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Symmetry reductions and exact solutions of the (2 + 1)-dimensional Camassa–Holm Kadomtsev–Petviashvili equation

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Abstract. In this paper, the classical Lie group method is employed to obtain exact travelling wave solutions of the generalized Camassa–Holm Kadomtsev–Petviashvili (g-CH–KP) equation. We give the conservation laws of the g-CH–KP equation. Using the symmetries, we find six classical similarity reductions of g-CH–KP equation. Many types of exact solutions of the g-CH–KP equation are derived by solving the reduced equations.

Keywords. Lie group method; g-CH–KP equation; symmetry reduction; conservation laws; exact solutions.

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1. Introduction

In order to understand the various physical mechanisms of nonlinear problems, many nonlinear models (nonlinear equations) were studied in succession. Nonlinear partial differential equations (PDEs) are widely used to describe the physical phenomena in various fields of sciences such as fluid mechanics, solid-state physics, plasma physics, plasma wave, chemical physics, condensed matter physics, optical fibres, biology, chemical kinematics and geochemistry. Searching for exact solutions of nonlinear evolution equations (NPDEs) becomes an important subject. In the numerical methods [1,2], stability and convergence should be considered, so as to avoid divergent or inappropriate results. However, in recent years, many effective analytical and semianalytical approaches have been suggested to obtain explicit travelling and solitary wave solutions of NLEEs, such as the variational iteration method [3–6], parameter-expansion method [7], sine–cosine method [8], tanh method [9,10], homotopy analysis method [11], homogeneous balance method [12], inverse scattering method, exp-function method [13–15] etc.

To understand the role of dispersion in the formation of patterns in liquid drops, Wazwaz [16] investigated the g-CH–KP equation given by

$$(u_t + au_x + cuu_x + bu_{xxt})_x + u_{yy} = 0, \tag{1}$$

where a, b, c are non-zero constants.

Many researchers have studied some types of solutions of the above equation [17]. Using the dynamical system theory and simulation method, the bounded travelling wave solutions of the g-CH–KP equation were studied [18], and some solutions were obtained using exp-function and G'/G-expansion methods [19].

2. Classical symmetry of g-CH–KP equation

First of all, let us consider a one-parameter Lie group of infinitesimal transformations:

$$\begin{aligned}
x &\to x + \varepsilon\xi(x,\ y,\ t,\ u),\\
y &\to y + \varepsilon\eta(x,\ y,\ t,\ u),\\
t &\to t + \varepsilon\tau(x,\ y,\ t,\ u),\\
u &\to u + \varepsilon\theta(x,\ y,\ t,\ u),
\end{aligned} \tag{2}$$

with a small parameter $\varepsilon \ll 1$. Thus, the vector field with the above group of transformations can be expressed as follows:

$$\begin{aligned}
V = {} & \xi(x,\ y,\ t,\ u)\frac{\partial}{\partial x} + \eta(x,\ y,\ t,\ u)\frac{\partial}{\partial y}\\
& + \tau(x,\ y,\ t,\ u)\frac{\partial}{\partial t} + \phi(x,\ y,\ t,\ u)\frac{\partial}{\partial u},
\end{aligned} \tag{3}$$

where the coefficient functions ξ, η, τ and ϕ are to be determined later. Obviously, its fourth prolongation can be given as

$$\overset{(4)}{\mathrm{Pr}}\, V = V + \phi^{x} + \phi^{xt} + \phi^{xx} + \phi^{yy} + \phi^{xxxt}, \tag{4}$$

where $\phi^x, \phi^{xt}, \phi^{xx}, \phi^{yy}, \phi^{xxxt}$ are given explicitly in terms of ξ, η, τ, ϕ and the derivatives of ϕ. Furthermore, one can get

$$\begin{aligned}
\phi^{x} &= D_x\big(\phi - u_x\xi - u_t\tau - u_y\eta\big) + \xi u_{xx} + \eta u_{xy} + \tau u_{xt},\\
\phi^{xx} &= D_{xx}\big(\phi - u_x\xi - u_y\eta - u_t\tau\big) + \xi u_{xxx} + \eta u_{xxy} + \tau u_{xxt},\\
\phi^{xt} &= D_{xt}\big(\phi - u_x\xi - u_y\eta - u_t\tau\big) + \xi u_{xxt} + \eta u_{xyt} + \tau u_{xtt},\\
\phi^{yy} &= D_{yy}\big(\phi - u_x\xi - u_y\eta - u_t\tau\big) + \xi u_{xyy} + \eta u_{yyy} + \tau u_{yyt},\\
\phi^{xxxt} &= D_{xxxt}\big(\phi - u_x\xi - u_y\eta - u_t\tau\big) + \xi u_{xxxxt} + \eta u_{xxxyt} + \tau u_{xxxtt}.
\end{aligned}$$

Here, D_i denotes the total derivative operator and is defined as

$$D_i = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2, 3$$

and $(x_1, x_2, x_3) = (t, x, y)$.

