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Stability analysis of fractional-order generalized chaotic susceptible–infected–recovered epidemic model and its synchronization using active control method

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Abstract. This paper presents the synchronization between a pair of identical susceptible–infected–recovered (SIR) epidemic chaotic systems and fractional-order time derivative using active control method. The fractional derivative is described in Caputo sense. Numerical simulation results show that the method is effective and reliable for synchronizing the fractional-order chaotic systems while it allows the system to remain in chaotic state. The striking features of this paper are: the successful presentation of the stability of the equilibrium state and the revelation that time for synchronization varies with the variation in fractional-order derivatives close to the standard one for different specified values of the parameters of the system.

Keywords. Susceptible–infected–recovered model; fractional time derivative; stability analysis; chaos; synchronization; active control method.

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1. Introduction

Mathematical modelling is widely used to analyse and gain insight to the spread of infectious diseases which can eventually be used to predict the future course of outbreak and to evaluate strategies to control an epidemic. Mathematical epidemiology and population dynamics are some suitable forms of describing the biological systems using the language of dynamical systems theory. In this connection we may refer to Buonomo *et al* [1], who stated the use of mathematical epidemiology in revealing valuable information regarding the spread and control of infectious diseases. In a given model, a person contracting the disease and then becoming immune to future infection after recovery is called susceptible–infected–recovered (SIR). This phenomenon of biological systems is directly connected to time evolution of population density of the interacting species or individuals in different states (susceptible and infected). Due to the complexity of interactions amongst species, it is difficult to solve the models describing such systems analytically. Therefore, once the model is formulated mathematically, it is necessary to solve it numerically using computer simulations to predict the response of biological systems. The study of dynamic implications of information-dependent vaccination for SIR vaccine used in preventable childhood diseases can be found in the work of D'Onofrio *et al* [2]. Simple epidemiological models with information-dependent vaccination functions give rise to sustained oscillations via Holf bifurcation as found in D'Onofrio *et al* [3]. Recently, the local and global stabilities of epidemic equilibrium have been studied well by Kar and Mondal [4].

Fractional-order modelling has been an active field of research currently both from the theoretical and applied perspectives. A wide range of problems in different branches of engineering and biology have already been studied by a number of researchers from different parts of the world to explore the potential of the fractional derivative. The usage of first-order time derivative with a fractional-order time derivative is not only applicable for non-Gaussian but also for non-Markovian systems. Fractional SIR epidemic model equations are obtained from the classical SIR epidemic equations in mathematical modelling by replacing first-order derivatives by fractional derivative of order α $(0 < \alpha \leq 1)$. Several universal phenomena can be modelled to a great degree of accuracy by using the property of these evolution equations. In contrast to integer-order differential operators, which are local operators, a fractional-order differential operator is non-local in the sense that it takes into account the fact that the future state not only depends upon the present state but also on the history of its previous states. For this realistic property, the usage of fractional-order systems is becoming popular to model the behaviour of real systems in various fields of science and engineering. It is to be noted that the present states of any real-life dynamic system are dependent upon the history of its past states. Such circumstances have motivated the authors to study the SIR epidemic model which has a great physical relevance from the perspective of public health policies and its consideration as fractional-order system in allied problems is valid.

Synchronization is a vital phenomenon of chaos that may occur when two or more chaotic systems are coupled. Synchronization between two structurally identical/non-identical systems with different initial conditions have attracted a great deal of interest in various fields due to its important applications in secure communication, system identification, pattern reorganization, vibration technology, economical system, ecological system, biology and biotechnology. After the pioneering investigation of Pecora and Carroll [5], that chaotic systems can be made to synchronize by linking them with common signals, lot of research works for synchronizing chaotic systems using linear and non-linear feedback control, adaptive control, active, time-delay feedback control, tracking control and sliding mode control methods have been completed [6–10], out of which active control method is very efficient and easy to use for the synchronization of a pair of identical or non-identical chaotic systems. Synchronization of fractional-order chaotic system was first studied by Li *et al* [11]. Synchronization between fractional-order chaotic systems is also being widely investigated [12–17]. In 2012, Agrawal *et al* [18] have successfully applied the active control method for synchronization of different pairs of fractional-order chaotic systems. It can be assumed that the occurrences of chaotic attractors for a fractional-order SIR model have not yet been explored by any researcher.

