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New exact wave solutions for Hirota equation

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Abstract. In this paper, we construct the topological or dark solitons of Hirota equation by using the first integral method. This approach provides first integrals in polynomial form with a high accuracy for two-dimensional plane autonomous systems. Exact soliton solution is constructed through the established first integrals. This method is a powerful tool for searching exact travelling solutions of nonlinear partial differential equations (NPDEs) in mathematical physics.

Keywords. Hirota equation; first integral method.

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1. Introduction

Nonlinear partial differential equations (NPDEs) of mathematical physics are major subjects in physical science. With the development of soliton theory, many useful methods for obtaining exact solutions of NPDEs have been presented. Some of them are: the (G'/G)-expansion method [1–4], the simplest equation method [5–7], the solitary wave ansatz method [8–10], the first integral method [11–14], the functional variable method [15–17] etc.

One of the most effective direct method to develop travelling wave solution of NPDEs is the first integral method. This method has been successfully applied to obtain exact solutions for a variety of NPDEs [11–14]. Contrary to other traditional methods, the first integral method has many advantages, it mainly avoids a great deal of complicated and tedious calculations and provides more exact and explicit travelling solitary solutions with high accuracy. In the present work, we would like to extend the first integral method to solve the Hirota equation [18]

$$iu_t + u_{xx} + 2|u|^2u + i\alpha u_{xxx} + 6i\alpha|u|^2u_x = 0. \tag{1}$$

When $\alpha = 0$, the equation becomes Schrödinger equation. It is well known that the nonlinear Schrödinger equation is widely used in basic models of nonlinear waves in many areas of physics. This arises from the study of nonlinear wave propagation in dispersive and inhomogeneous media, such as plasma phenomena and nonuniform dielectric media. This is a generic equation describing the evolution of the slowly varying amplitude of a nonlinear wave train in weakly nonlinear, strongly dispersive and hyperbolic systems.

2. Hirota equation

We look for the travelling wave solutions of eq. (1) in the form of

$$u(x,t) = \mathrm{e}^{i(px+qt)} f(\xi), \quad \xi = x + \omega t, \tag{2}$$

where p, q, ω are constants and $f(\xi)$ is a real function.

By substituting eq. (2) into eq. (1) and separating the real and imaginary parts of the result, we obtain the following two ordinary differential equations:

$$(p^3\alpha - p^2 - q) f(\xi) + (1 - 3\alpha p)\, f''(\xi) + 2\,(1 - 3\alpha p)\, f^3(\xi) = 0, \tag{3}$$

$$(\omega + 2p - 3\alpha p^2) f'(\xi) + \alpha f'''(\xi) + 6\alpha f^2(\xi) f'(\xi) = 0. \tag{4}$$

Integrating eq. (4) once, with respect to ξ, yields

$$(\omega + 2p - 3\alpha p^2) f(\xi) + \alpha f''(\xi) + 2\alpha f^3(\xi) = R, \tag{5}$$

where R is an integration constant.

As the same function $f(\xi)$ satisfies both eqs (3) and (5), we obtain the following constraint condition:

$$\frac{p^3\alpha - p^2 - q}{\omega + 2p - 3\alpha p^2} = \frac{1 - 3\alpha p}{\alpha}, \quad R = 0. \tag{6}$$

By using condition (6), we have

$$\omega = \frac{p^3\alpha^2 - p^2\alpha - q\alpha}{1 - 3\alpha p} + 3\alpha p^2 - 2p. \tag{7}$$

We can rewrite second-order ordinary differential eq. (3) as

$$f''(\xi) + \left(\frac{p^3\alpha - p^2 - q}{1 - 3\alpha p}\right) f(\xi) + 2f^3(\xi) = 0. \tag{8}$$

If we let $X(\xi) = f(\xi)$, $Y(\xi) = \mathrm{d}f(\xi)/\mathrm{d}\xi$, eq. (8) is equivalent to the two-dimensional autonomous system

$$\begin{aligned} \dot{X}(\xi) &= Y(\xi), \\ \dot{Y}(\xi) &= \left(\frac{q + p^2 - p^3\alpha}{1 - 3\alpha p}\right) X(\xi) - 2X^3(\xi). \end{aligned} \tag{9}$$

In order to find the travelling wave solutions of eq. (1), we now apply the first integral method, the key idea of which is to utilize the so-called Division Theorem which is based on the ring theory of commutative algebra and to obtain the first integrals to system (9) under various parameter conditions. Then using these first integrals, the two-dimensional autonomous system (9) can be reduced to different first-order integrable differential equations. Finally, by solving these first-order differential equations directly, travelling wave solutions for eq. (1) can be established easily. Next, let us recall the Division Theorem for two variables in the complex domain C [19].