From $\mathrm{Pr}^{(4)}(\Delta)|_{\Delta=0} = 0$, applying the fourth prolongation $\mathrm{Pr}^{(4)}(V)$ to eq. (1), we find the following system of symmetry equations. Then the invariant condition reads as

$$\phi^{xt} + a\phi^{xx} + b\phi^{xxxt} + 2c\phi^{x}u_x + c\phi u_{xx} + cu\phi^{xx} + \phi^{yy} = 0. \tag{5}$$

By Lie symmetry analysis, setting the coefficients of the polynomial to zero, yields many differential equations about the functions ξ, η, τ and ϕ as follows:

$$\xi = c_3, \quad \eta = c_1 y + c_2, \quad \tau = 2c_1 t + c_4, \quad \phi = \frac{2c_1(a+cu)}{c}, \tag{6}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Therefore, we obtain the following symmetries of eq. (1):

$$\sigma = (2c_1 t + c_4)u_t + c_3 u_x + (c_1 y + c_2)u_y + \frac{2c_1(a+cu)}{c}. \tag{7}$$

Then, equivalent vector expression of the above symmetry is

$$V = (2c_1 t + c_4)\frac{\partial}{\partial t} + c_3\frac{\partial}{\partial x} + (c_1 y + c_2)\frac{\partial}{\partial y} - \frac{2c_1\,(a+cu)}{c}\frac{\partial}{\partial u}. \tag{8}$$

From eq. (8), the corresponding four-dimensional Lie algebra $= \{V_1, V_2, V_3, V_4\}$ is reached, where

$$\begin{aligned} V_1 &= 2t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} - \left(\frac{2a}{c} + 2u\right)\frac{\partial}{\partial u}, \quad V_2 = \frac{\partial}{\partial t}, \\ V_3 &= \frac{\partial}{\partial x}, \quad V_4 = \frac{\partial}{\partial y}. \end{aligned} \tag{9}$$

Their commutator table is given in table 1.

The one-parameter group g_i generated by $V_i (i = 1, \ldots, 4)$ is given as

$$\begin{aligned} &g_1\colon \quad \left(t\mathrm{e}^{2\varepsilon}, x, y\mathrm{e}^{\varepsilon}, u\mathrm{e}^{-2\varepsilon} - \frac{2a\varepsilon}{c}\right), \\ &g_2\colon \quad (x, t+\varepsilon, y, u), \\ &g_3\colon \quad (x+\varepsilon, t, y, u), \\ &g_4\colon \quad (t, x, y+\varepsilon, u). \end{aligned} \tag{10}$$

The symmetry groups g_2, g_3 and g_4 demonstrate the time and space invariance of the equation, while g_1 represents a kind of Galilean boost to a moving coordinate frame. Its appearance is far from obvious from the basic physical principles, but it is important to

Table 1. Lie bracket.

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	$-V_1$	V_3	V_4
V_2	V_1	0	0	0
V_3	$-V_3$	0	0	0
V_4	$-V_4$	0	0	0

study the properties and exact solutions of eq. (1). From the above groups g_1, g_2, g_3 and g_4 one can get

$$\begin{aligned} u_1 &= \mathrm{e}^{2\varepsilon} f(t\mathrm{e}^{-2\varepsilon}, x, y\mathrm{e}^{-\varepsilon}) + \frac{2a\varepsilon}{c}, \\ u_2 &= f(t-\varepsilon, x, y), \\ u_3 &= f(t, x-\varepsilon, y), \\ u_4 &= f(t, x, y-\varepsilon), \end{aligned} \tag{11}$$

where ε is a parameter.

By taking the following soliton solution [20] of eq. (1)

$$u(x, y, t) = \frac{A}{\cosh^2(B_1 x + B_2 y - vt)}$$

one can obtain new exact solutions of eq. (1) by applying u_1 and u_2 as follows:

$$u(x, y, t) = \mathrm{e}^{2\varepsilon} \frac{A}{\cosh^2(B_1 x + B_2 y\mathrm{e}^{-\varepsilon} - vt\mathrm{e}^{-2\varepsilon})} + \frac{2a\varepsilon}{c}$$

and

$$u(x, y, t) = \frac{A}{\cosh^2[B_1 x + B_2 y - v(t-\varepsilon)]},$$

where A, B_1, B_2 are arbitrary constants

Remark 1. A number of new invariant solutions can be found from the given solutions in [17] for the g-CH–KP equation. Thus, the corresponding results in [20] are generalized.