In this paper, the authors have performed synchronization between a pair of identical fractional-order chaotic SIR model using the active control method. Numerical simulations have been carried out for different order fractional derivatives close to the standard one which are depicted through graphs for particular cases. The aim of this study is to investigate the minimum time required for synchronization when the fractional-order time derivative approaches the standard order.

2. System description

Let us consider the following SIR epidemic model:

$$\begin{aligned} D_t^\alpha S &= rS\left(1-\frac{S}{k}\right)-\frac{\beta SI}{1+aS},\\ D_t^\alpha I &= \frac{\beta SI}{1+aS}-\mu I-\gamma I,\\ D_t^\alpha R &= \gamma I-\lambda R, \end{aligned} \tag{1}$$

where $0 < \alpha \leq 1$, S, I and R are the densities of susceptible, infected and recovered within the population, respectively, and the parameters viz., r is the intrinsic growth rate of susceptible, k is the carrying capacity of susceptible, a is the saturation factor that measures the inhibitory effect, β is the transmission or contact rate, γ is the rate of recovery from infection, and μ and λ are the death rates. We consider a new variable Z,

$$Z(t) = \int_{-\infty}^{t} g(S(\tau), I(\tau))K(t-\tau)\mathrm{d}\tau, \tag{2}$$

known as information variable [1–4], which depends on current values of state variables and also summarizes information about past values of state variables. Here, $K(t-\tau)$ is the delaying kernel and τ is the distributed delay with $\tau \leq t$. Assuming that $g(S, I) = S$ and $K(t-\tau) = \frac{1}{T}\exp(-\frac{1}{T}(t-\tau))$, where T is the average delay of the collected information on the disease, as well as the average length of the historical memory concerning the disease, the model (1) reduces to

$$\begin{aligned} D_t^\alpha S &= rS\left(1-\frac{S}{k}\right)-\frac{\beta SI}{1+aS},\\ D_t^\alpha I &= \frac{\beta IZ}{1+aZ}-\mu I-\gamma I,\\ D_t^\alpha Z &= \frac{1}{T}(S-Z),\\ D_t^\alpha R &= \gamma I-\lambda R. \end{aligned} \tag{3}$$

The last equation in eq (3) can be ignored as here the dynamics of R depends only on the dynamics of I. Therefore, we shall study the following fractional-order non-linear system:

$$\begin{aligned}
D_t^{\alpha} S &= rS\left(1-\frac{S}{k}\right)-\frac{\beta SI}{1+aS},\\
D_t^{\alpha} I &= \frac{\beta IZ}{1+aZ}-\mu I-\gamma I,\\
D_t^{\alpha} Z &= \frac{1}{T}(S-Z),
\end{aligned} \tag{4}$$

where $r, k, a, \beta, \mu, \gamma, T > 0$.

2.1 *Equilibrium points and their asymptotic stabilities*

To evaluate the equilibrium points, let

$$D_t^{\alpha} S = 0, \quad D_t^{\alpha} I = 0, \quad D_t^{\alpha} Z = 0. \tag{5}$$

The system (4) has trivial equilibrium at $E_0 = (0, 0, 0)$, disease-free equilibrium at $E_1 = (k, 0, k)$ and endemic equilibrium at the point $E_2 = (\bar{S}, \bar{I}, \bar{Z})$, where

$$\bar{S}=\frac{\mu+\gamma}{\beta-a(\mu+\gamma)}, \qquad \bar{I}=\frac{r}{\beta k}(1+a\bar{S})(k-\bar{S}), \qquad \bar{Z}=\frac{\mu+\gamma}{\beta-a(\mu+\gamma)}. \tag{6}$$

Jacobian matrix of system (4) is given by

$$J=\begin{bmatrix} r-(2rS/k)-(\beta I/(1+aS)^2) & -\beta S/(1+aS) & 0\\ 0 & \beta Z/(1+aZ)-\mu-\gamma & \beta I/(1+aZ)^2\\ 1/T & 0 & -1/T \end{bmatrix}. \tag{7}$$

