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Spin–spin entanglement in moving frames: Properties of negativity

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Abstract. In the present article, we use negativity to investigate the entanglement between two massive particles in the spin degrees of freedom, as seen by moving observers. Assuming that the occurrence of spin-momentum states is determined by Gaussian probability distributions, we show that the degree of entanglement monotonically descends to a diminishingly small value at high rapidities. We further report, how the characteristics of this behaviour vary as the widths of distributions change. In particular, the degree of maximally entangled spin–spin states, resulting from equal distribution widths, is shown to exhibit extrema, depending on the width, at certain rapidities. The material presented in this paper then supports the idea that, for relativistic particles, a consistent reduced spin density (from which the negativity is derived) is impossible to construct.

Keywords. Relativistic entanglement; negativity; Gaussian distribution.

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1. Introduction

It is by now well established and experimentally verified that quantum mechanical concepts form the milestone of technological implementation in information processing schemes [1–3]. Specifically, the concept of entangled states [4,5], with no classical counterparts, is crucial in the development of quantum teleportation [6,7], communication [8,9], cryptography [10,11], etc. In all such applications, generation and quantification of entanglement are of fundamental importance. To this end, quantification of entangled pure states (ensembles) is settled; for instance, through the von Neumann entropy [12–16]. The von Neumann entropy has indeed the advantage of being easily computable. For the entanglement of mixed states, several measures, each having both advantages and disadvantages, have been proposed. These proposals include the entanglement of formation (concurrence) [17,18], entanglement of distillation (negativity, at least for 2×2 or 2×3 bipartite systems) [19–21], relative entropy of entanglement [22,23], etc. Amongst these proposals, especially for bipartite systems, the negativity forms a computable measure [19–21] and has been extensively used in connection with teleportation capacities [19]. Thus, the main aim of the present article is to exploit the properties of negativity, as a measure of spin–spin entanglement, under Lorentz transformations. The results of such a study provide deeper understanding of negativity and entanglement, especially in connection with teleportation where information is stored in spin states and (desirably) transported at high speeds.

As conventionally interpreted, the entanglement, or disentanglement, between spin degrees of freedom, for massive particles, depend upon the frame in which it is measured. This is due to the fact that under a Lorentz transformation the momenta are boosted, while the spin states are linearly Wigner-rotated [24]. As the linear application of the Wigner rotations of spin states depend on the momentum states, they rotate differently. Moreover, in looking at entanglement between spins, one has to trace over the momentum states which, in turn, gives rise to spin–spin entanglement [25,26]. This conclusion has been mostly verified by calculating the Wootter's [17] concurrence for single and two particles [27,28]. However, systematic studies of the negativity, as a measure of spin–spin entanglement, has rarely been attempted [29]. Particularly, for a system of two massive spin-$\frac{1}{2}$ particles in which the spin-momentum states occur with specific probability distributions, such a study has not been reported. In what follows, therefore, we present a thorough examination of the negativity for such a system under the assumption that spin-momentum states occur with Gaussian probability distributions. Since the width (second moment) of each probability distribution is directly proportional to the number of momentum states available to that particle, the variation in the ratio of widths is expected to drastically change the spin–spin entanglement. We shall elaborate more on this point in the concluding section. It is worth mentioning that in refs [30,31], the authors use Gaussian probability distributions of equal widths, while in refs [25,32–34] sharp distributions (single momentum states) are employed. We begin our examination of the negativity by assuming that spin–spin entanglement is impartial to the momenta and perform a tracing over momentum states. This operation then leaves the spin–spin part in a mixed state [30] for which the negativity is an appropriate measure. We then proceed by Lorentz transforming the stateket and calculate the degree of entanglement as seen by a moving observer. In doing so, we obtain relatively simple expressions for the eigenvalues of the partially transposed density matrix which allows us to investigate the effect of rapidity as well as the distributions' widths on the entanglement. It is thus shown that for nonequal distribution widths, the negativity is a descending function of rapidity and asymptotically diminishes (similar to concurrence [30,31]). From our calculations, we further demonstrate that, for equal widths, the negativity starts from a maximal value in the fixed frame, exhibits extrema and asymptotically approaches the same maximal value. The number of extrema, as will be seen, indeed depends upon the ratio of widths (contrary to the concurrence [30]). The foregoing observations indicate that the negativity is a frame-dependent measure and supports the conclusions in refs [35–37] where appropriate interpretations of the reduced density matrix are discussed. Moreover, the theorems on the connections between concurrence and negativity (such as the former being an upper bound on the latter [38,39]) may fail to hold under Lorentz transformations.