3. Conservation laws of g-CH–KP equation

In this section, we shall study the conservation laws by using the adjoint equation and symmetries of the g-CH–KP equation. For eq. (1), the adjoint equation has the form

$$v_{xt} + av_{xx} + cuv_{xx} + bv_{xxxt} + v_{yy} = 0 \tag{12}$$

and the Lagrangian in the symmetrized form

$$L = v(u_{tx} + au_{xx} + cu_x^2 + cuu_{xx} + bu_{xxxt} + u_{yy}). \tag{13}$$

Theorem 1. *Every Lie point, Lie–Bäcklund and non-local symmetry of eq.* (1) *provides a conservation law for eq.* (1) *and the adjoint equation* [20–22]. *Then the elements of conservation vector* (C^1, C^2, C^3) *are defined by the following expression*:

$$\begin{aligned} C^i = {} & \xi^i L + W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j \left(\frac{\partial L}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) - D_j D_k D_r \left(\frac{\partial L}{\partial u_{ijkr}^\alpha} \right) + \cdots \right] \\ & + D_j(W) \left[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) + \cdots \right] \\ & + D_j D_k(W) \left[\frac{\partial L}{\partial u_{ijk}^\alpha} - D_r \left(\frac{\partial L}{\partial u_{ijkr}^\alpha} \right) + \cdots \right], \end{aligned} \tag{14}$$

where W is the Lie characteristic function and

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j.$$

The conserved vector corresponding to an operator is

$$v = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} + \xi_3 \frac{\partial}{\partial y} + \xi_4 \frac{\partial}{\partial u}.$$

The operator v yields the conservation laws $D_t(C^1) + D_x(C^2) + D_y(C^3) = 0$, where the conserved vector $C = (C^1, C^2, C^3)$ is given by eq. (14) and has the components

$$C^1 = \xi_1 L + W\left(\frac{\partial L}{\partial u_t} - D_x\left(\frac{\partial L}{\partial u_{xt}}\right)\right) + W_x\left(\frac{\partial L}{\partial u_{xt}}\right), \tag{15}$$

$$\begin{aligned} C^2 =& \ \xi_2 L + W\left(\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right) - D_t\left(\frac{\partial L}{\partial u_{xt}}\right) - D_y\left(\frac{\partial L}{\partial u_{xy}}\right)\right. \\ & \left. + D_{xx}\left(\frac{\partial L}{\partial u_{xxx}}\right) - D_{xxt}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right) \\ & + W_x\left(\frac{\partial L}{\partial u_{xx}} - D_x\left(\frac{\partial L}{\partial u_{xxx}}\right) + D_{xt}\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right) + W_t\left(\frac{\partial L}{\partial u_{xt}}\right) \\ & + W_{xx}\left(\frac{\partial L}{\partial u_{xxx}} - D_t\left(\frac{\partial L}{\partial u_{xxxt}}\right)\right) + W_{xxt}\left(\frac{\partial L}{\partial u_{xxxt}}\right) \end{aligned} \tag{16}$$

$$C^3 = \xi_3 L + W\left(\frac{\partial L}{\partial u_y} - D_y\left(\frac{\partial L}{\partial u_{yy}}\right)\right) + W_y \frac{\partial L}{\partial u_{yy}}. \tag{17}$$

So, eqs (15)–(17) define the corresponding components of non-local conservation law for the system of eq (1) and (12) corresponding to any operator v admitted by eq. (1).

Let us make calculations for the operator

$$V = 2t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} - \frac{2(a+cu)}{c}\frac{\partial}{\partial u}. \tag{18}$$

For this operator, one can get

$$W = -\frac{2(a+cu)}{c} - yu_y - 2tu_t. \tag{19}$$

We can get the conservation vector of eq. (1)

$$\begin{aligned} C^1 =& -2tv(u_{tx} + au_{xx} + cu_x^2 + cuu_{xx} + bu_{xxxt} + u_{yy}) \\ & + \left(\frac{2(a+cu)}{c} + yu_y + 2tu_t\right) \\ & \times (v_x + bv_{xxt}) - (2u_x + yu_{xy} + 2tu_{xt})(v + bv_{xx}) \\ & + (2u_{xx} + yu_{xxy} + 2tu_{xxt}) \times bv_x \\ & - (2u_{xxx} + yu_{xxxy} + 2tu_{xxxt}) \times bv, \end{aligned}$$