Division Theorem. *Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials in $C[w, z]$; and $P(w, z)$ is irreducible in $C[w, z]$. If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $C[w, z]$ such that*

$$Q(w, z) = P(w, z)G(w, z).$$

The Division Theorem follows immediately from the Hilbert–Nullstellensatz Theorem [20].

Hilbert–Nullstellensatz Theorem. *Let k be a field and L an algebraic closure of K.*
(i) *Every ideal γ* of *$K[X_1, ..., X_n]$ not containing 1 admits at least one zero in L^n.*
(ii) *Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ be two elements of L^n; for the set of polynomials of $K[X_1, ..., X_n]$ zero at x to be identical with the set of polynomials of $K[X_1, ..., X_n]$ zero at y, it is necessary and sufficient that there exists a K-automorphism s of L such that $y_i = s(x_i)$ for $1 \leq i \leq n$.*
(iii) *For an ideal α of $K[X_1, ..., X_n]$ to be maximal, it is necessary and sufficient that there exists x in L^n such that α is the set of polynomials of $K[X_1, ..., X_n]$ zero at x.*
(iv) *For a polynomial Q of $K[X_1, ..., X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $K[X_1, ..., X_n]$, it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.*

Now, we apply the Division Theorem to look for the first integral of system (9). Suppose $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of system (9), and

$$q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0$$

is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$q\left(X(\xi), Y(\xi)\right) = \sum_{i=0}^{m} a_i\left(X(\xi)\right) Y^i(\xi) = 0, \tag{10}$$

where $a_i(X)$ $(i = 0, 1, ..., m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (10) is called the first integral to system (9). According to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$, in the complex domain $C[X, Y]$ such that

$$\frac{\mathrm{d}Q}{\mathrm{d}\xi} = \frac{\mathrm{d}Q}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}\xi} + \frac{\mathrm{d}Q}{\mathrm{d}Y}\frac{\mathrm{d}Y}{\mathrm{d}\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i. \tag{11}$$

In this example, we take two different cases, assuming that $m = 1$ and $m = 2$ in (10).

Case A

Suppose that $m = 1$, by comparing with the coefficients of Y^i $(i = 2, 1, 0)$ on both sides of (11), we have

$$\dot{a_1}(X) = h(X)a_1(X), \tag{12}$$

$$\dot{a_0}(X) = g(X)a_1(X) + h(X)a_0(X), \tag{13}$$

$$a_1(X)\left[\left(\frac{q + p^2 - p^3\alpha}{1 - 3\alpha p}\right) X - 2X^3\right] = g(X)a_0(X). \tag{14}$$

As $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (12) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$. Suppose that $g(X) = A_1X + B_0$, then we find $a_0(X)$.

$$a_0(X) = A_0 + B_0X + \frac{1}{2}A_1X^2, \tag{15}$$

where A_0 is an arbitrary integration constant.

Substituting $a_0(X)$ and $g(X)$ into (14) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and on solving it, we obtain

$$B_0 = 0, \quad A_1 = 2i, \quad q = -p^2 + p^3\alpha + 2iA_0(1 - 3\alpha p), \tag{16}$$

$$B_0 = 0, \quad A_1 = -2i, \quad q = -p^2 + p^3\alpha - 2iA_0(1 - 3\alpha p), \tag{17}$$

where A_0, p and q are arbitrary constants.

Substituting conditions (16) in (10), we obtain

$$Y(\xi) = -A_0 - iX^2(\xi). \tag{18}$$

Combining (18) with (9), we obtain exact solution to eq. (8) and the exact solution to the Hirota equation can be written as

$$u(x, t) = \sqrt{-iA_0}\mathrm{e}^{i\left(px + \left(-p^2 + p^3\alpha + 2iA_0(1-3\alpha p)\right)t\right)} \times \tan\left[\sqrt{iA_0}\left(x + \left(\frac{p^3\alpha^2 - p^2\alpha - q\alpha}{1 - 3\alpha p} + 3\alpha p^2 - 2p\right)t + \xi_0\right)\right], \tag{19}$$

where ξ_0 is an arbitrary constant.

