and

$$a\left(l^2 - \lambda^2\right) > 0, \tag{46}$$

respectively. Thus, the topological 1-soliton solution to eq. (1) is given by

$$u(x,t) = \pm\sqrt{-\frac{a}{b}}\tanh\left[\pm\sqrt{\frac{a}{2\left(l^2 - \lambda^2\right)}}\left(lx - \frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}\right)\right]. \tag{47}$$

3.2.3 *Singular soliton solution.* For singular soliton, the hypothesis is

$$U(\tau) = A\,\mathrm{csch}^p\tau, \tag{48}$$

where τ is the same as (26). The value of the unknown exponent p differs during the course of derivation of the soliton solutions. Also A and B are free parameters, while λ is the speed of the soliton. Substitution of (48) into eq. (24) leads to

$$B^2\left(\lambda^2 - l^2\right)\left\{Ap^2\,\mathrm{csch}^p\tau + Ap(p+1)\,\mathrm{csch}^{p+2}\tau\right\}$$
$$-aA\,\mathrm{csch}^p\tau - bA^3\,\mathrm{csch}^{3p}\tau = 0. \tag{49}$$

From (49), the balancing principle yields

$$p = 1. \tag{50}$$

Next, from (49) setting the coefficients of the linearly independent functions to zero implies

$$A = \pm\sqrt{\frac{2a}{b}} \tag{51}$$

and

$$B = \pm\sqrt{\frac{a}{\left(\lambda^2 - l^2\right)}}. \tag{52}$$

Equations (51) and (52) prompt the constraints

$$ab > 0 \tag{53}$$

and

$$a\left(\lambda^2 - l^2\right) > 0. \tag{54}$$

Thus, the singular 1-soliton solution to eq. (1) is given by

$$u(x,t) = \pm\sqrt{\frac{2a}{b}}\,\operatorname{csch}\left[\pm\sqrt{\frac{a}{\left(\lambda^2 - l^2\right)}}\left(lx - \frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}\right)\right]. \tag{55}$$

3.3 *Application of SVP method to nonlinear fractional generalized Hirota–Satsuma coupled KdV system*

In order to solve eq. (1) by He's semi-inverse method, we use the following wave transformations [1]:

$$u(x,t) = \frac{1}{\lambda}U^2(\xi), \quad v(x,t) = -\lambda + U(\xi), \quad w(x,t) = 2\lambda^2 - 2\lambda U(\xi), \tag{56}$$

where

$$\xi = x - \frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}.$$

By replacing eq. (56) into eq. (2), we have

$$\lambda U'' - 2\lambda^2 U + 2U^3 = 0. \tag{57}$$

By He's semi-inverse principle [19,20], we can obtain the following variational formulation:

$$J = \int_0^{\infty}\left[-\frac{\lambda}{2}\left(U'\right)^2 - \lambda^2\, U^2 + \frac{1}{2}\, U^4\right]\mathrm{d}\xi. \tag{58}$$

By a Ritz-like method, we search a solitary wave solution in the form

$$U(\xi) = A\,\operatorname{sech}(B\xi)\,, \tag{59}$$

where A and B are unknown constants to be determined later. Substituting eq. (59) into eq. (58), we have

$$J = -\frac{A^2 B\lambda}{6} - \frac{\lambda^2 A^2}{B} + \frac{A^4}{3B}. \tag{60}$$

Making J stationary with A and B yields

$$\frac{\partial J}{\partial A} = -\frac{AB\lambda}{3} - \frac{2\lambda^2 A}{B} + \frac{4A^3}{3B} = 0, \tag{61}$$

$$\frac{\partial J}{\partial B} = -\frac{A^2\lambda}{6} + \frac{\lambda^2 A^2}{B^2} - \frac{A^4}{3B^2} = 0. \tag{62}$$

From eqs (61) and (62), we have

$$A = \pm\sqrt{2}\lambda, \qquad B = \pm\sqrt{2\lambda}. \tag{63}$$

Using the travelling wave transformation (56), we have the following bright (bell-shaped) soliton solutions of eq. (2):

$$\begin{aligned} u(x,t) &= 2\lambda\,\mathrm{sech}^2\left[\pm\sqrt{2\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right)\right], \\ v(x,t) &= -\lambda\left\{1 \mp \sqrt{2}\,\mathrm{sech}\left[\pm\sqrt{2\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right)\right]\right\}, \\ w(x,t) &= 2\lambda^2\left\{1 \mp \sqrt{2}\,\mathrm{sech}\left[\pm\sqrt{2\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right)\right]\right\}. \end{aligned} \tag{64}$$

3.4 *Application of ansatz method to nonlinear fractional generalized Hirota–Satsuma coupled KdV system*

In order to solve eq. (2) by ansatz method, we use the following wave transformations:

$$u(x,t) = \frac{1}{\lambda}U^2(\xi), \quad v(x,t) = -\lambda + U(\xi), \quad w(x,t) = 2\lambda^2 - 2\lambda U(\xi), \tag{65}$$

where

$$\xi = B\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right).$$

By replacing eq. (65) in eq. (1), we have

$$B^2\lambda U'' - 2\lambda^2 U + 2U^3 = 0. \tag{66}$$

3.4.1 *Bright soliton solution.*

For bright soliton, the hypothesis is

$$U(\tau) = A\,\mathrm{sech}^p\tau, \tag{67}$$

where

$$\tau = B\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right). \tag{68}$$

The value of the unknown exponent p will differ during the course of derivation of the soliton solutions. Also A and B are free parameters, while λ is the speed of the soliton. Substitution of (67) into eq. (66) leads to

$$B^2\lambda\left\{Ap^2\,\mathrm{sech}^p\tau - Ap(p+1)\,\mathrm{sech}^{p+2}\tau\right\}$$
$$-2\lambda^2 A\,\mathrm{sech}^p\tau + 2A^3\,\mathrm{sech}^{3p}\tau = 0. \tag{69}$$