$$
\begin{aligned}
C^2 &= -\left(\frac{2(a+cu)}{c} + yu_y + 2tu_t\right) \times (cu_x v - cuv_x - v_x + bv_{xxt}) \\
&\quad + \left(2u_x + yu_{xy} + 2tu_{xt}\right) \\
&\quad \times (2cv + bv_{xt}) + \left(2u_{xx} + yu_{xxy} + 2tu_{xxt}\right) \times bv_t \\
&\quad + \left(2u_{xxt} + yu_{xxyt} + 2tu_{xxtt}\right) \times bv, \\
C^3 &= yv(u_{tx} + au_{xx} + cu_x^2 + cuu_{xx} + bu_{xxxt} + u_{yy}) \\
&\quad - \left(\frac{2(a+cu)}{c} + yu_y + 2tu_t\right) v_y \\
&\quad - \left(2u_y + yu_{yy} + 2tu_{ty}\right) v.
\end{aligned}
$$

This vector involves an arbitrary solution v of the adjoint equation (eq. (12)).

Remark 2. With the aid of Maple16, we have checked that the above vector $(C^1,\ C^2,\ C^3)$ is the conservation vector of eq. (1).

4. Reductions and exact solutions of g-CH–KP equation

Next, the similarity reductions [23] and similarity solutions of eq. (1) will be discussed by using the compatibility of $\sigma = 0$. To get the solutions of eq. (1), the characteristic equation has been written from eq. (7) as follows:

$$
\frac{\mathrm{d}t}{2c_1t + c_4} = \frac{\mathrm{d}x}{c_3} = \frac{\mathrm{d}y}{c_1y + c_2} = \frac{\mathrm{d}u}{-\frac{2c_1(a+cu)}{c}}. \tag{20}
$$

Six cases will be discussed.

Case 1. $c_1 = 0, c_3 = 0, c_2 \neq 0, c_4 \neq 0$

Solving $\sigma = 0$, the similarity reductions can be obtained as

$$
u = f(\xi, \eta),
$$

where $\xi = x, \eta = c_4y - c_2t$, and $f(\xi, \eta)$ needs to satisfy

$$
-bc_2 f_{\xi\xi\xi\eta} - c_2 f_{\xi\eta} + af_{\xi\xi} + cf_\xi^2 + cf_{\xi\xi} f + c_4^2 f_{\eta\eta} = 0. \tag{21}
$$

Case 2. $c_1 = 0, c_3 = 0, c_2 = 0, c_4 \neq 0$

From $\sigma = 0$, the similarity reductions can be derived as

$$
u = f(\xi, \eta), \tag{22}
$$

where $\xi = x, \eta = c_4y - t$, and $f(\xi, \eta)$ needs to satisfy

$$
c_4^2 f_{\eta\eta} - f_{\xi\eta} + af_{\xi\xi} + cf_\xi^2 + cf_{\xi\xi} f - bf_{\xi\xi\xi\eta} = 0. \tag{23}
$$

Case 3. $c_1 = 0, c_4 = 0, c_2 \neq 0, c_3 \neq 0$

From $\sigma = 0$, the similarity reductions can be obtained as

$$u = f(\xi, \eta),$$

where $\xi = t$, $\eta = c_2 x - c_3 y$, and $f(\xi, \eta)$ needs to satisfy

$$bc_2^3 f_{\eta\eta\eta\xi} + c_2 f_{\xi\eta} + (ac_2^2 + c_3^2) f_{\eta\eta} + cc_2^2 f_\eta^2 + cc_2^2 f_{\eta\eta} f = 0. \tag{24}$$

Case 4. $c_1 = 0, c_2 = 0, c_4 = 0, c_3 \neq 0$

From $\sigma = 0$, the similarity reductions can be obtained as

$$u = f(\xi, \eta),$$

where $\xi = t$, $\eta = c_3 y - x$, and $f(\xi, \eta)$ needs to satisfy

$$(c_3^2 + a) f_{\eta\eta} - f_{\xi\eta} + cf_\eta^2 + cf_{\eta\eta} f - bf_{\xi\eta\eta\eta} = 0. \tag{25}$$

Case 5. $c_1 = 0, c_3 = 0, c_4 = 0, c_2 \neq 0$

From $\sigma = 0$, the similarity reductions can be obtained as

$$u = f(\xi, \eta),$$

where $\xi = t$, $\eta = c_2 x - y$, and $f(\xi, \eta)$ needs to satisfy

$$(ac_2^2 + 1) f_{\eta\eta} + c_2 f_{\xi\eta} + cc_2^2 f_\eta + cc_2^2 f_{\eta\eta} f - bc_2^2 f_{\xi\eta\eta\eta} = 0. \tag{26}$$