The eigenvalues of the equilibrium points can be determined by solving the characteristic equation

$$|J-\lambda I| = 0 \tag{8}$$

i.e.,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0. \tag{9}$$

The equilibrium point is locally, asymptotically stable if all the eigenvalues λ of Jacobian matrix $J(E)$ satisfy the following condition [19,20]:

$$|\arg(\lambda)| > \frac{\alpha\pi}{2}. \tag{10}$$

At trivial equilibrium point $E_0 = (0, 0, 0)$, the Jacobian matrix is given by

$$J(E_0) = \begin{bmatrix} r & 0 & 0\\ 0 & -(\mu+\gamma) & 0\\ 1/T & 0 & -1/T \end{bmatrix}.$$

Characteristic equation is given by

$$\lambda^3+\left(\gamma+\mu+\frac{1}{T}-r\right)\lambda^2+\left(\frac{1}{T}(\gamma+\mu)-r\left(\gamma+\mu+\frac{1}{T}\right)\right)\lambda-\frac{r}{T}(\gamma+\mu)=0.$$

At disease-free equilibrium point $E_1=(k,0,k)$,

$$J=\begin{bmatrix} -r & -\beta k/(1+ak) & 0 \\ 0 & -(\mu+\gamma-(\beta k/1+ak)) & 0 \\ 1/T & 0 & -1/T \end{bmatrix}.$$

Characteristic equation is given by

$$\lambda^3+\left(r+\frac{1}{T}+M\right)\lambda^2+\left(\frac{r}{T}+M\left(r+\frac{1}{T}\right)\right)\lambda+\frac{Mr}{T}=0,$$

where

$$M=\left(\mu+\gamma-\frac{\beta k}{1+ak}\right).$$

At endemic equilibrium point $E_2=(\bar{S},\bar{I},\bar{Z})$,

$$J(E_2)=\begin{bmatrix} r-(2r\bar{S}/k)-(\beta\bar{I}/(1+a\bar{S})^2) & -\beta\bar{S}/(1+a\bar{S}) & 0 \\ 0 & \beta\bar{Z}/(1+a\bar{Z})-\mu-\gamma & \beta\bar{I}/(1+a\bar{Z})^2 \\ 1/T & 0 & -1/T \end{bmatrix},$$

where

$$\bar{S}=\frac{\mu+\gamma}{\beta-a(\mu+\gamma)},\qquad \bar{I}=\frac{r}{\beta k}(1+a\bar{S})(k-\bar{S}),\qquad \bar{Z}=\frac{\mu+\gamma}{\beta-a(\mu+\gamma)}.$$

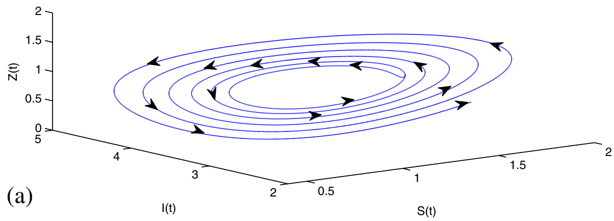
Characteristic equation is given by

$$\lambda^3+\left(M+N+\frac{1}{T}\right)\lambda^2+\left(MN+\frac{M}{T}+\frac{N}{T}\right)\lambda + \frac{MN}{T}+\frac{\beta^2\bar{S}\bar{I}}{T(1+a\bar{S})(1+a\bar{Z})^2}=0,$$

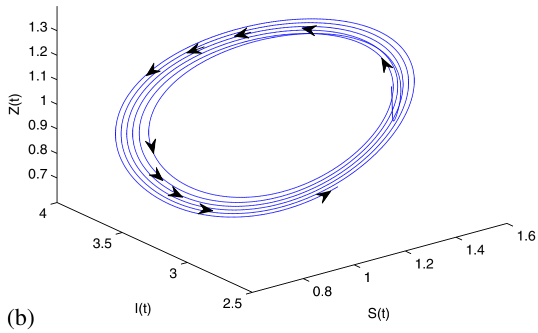
where

$$M=\left(\mu+\gamma-\frac{\beta\bar{S}}{1+a\bar{S}}\right),\qquad N=\left(\frac{\beta\bar{I}}{(1+a\bar{S})^2}+\frac{2r\bar{S}}{k}-r\right).$$