By virtue of the balancing principle, on equating the exponents $3p$ and $p+2$, from (69), we get

$$p = 1. \tag{70}$$

Next, from (69) setting the coefficients of the linearly independent functions to zero implies

sech^1 coeff.:

$$AB^2\lambda - 2\lambda^2 A = 0,$$

sech^3 coeff.:

$$2A^3 - 2AB^2\lambda = 0. \tag{71}$$

Solving the above equations yields

$$A = \pm\sqrt{2}\lambda,\quad B = \pm\sqrt{2\lambda}. \tag{72}$$

Using the travelling wave transformation (65), we have the following bright 1-soliton solutions of eq. (2):

$$u(x,t) = 2\lambda\,\mathrm{sech}^2\left[\pm\sqrt{2\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right)\right],$$
$$v(x,t) = -\lambda\left\{1 \mp \sqrt{2}\,\mathrm{sech}\left[\pm\sqrt{2\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right)\right]\right\},$$
$$w(x,t) = 2\lambda^2\left\{1 \mp \sqrt{2}\,\mathrm{sech}\left[\pm\sqrt{2\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right)\right]\right\}. \tag{73}$$

3.4.2 *Topological (dark) soliton solution.* The initial hypothesis for dark 1-soliton solution to eq. (66) is

$$U(\tau) = A\,\tanh^p\tau, \tag{74}$$

where τ is the same as (68). However, for dark solitons the parameters A and B are indeed free soliton parameters, although λ still represents the velocity of the dark soliton. In this case, substituting this hypothesis (74) into eq. (66) leads to

$$B^2\lambda\left\{Ap(p-1)\tanh^{p-2}\tau - 2Ap^2\tanh^p\tau + Ap(p+1)\tanh^{p+2}\tau\right\}$$
$$-2\lambda^2 A\,\tanh^p\tau + 2A^3\,\tanh^{3p}\tau = 0. \tag{75}$$

By equating the power of $\tanh^{p+2}$ and $\tanh^{3p}$ in (75) we have

$$p = 1. \tag{76}$$

Now, from (75), setting the coefficients of the linearly independent functions $\tanh^{(p+j)}\tau$ to zero, where $j = 0,\ 2$, gives

$\tanh^1$ coeff.:

$$-2A\lambda B^2 - 2\lambda^2 A = 0,$$

$\tanh^3$ coeff.:

$$2\lambda AB^2 + 2A^3 = 0. \tag{77}$$

Solving the above equations yields

$$A = \pm\lambda, \quad B = \pm\sqrt{-\lambda}. \tag{78}$$

Using the travelling wave transformation (65), we have the following topological 1-soliton solution of eq. (2):

$$\begin{aligned} u(x,t) &= \lambda \tanh^2\left[\pm\sqrt{-\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right)\right], \\ v(x,t) &= -\lambda\left\{1 \mp \tanh\left[\pm\sqrt{-\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right)\right]\right\}, \\ w(x,t) &= 2\lambda^2\left\{1 \mp \tanh\left[\pm\sqrt{-\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^\alpha\right)\right]\right\}. \end{aligned} \tag{79}$$

Remark. In this case, for $\lambda = -4/A_1^2$, comparing our results with Lu's results [1], it can be seen that both are same.

3.4.3 *Singular soliton solution.* For singular soliton, the hypothesis is

$$U(\tau) = A\,\text{csch}^p\tau, \tag{80}$$

where τ is the same as (68). The value of the unknown exponent p will differ during the course of derivation of the soliton solutions. Also A and B are free parameters, while λ is the speed of the soliton. Substitution of (80) into eq. (66) leads to

$$\begin{aligned} &B^2\lambda\left\{Ap^2\,\text{csch}^p\tau + Ap(p+1)\,\text{csch}^{p+2}\tau\right\} \\ &\quad -2\lambda^2 A\,\text{csch}^p\tau + 2A^3\,\text{csch}^{3p}\tau = 0. \end{aligned} \tag{81}$$

From (81), the balancing principle yields

$$p = 1. \tag{82}$$

Next, from (81) setting the coefficients of the linearly independent functions to zero implies

$$A = \pm\sqrt{2}i\lambda, \quad B = \pm\sqrt{2\lambda}. \tag{83}$$

Using the travelling wave transformation (65), we have the following singular 1-soliton solution of eq. (2):

$$\begin{aligned} u(x,t) &= -2\lambda\, \mathrm{csch}^2 \left[\pm\sqrt{2\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}\right)\right], \\ v(x,t) &= -\lambda\left\{1 \mp i\sqrt{2}\, \mathrm{csch}\left[\pm\sqrt{2\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}\right)\right]\right\}, \\ w(x,t) &= 2\lambda^2\left\{1 \mp i\sqrt{2}\, \mathrm{csch}\left[\pm\sqrt{2\lambda}\left(x - \frac{\lambda}{\Gamma(1+\alpha)}t^{\alpha}\right)\right]\right\}. \end{aligned} \tag{84}$$

4. Conclusions

In this paper, He's semi-inverse variational principle method and the ansatz method have been applied to obtain exact solutions of nonlinear fractional Klein–Gordon equation and nonlinear fractional generalized Hirota–Satsuma coupled KdV system. The results show that these methods are powerful tools for obtaining exact solutions of fractional nonlinear partial differential equations. We have predicted that He's semi-inverse variational principle method and the ansatz method can be extended to solve many systems of nonlinear fractional partial differential equations in mathematical and physical sciences.

Acknowledgement

The author is very grateful to the referees for their detailed comments and kind help.

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