Case 6. $c_1 = 0, c_2 \neq 0, c_3 \neq 0, c_4 \neq 0$

By solving the characteristic equation, we get the following expression of u:

$$u = f(\xi, \eta),$$

where $\xi = c_3 t - c_4 x$, $\eta = c_2 t - c_4 y$, and $f(\xi, \eta)$ needs to satisfy

$$\begin{aligned} &-bc_4^3 c_3 f_{\xi\xi\xi\xi} - bc_4^3 c_2 f_{\xi\xi\xi\eta} + (ac_4^2 - c_3 c_4) f_{\xi\xi} - c_2 c_4 f_{\xi\eta} \\ &\quad + cc_4^2 f_\xi^2 + cc_4^2 f_{\xi\xi} f + c_4^2 f_{\eta\eta} = 0. \end{aligned} \tag{27}$$

Here, we obtain the new exact solutions of Cases 1, 2 and 3 of eq. (1).

Case 1. We shall obtain the travelling wave solutions by using the (G'/G)-expansion method. Suppose that the solution of eq. (21) has the following form:

$$f = \sum_{i=-n}^{n} p_i \left(\frac{G'(\omega)}{G(\omega)}\right)^i + p_0, \quad p_n \neq 0. \tag{28}$$

Substituting $f = f(\omega)$ with the transformation $\omega = k\xi + l\eta$ into eq. (21), it follows that

$$-bc_2 k^3 l f'''' + (ak^2 - c_2 kl + c_4^2 l^2) f'' + ck^2 f'^2 + ck^2 ff'' = 0. \tag{29}$$

Balancing f'''' with ff'' in eq. (29) gives $M + 4 = M + M + 2$. So $M = 2$. Set the solution of eq. (21) in the following form:

$$\begin{aligned} f \;=\; & p_2 \left(\frac{G'(\omega)}{G(\omega)}\right)^2 + p_1 \left(\frac{G'(\omega)}{G(\omega)}\right) + p_0 + p_{-1} \left(\frac{G'(\omega)}{G(\omega)}\right)^{-1} \\ & + p_{-2} \left(\frac{G'(\omega)}{G(\omega)}\right)^{-2}, \; p_2 \neq 0, \end{aligned} \tag{30}$$

where $G(\omega)$ satisfies the second-order linear ODE

$$G''(\omega) + \lambda G'(\omega) + \mu G(\omega) = 0. \tag{31}$$

By substituting eqs (30) and (31) in eq. (29) and collecting all terms with the same power of (G'/G) together, the left-hand side of eq. (29) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for p_2, p_1, p_0, p_{-1}, p_{-2}, k, l. We get the following solutions:

$$\begin{aligned} p_{-2} &= 0, \quad p_{-1} = 0, \\ p_0 &= \frac{8bc_2k^3l\mu + bc_2k^3l\lambda^2 - c_4^2l^2 + c_2kl - ak^2}{ck^2}, \\ p_1 &= \frac{12bc_2k\lambda l}{c}, \quad p_2 = \frac{12bc_2kl}{c}. \end{aligned} \tag{32}$$

So we have three types of travelling wave solutions of eq. (1) as follows:

When $\lambda^2 - 4\mu > 0$,

$$\begin{aligned} u = {} & \frac{12bc_2kl}{c}\left(-\frac{\lambda}{2} + \frac{d_1 \sinh\sqrt{\lambda^2-4\mu}\omega + d_2\cosh\sqrt{\lambda^2-4\mu}\omega}{d_1\cosh\sqrt{\lambda^2-4\mu}\omega + d_2\sinh\sqrt{\lambda^2-4\mu}\omega}\right)^2 \\ & + \frac{12bc_2k\lambda l}{c}\left(-\frac{\lambda}{2} + \frac{d_1 \sinh\sqrt{\lambda^2-4\mu}\omega + d_2\cosh\sqrt{\lambda^2-4\mu}\omega}{d_1\cosh\sqrt{\lambda^2-4\mu}\omega + d_2\sinh\sqrt{\lambda^2-4\mu}\omega}\right) \\ & + \frac{8bc_2k^3l\mu + bc_2k^3l\lambda^2 - c_4^2l^2 + c_2kl - ak^2}{ck^2}. \end{aligned} \tag{33}$$