It is known that an equilibrium point E is said to be a saddle point of index 1 if the Jacobian matrix at E has one eigenvalue with a non-negative real part and a saddle point of index 2 if the Jacobian matrix at E has two unstable eigenvalues. It is noticed that the scrolls are generated only around the saddle points of index 2, whereas saddle points of index 1 are responsible only for connecting scrolls [21–25]. With this idea we proceed to find the eigenvalues for predicting a stable or unstable point taking the parameters as $r=2, k=5, a=0.01, \beta=0.5, \mu=0.3, \gamma=0.2$ and $T=0.85$. We see that at the point $E_0=(0,0,0)$, the eigenvalues are $2, -0.5000, -1.7647$, which clearly show that the point E_0 is unstable. At point $E_1=(k,0,k)=(5,0,5)$, the



(a)



(b)

Figure 1. The chaotic attractors of the generalized SIR model (4) for (**a**) fractional-order $\alpha = 0.99$ and (**b**) $\alpha = 0.975$.

eigenvalues are $1.8810, -1.7647, -2$, which imply that the point E_1 is unstable. At the point $E_2 = (\bar{S}, \bar{I}, \bar{Z})$, i.e., (1.0101, 3.22416, 1.0101) the eigenvalues are -1.63339, $0.03442 + 0.75354i$, $0.03442 - 0.75354i$, which clearly exhibit that E_2 is a saddle point of index 2 for $\alpha > 0.9704$ satisfying the condition (10). The chaotic attractors of system (4) for different values of α are depicted in figure 1.

2.2 *Synchronization of the fractional-order SIR model using active control method*

Here the drive system is described by eq. (4) as

$$\begin{aligned} D_t^{\alpha} S_1 &= r S_1 \left(1 - \frac{S_1}{k}\right) - \frac{\beta S_1 I_1}{1 + a S_1}, \\ D_t^{\alpha} I_1 &= \frac{\beta I_1 Z_1}{1 + a Z_1} - \mu I_1 - \gamma I_1, \\ D_t^{\alpha} Z_1 &= \frac{1}{T}(S_1 - Z_1), \end{aligned} \tag{11}$$

and the response system as

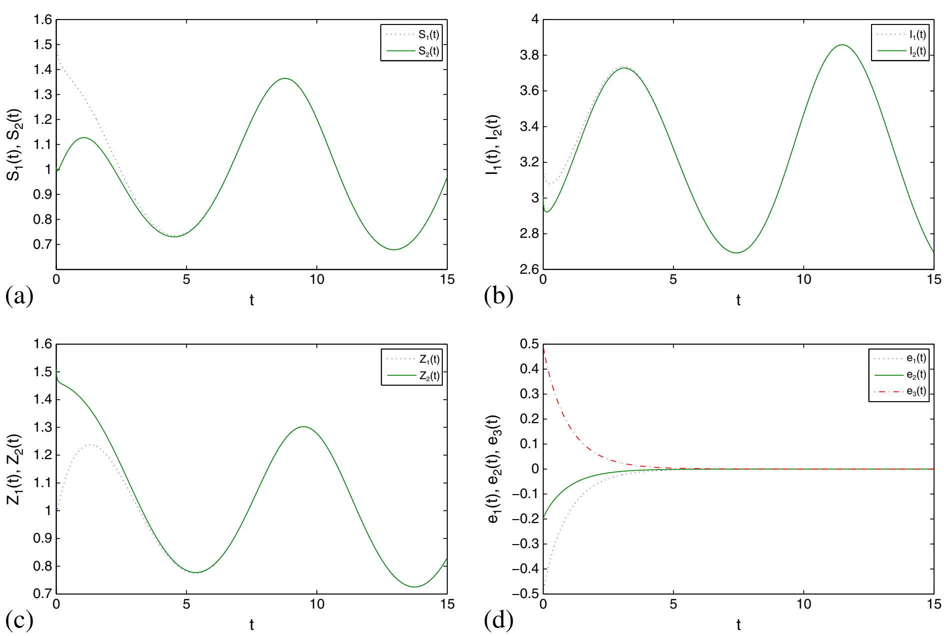
$$\begin{aligned} D_t^{\alpha} S_2 &= r S_2 \left(1 - \frac{S_2}{k}\right) - \frac{\beta S_2 I_2}{1 + a S_2} + \mu_1(t), \\ D_t^{\alpha} I_2 &= \frac{\beta I_2 Z_2}{1 + a Z_2} - \mu I_2 - \gamma I_2 + \mu_2(t), \\ D_t^{\alpha} Z_2 &= \frac{1}{T}(S_2 - Z_2) + \mu_3(t), \end{aligned} \tag{12}$$