If $k = iA_0$, then

$$u(x, t) = \sqrt{-k}\mathrm{e}^{i\left(px + \left(-p^2 + p^3\alpha + 2k(1-3\alpha p)\right)t\right)} \times \tan\left[\sqrt{k}\left(x + \left(\frac{p^3\alpha^2 - p^2\alpha - q\alpha}{1 - 3\alpha p} + 3\alpha p^2 - 2p\right)t + \xi_0\right)\right]. \tag{20}$$

Similarly, in the case of (17), from (10), we obtain

$$Y(\xi) = -A_0 + iX^2(\xi), \tag{21}$$

and then the exact solution of the Hirota equation can be written as

$$u(x,t) = \sqrt{-k}\mathrm{e}^{i\left(px+(-p^2+p^3\alpha-2k(1-3\alpha p))t\right)} \times \tanh\left[\sqrt{k}\left(x+\left(\frac{p^3\alpha^2-p^2\alpha-q\alpha}{1-3\alpha p}+3\alpha p^2-2p\right)t+\xi_0\right)\right], \tag{22}$$

where ξ_0 is an arbitrary constant and $k = iA_0$.

Case B

If $m = 2$, by equating the coefficients of Y^i $(i = 3, 2, 1, 0)$ on both sides of (11), we have

$$\dot{a_2}(X) = h(X)a_2(X), \tag{23}$$

$$\dot{a_1}(X) = g(X)a_2(X) + h(X)a_1(X), \tag{24}$$

$$\dot{a_0}(X) = -2a_2(X)\left[\left(\frac{q+p^2-p^3\alpha}{1-3\alpha p}\right)X - 2X^3\right] + g(X)a_1(X) + h(X)a_0(X), \tag{25}$$

$$a_1(X)\left[\left(\frac{q+p^2-p^3\alpha}{1-3\alpha p}\right)X - 2X^3\right] = g(X)a_0(X). \tag{26}$$

As $a_i(X)$ $(i = 0, 1, 2)$ are polynomials, then from (23) we deduce that $a_2(X)$ is constant and $h(X) = 0$. For simplicity, take $a_2(X) = 1$. Balancing the degrees of $g(X)$, $a_1(X)$ and $a_2(X)$, we conclude that $\deg(g(X)) = 1$. Suppose that $g(X) = A_1X + B_0$, then we find $a_1(X)$ and $a_0(X)$ as follows:

$$a_1(X) = A_0 + B_0X + \frac{1}{2}A_1X^2, \tag{27}$$

$$a_0(X) = d + B_0A_0X + \frac{1}{2}\left(\frac{2(q+p^2-p^3\alpha)}{1-3\alpha p} + B_0^2 + A_0A_1\right)X^2 + \frac{1}{2}A_1B_0X^3 + \frac{1}{4}\left(4+\frac{1}{2}A_1^2\right)X^4. \tag{28}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in eq. (26) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and on solving it with the aid of *Maple*, we obtain

$$B_0 = 0, \quad A_0 = \pm\frac{i(q+p^2-p^3\alpha)}{3\alpha p-1}, \quad A_1 = \pm 4i,$$
$$d = -\frac{1}{4}\frac{\left(q^2+2qp^2-2qp^3\alpha+p^4-2p^5\alpha+p^6\alpha^2\right)}{1-6\alpha p+9\alpha p^2}, \tag{29}$$

where p and q are arbitrary constants.

Substituting condition (29) in (10), we get

$$Y(\xi) = \pm\frac{i}{2}\left[\frac{q+p^2-p^3\alpha}{3\alpha p-1} + 2X^2(\xi)\right]. \tag{30}$$

Combining (30) with (9), we obtain exact solution to eq. (8) and the exact solution to Hirota equation can be written as

$$u(x,t) = \pm\sqrt{\frac{(p\alpha^3 - q - p^2)}{2\,(3\alpha p - 1)}}\mathrm{e}^{i(px+qt)} \times \tanh\left[\sqrt{\frac{(q + p^2 - p\alpha^3)}{2(3\alpha p - 1)}} \times \left(x + \left(\frac{p^3\alpha^2 - p^2\alpha - q\alpha}{1 - 3\alpha p} + 3\alpha p^2 - 2p\right)t + \xi_0\right)\right], \quad (31)$$

where ξ_0 is an arbitrary constant.

3. Conclusions

In this paper, the first integral method was directly applied to obtain exact wave solutions for Hirota equation. It can be easily seen that the first integral method is easier and quicker than other traditional techniques. The study indicates the validity and great potential of the first integral method in solving complicated solitary wave problems.

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