When $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} u = {} & \frac{12bc_2kl}{c}\left(-\frac{\lambda}{2} + \frac{-d_1 \sin\sqrt{4\mu-\lambda^2}\omega + d_2\cos\sqrt{4\mu-\lambda^2}\omega}{d_1\cos\sqrt{\lambda^2-4\mu}\omega + d_2\sin\sqrt{4\mu-\lambda^2}\omega}\right)^2 \\ & + \frac{12bc_2k\lambda l}{c}\left(-\frac{\lambda}{2} + \frac{d_1 \sin\sqrt{4\mu-\lambda^2}\omega + d_2\cos\sqrt{4\mu-\lambda^2}\omega}{d_1\cos\sqrt{4\mu-\lambda^2}\omega + d_2\sin\sqrt{4\mu-\lambda^2}\omega}\right) \\ & + \frac{8bc_2k^3l\mu + bc_2k^3l\lambda^2 - c_4^2l^2 + c_2kl - ak^2}{ck^2}. \end{aligned} \tag{34}$$

When $\lambda^2 - 4\mu = 0$,

$$\begin{aligned} u = {} & \frac{12bc_2kl}{c}\left(-\frac{\lambda}{2} + \frac{d_2}{d_1 + d_2\omega}\right)^2 + \frac{12bc_2k\lambda l}{c}\left(-\frac{\lambda}{2} + \frac{d_2}{d_1 + d_2\omega}\right) \\ & + \frac{8bc_2k^3l\mu + bc_2k^3l\lambda^2 - c_4^2l^2 + c_2kl - ak^2}{ck^2}, \end{aligned} \tag{35}$$

where d_1 and d_2 are arbitrary constants.

Case 2. We shall obtain the travelling wave solutions by using the extended tanh method. Suppose that the solution of eq. (23) has the following form:

$$u(\omega) = \sum_{k=0}^{n} a_k Y^k. \tag{36}$$

Substituting $f = f(\omega)$ with the transformation $\omega = k\xi + l\eta$ into eq. (23), it follows that

$$-bk^3 l f'''' + (ak^2 - kl + c_4^2 l^2) f'' + ck^2 f'^2 + ck^2 f f'' = 0. \tag{37}$$

Balancing f'''' with ff'' in eq. (37) gives $M + 4 = M + M + 2$. So $M = 2$. Set the solution of eq. (23) in the following form:

$$u(\mu\omega) = a_0 + a_1 Y + a_2 Y^2, \tag{38}$$

$$\begin{aligned} &Y = \tanh(\mu\omega),\\ &\frac{\mathrm{d}}{\mathrm{d}\omega} = \mu(1 - Y^2)\frac{\mathrm{d}}{\mathrm{d}Y},\\ &\frac{\mathrm{d}^2}{\mathrm{d}\omega^2} = -2\mu^2 Y(1 - Y^2)\frac{\mathrm{d}}{\mathrm{d}Y} + \mu^2(1 - Y^2)^2\frac{\mathrm{d}^2}{\mathrm{d}Y^2} \quad . \end{aligned} \tag{39}$$

Substitute eq. (38) into eq. (37) and use eq. (39) while collecting the coefficients of Y.

Solving this system gives the following sets of solutions:

$$a_0 = -\frac{8bk^3 l\mu^2 - kl + c_4^2 l^2 + ak^2}{ck^2}, \quad a_1 = 0, \quad a_2 = \frac{12bkl\mu^2}{c}, \tag{40}$$

where c_2, a, b, k and l are non-zero arbitrary constants.

Thus, some solutions can be derived as follows:

$$u = -\frac{8bk^3 l\mu^2 - kl + c_4^2 l^2 + ak^2}{ck^2} + \frac{12bkl\mu^2}{c}\tanh^2[\mu(k\xi + l\eta)]. \tag{41}$$

Case 3. Substituting $f = f(\omega)$ with the transformation $\omega = k\xi + l\eta$ into eq. (24), it follows that

$$bc_2^3 kl^2 f'''' + (ac_2^2 l + c_2 k + c_3^2 l) f'' + cc_2^2 l f'^2 + cc_2^2 l f f'' = 0. \tag{42}$$

Suppose that the solution of eq. (42) has the following form:

$$f = \sum_{i=1}^{n} p_i(\phi)^i + p_0, \quad p_n \neq 0, \tag{43}$$

where p_i are constants to be determined later. By balancing f'''' and ff'' in eq. (42), one can get $M = 2$.