where $\mu(t) = [\mu_1(t)\mu_2(t)\mu_3(t)]^T$ is the controller to be designed. To investigate the synchronization of systems (11) and (12), we define the error states as $e_1 = S_2 - S_1$, $e_2 = I_2 - I_1$, $e_3 = Z_2 - Z_1$. The corresponding error dynamical system can be obtained by subtracting eq. (11) from eq. (12), which is given by

$$\begin{aligned} D_t^\alpha e_1 &= re_1 - \frac{r}{k}(S_2^2 - S_1^2) - \frac{\beta S_2 I_2}{1+aS_2} + \frac{\beta S_1 I_1}{1+aS_1} + \mu_1(t), \\ D_t^\alpha e_2 &= -\mu e_2 - \gamma e_2 + \frac{\beta I_2 Z_2}{1+aZ_2} - \frac{\beta I_1 Z_1}{1+aZ_1} + \mu_2(t), \\ D_t^\alpha e_3 &= \frac{1}{T}(e_1 - e_3) + \mu_3(t). \end{aligned} \tag{13}$$

The two systems (11) and (12) are realized to synchronize if system (13) is globally, asymptotically stable under a suitable controller. We define active control functions $\mu(t)$ as

$$\begin{aligned} \mu_1(t) &= V_1(t) + \frac{r}{k}(S_2^2 - S_1^2) + \frac{\beta S_2 I_2}{1+aS_2} - \frac{\beta S_1 I_1}{1+aS_1}, \\ \mu_2(t) &= V_2(t) - \frac{\beta I_2 Z_2}{1+aZ_2} + \frac{\beta I_1 Z_1}{1+aZ_1}, \\ \mu_3(t) &= V_3(t) \end{aligned} \tag{14}$$



(a) (b) (c) (d)

Figure 2. The state trajectories of systems (11) and (12) at $\alpha = 0.99$ (**a**) between S_1 and S_2, (**b**) between I_1 and I_2, (**c**) between Z_1 and Z_2 and (**d**) the evolution of the error functions $e_1(t), e_2(t), e_3(t)$.

leading to the error function as

$$D_t^{\alpha} e_1 = re_1 + V_1(t),$$
$$D_t^{\alpha} e_2 = -\mu e_2 - \gamma e_2 + V_2(t),$$
$$D_t^{\alpha} e_3 = \frac{1}{T}(e_1 - e_3) + V_3(t), \tag{15}$$

where $V_1(t), V_2(t), V_3(t)$ are the linear control inputs chosen such that system (15) becomes stable. Now consider

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = M \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where M is a 3×3 constant matrix. In order to make the closed loop system stable, matrix M should be selected in such a way that the feedback system has eigenvalues λ_i which satisfy the condition $|\arg(\lambda_i)| > (\alpha\pi/2)$, $i = 1, 2, 3$. There is no unique choice for matrix M, but a good choice can be

$$M = \begin{bmatrix} -(r+1) & 0 & 0 \\ 0 & (\mu+\gamma-1) & 0 \\ -1/T & 0 & (1/T)-1 \end{bmatrix}$$