Suppose that the solutions of eq. (42) can be expressed as follows:

$$f = p_2\phi^2 + p_1\phi + p_0, \quad p_2 \neq 0, \tag{44}$$

where $p_i (i = 0, 1, 2)$ are constants to be determined, $\phi(\omega)$ is a solution of the following first-order nonlinear ordinary differential equation [24,25]:

$$\phi' = \pm\sqrt{h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4}. \tag{45}$$

From eq. (45), we can get

$$\phi'^2 = h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4, \tag{46}$$

$$\phi'' = \frac{1}{2}h_1 + h_2\phi + \frac{3}{2}h_3\phi^2 + 2h_4\phi^3, \tag{47}$$

$$\phi''' = \phi'\left(h_2 + 3h_3\phi + 6h_4\phi^2\right) \tag{48}$$

and

$$\phi'^3 = \phi'\left(h_0 + h_1\phi + h_2\phi^2 + h_3\phi^3 + h_4\phi^4\right). \tag{49}$$

Substituting eqs (46–49) and eq. (44) into eq. (42) and collecting all terms with the same power of ϕ together, one can get a set of simultaneous algebraic equations. Solving these equations, one can get the following solutions:

$$\begin{aligned} h_1 &= h_3 = 0, \quad p_0 = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l}, \\ p_1 &= 0, \quad p_2 = -\frac{12bc_2klh_4}{c}, \end{aligned}$$

where c is a non-zero constant.

By substituting different values of $h_i\,(i = 0, ..., 4)$ in eq. (45) we find eq. (43) to have a series of fundamental solutions. It is very tedious to record all possible solutions of eq. (1). For simplicity, we formulate a few important ones. Some solutions derived are as follows:

Case 3a. Rational solutions

When $h_4 = h_0 = 0$, two rational solutions can be derived

$$u_{a.1} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} \pm \frac{12bc_2klh_4}{c}\omega^2, \quad h_2 > 0,$$

$$u_{a.1} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} \mp i\,\frac{12bc_2klh_4}{c}\omega^2, \quad h_2 < 0.$$

Case 3b. Jacobi elliptic function solutions and combined Jacobi elliptic function solutions

When $h_2 = -(1 + m^2)$, there are four Jacobi elliptic function solutions:

$$\begin{aligned} u_{b.1} &= -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} \\ &\quad -\frac{12bc_2klh_4}{c}cd^2(\omega), \quad h_0 = 0, h_4 = m^2, \\ u_{b.2} &= -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} \\ &\quad -\frac{12bc_2klh_4}{c}sn^2(\omega), \quad h_0 = 0, h_4 = m^2, \end{aligned}$$

$$u_{b.3} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}ns^2(\omega), \quad h_0 = m^2, h_4 = 1,$$

$$u_{b.4} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}dc^2(\omega), \quad h_0 = m^2, h_4 = 1.$$

When $h_2 = \left(1 - 2m^2\right)/2$, there are five combined Jacobi elliptic function solutions

$$u_{b.5} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}(ns(\omega) \pm cs(\omega))^2, \quad h_0 = h_4 = \frac{1}{4},$$

$$u_{b.7} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}(nc(\omega) \pm sc(\omega))^2, \quad h_0 = h_4 = \frac{1-m^2}{4},$$

$$u_{b.9} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}(ns(\omega) + ds(\omega))^2, \quad h_0 = \frac{m^2}{4}, h_4 = \frac{1}{4}.$$

Case 3c. Bell profile solution (figure 1)

When $h_0 = 0, h_2 > 0, h_4 < 0$, the bell profile solution is

$$u_{c.1} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}\sqrt{-\frac{h_2}{h_4}}\,\mathrm{sech}^2(\sqrt{h_2}\omega).$$

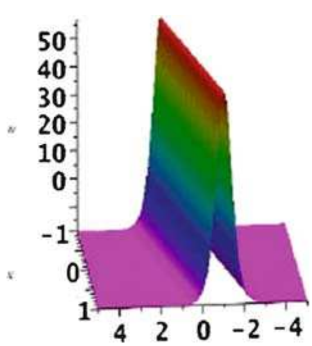

Figure 1. Solution of $u_{c.1}$ is shown at $h_2 = 2, h_4 = -5, c_1 = c_2 = c_3 = a = b = k = 1, t = 0$ (bell profile solution).

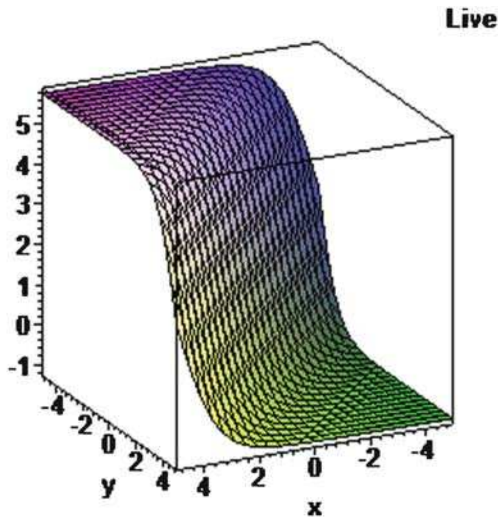

Figure 2. Solution of $u_{d.1}$ is shown at $h_2 = -1, h_4 = 1, c_1 = c_2 = c_3 = a = h_0 = b = k = 1, t = 0$ (kink profile solution).

Case 3d. Kink profile solution (figure 2)

When $h_0 = h_2^2/4h_4, h_2 < 0, h_4 > 0$, the kink profile solution is

$$u_{d.1} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}\sqrt{-\frac{h_2}{h_4}}\tanh^2(\sqrt{h_2}\omega).$$

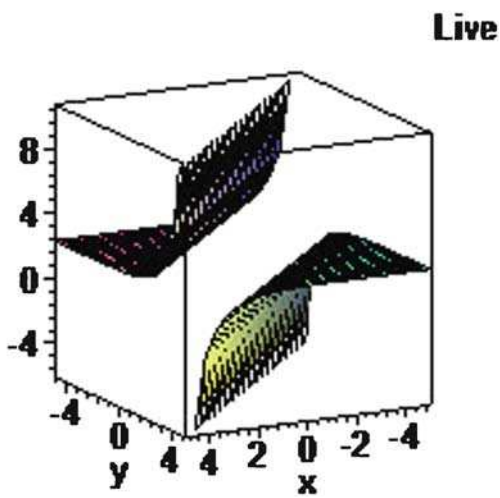

Figure 3. Solution of $u_{e.1}$ is shown at $h_2 = h_4 = c_1 = c_2 = c_3 = a = b = k = 1, t = 0$ (singular profile solution).

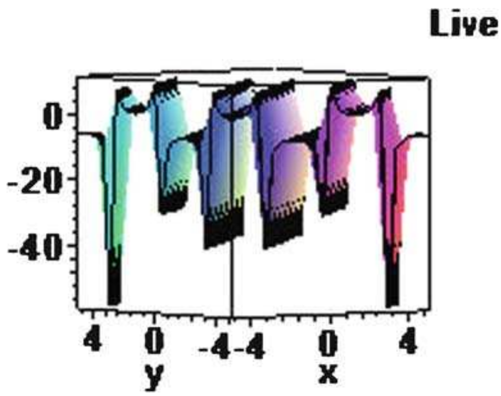

Figure 4. Solution of $u_{f,1}$ is shown at $h_2 = -4, h_4 = 1, c_1 = c_2 = c_3 = a = b = k = 1, t = 0$ (triangular periodic solutions).

Case 3e. Singular solution (figure 3)

When $h_0 = 0, h_2 > 0, h_4 > 0$, the singular profile solution is

$$u_{e.1} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}\sqrt{-\frac{h_2}{h_4}}\csc h^2(\sqrt{h_2}\omega).$$

Case 3f. Triangular periodic solutions (figure 4)

When $h_0 = 0, h_2 < 0, h_4 > 0$, there are two singular profile solutions:

$$u_{f.1} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}\sqrt{-\frac{h_2}{h_4}}\sec^2(\sqrt{-h_2}\omega),$$

$$u_{f.2} = -\frac{c_2k + c_3^2l + ac_2^2l + 4bc_2^3kl^2h_2}{cc_2^2l} - \frac{12bc_2klh_4}{c}\sqrt{-\frac{h_2}{h_4}}\csc^2(\sqrt{-h_2}\omega),$$

where $i^2 = -1, \omega = k\xi + l\eta$ and $m(0 < m < 1)$ denotes the modulus of the Jacobi elliptic function.

Remark 3. All the solutions for eq. (1) obtained in this paper have been checked by Maple software.

Remark 4. The above-mentioned solutions including the hyperbolic functions, the trigonometric functions and the rational functions are more extensive than those in [17], and the elliptic functions have not been recorded in previous papers till date.

5. Conclusions

By applying the direct symmetry method to the modified g-CH–KP equation, we have obtained the classical Lie point symmetry of the equation. We have also derived some exact solutions of the g-CH–KP equation by using the relationship between the new solutions and the old ones. Also, we have obtained some new exact analytic solutions and given the conservation laws of g-CH–KP equation. These conclusions may be useful for explaining some practical physical problems.

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