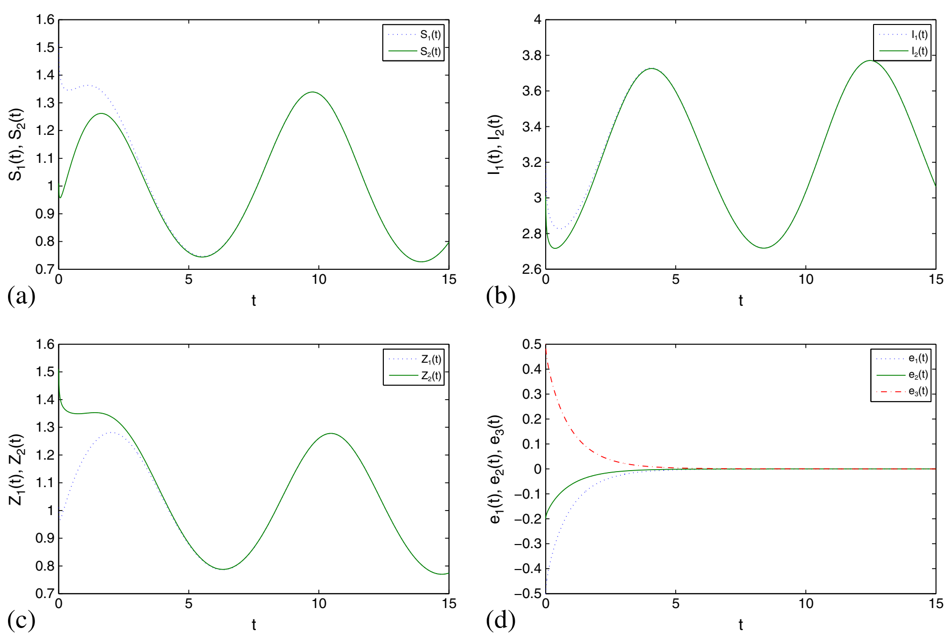


Figure 3. The state trajectories of systems (11) and (12) at $\alpha = 0.975$ (**a**) between S_1 and S_2, (**b**) between I_1 and I_2, (**c**) between Z_1 and Z_2 and (**d**) the evolution of the error functions $e_1(t), e_2(t), e_3(t)$.

so that the error system (15) reduces to

$$D_t^{\alpha} e_1 = -e_1, \qquad D_t^{\alpha} e_2 = -e_2, \qquad D_t^{\alpha} e_3 = -e_3. \tag{16}$$

Here all eigenvalues λ_i are -1, which satisfy the condition $|\arg(\lambda_i)| > (\alpha\pi/2)$, for $0 < \alpha \leq 1$. Therefore, the linear system (16) is stable and the required synchronization is obtained.

3. Numerical simulations and results

In numerical simulations, the parameters of the SIR system are taken as before, i.e., $r = 2, k = 5, a = 0.01, \beta = 0.5, \mu = 0.3, \gamma = 0.2$ and $T = 0.85$. Time step size is taken as 0.005. State trajectories of drive system (11) and response system (12) are shown in figure 2 for the order of the fractional derivative $\alpha = 0.99$ and in figure 3 for $\alpha = 0.975$.

Figures 2 and 3 demonstrate that the system is synchronized after a small duration of time for the considered fractional-order time derivatives $\alpha = 0.99$ and 0.975. It is also seen from the figures that the time taken for synchronization of the system decreases with the increase in fractional orders approaching the standard order system.

4. Conclusion

In brief, the authors have achieved four important objective in this paper: (1) studied stability analysis of fractional-order SIR model, (2) studied the dynamical behaviour of two identical fractional-order chaotic systems, (3) successfully implemented the powerful active control method which provides a simple way to synchronize coupled chaotic systems and (4) observed that the synchronization time increases when the system pair approaches standard order from fractional order, which is a major outcome of this study. Numerical simulation results demonstrate the ease of implementation, the reliability and the effectiveness of the proposed control technique even for the synchronization of fractional-order chaotic systems. The authors affirm that the present study will be appreciated and can be utilized by those researchers who are involved in the field of mathematical modelling of fractional-order dynamical systems.

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