This work is organized as follows. After the introduction, §2 is devoted to the calculation of the partially transposed density matrix and the corresponding eigenvalues, as seen by a moving observer. We then adopt the results of §2 to the case of Gaussian probability distributions in §3. In this section, we also discuss the effect of rapidity along with the distributions' widths on the spin–spin entanglement. Finally, concluding remarks are given in §4.

2. Negativity in moving frames

Our investigation of negativity, and consequently the entanglement, begins by considering two spin-1/2 particles with spin states $|s\rangle$ and $|\sigma\rangle$. Such a bipartite state, as viewed in a fixed (laboratory) frame, may be expressed as

$$|\Psi\rangle = \sum_{s,\sigma} \int \int \mathrm{d}^3 p \; \mathrm{d}^3 q \; a_{s,\sigma}(\mathbf{p}, \mathbf{q}) |\mathbf{p}, \mathbf{q}, s, \sigma\rangle \; , \tag{1}$$

where $\mathbf{p}$ and $\mathbf{q}$ represent the 3-momenta of the two particles and momentum representation has been assumed. Physically, eq. (1) describes a state in which the spin states occur with momentum-dependent probabilities, $a_{s,\sigma}(\mathbf{p}, \mathbf{q})$, such that $\Sigma_{s,\sigma} \int \int \mathrm{d}^3 p \; \mathrm{d}^3 q |a_{s,\sigma}(\mathbf{p}, \mathbf{q})|^2 = 1$. In this normalization condition, we have deliberately omitted factors of the form $\sqrt{p^2c^2 + m^2c^4}$ which we shall include, as we proceed. In a frame moving relative to the fixed one, with a velocity $\vec{v} = v\hat{n}$ (rapidity ζ), the state of eq. (1) experiences a boost and Wigner rotation according to

$$|\Psi\rangle^{\Lambda} = \sqrt{\frac{p_0^{\Lambda}}{p_0}} \sqrt{\frac{q_0^{\Lambda}}{q_0}} \times \sum_{s,\sigma} \int \int \mathrm{d}^3 p \; \mathrm{d}^3 q \; a_{s,\sigma}(\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda}) W(\mathbf{p}, \mathbf{q}, s, \sigma) |\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda}, s, \sigma\rangle, \tag{2}$$

where the boosted momenta are

$$P_0^{\Lambda} = P_0 \cosh(\zeta) + (\mathbf{P} \cdot \mathbf{n}) \sinh(\zeta) \tag{3}$$

and

$$\mathbf{P}^{\Lambda} = \mathbf{P} + [(\mathbf{P} \cdot \mathbf{n}) \cosh(\zeta) + P_0 \sinh(\zeta) - (\mathbf{P} \cdot \mathbf{n})]\mathbf{n} \tag{4}$$

for $\mathbf{P} = \mathbf{p}, \mathbf{q}$. The Wigner rotation $W(\mathbf{p}, \mathbf{q}, s, \sigma)$ in eq. (2), which acts upon the spin states, is given by

$$W(\mathbf{p}, \mathbf{q}, s, \sigma) = W(\mathbf{p}, s) \otimes W(\mathbf{q}, \sigma), \tag{5}$$

where

$$W(\mathbf{P}, \eta) = \frac{1}{\sqrt{(P_0^{\Lambda} + mc)(P_0 + mc)}} [(P_0 + mc) \cosh(\zeta/2) + [\mathbf{P} \cdot \mathbf{n} - i\eta(\mathbf{P} \times \mathbf{n})] \sinh(\zeta/2)] \tag{6}$$

for $\eta = s, \sigma$. The spin–spin entanglement is then formed by tracing over the momentum states of the density operator, $\rho^{\Lambda} = |\Psi\rangle^{\Lambda\Lambda}\langle\Psi| (= |\Psi\rangle\langle\Psi|$ in the fixed frame) one finds,

$$\rho_{\mathrm{spin}}^{\Lambda} = \left(\frac{p_0^{\Lambda} q_0^{\Lambda}}{p_0 q_0}\right) \sum_{s,s'} \sum_{\sigma,\sigma'} \int \int \mathrm{d}^3 p \; \mathrm{d}^3 q \; a_{s,\sigma}(\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda}) a^*_{s',\sigma'}(\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda}) \times W(\mathbf{p}, \mathbf{q}, s, \sigma) |s, \sigma\rangle \langle s', \sigma'| W^{\dagger}(\mathbf{p}, \mathbf{q}, s', \sigma'). \tag{7}$$

The negativity, as a measure of entanglement, is then obtained from the eigenvalues of the partial transposition of eq. (7), $(\rho^{\Lambda}_{\text{spin}})^{\text{TP}}$, through $N = \sum(\max(0, -2\lambda_i)$, where λ_is are the eigenvalues of $(\rho^{\Lambda}_{\text{spin}})^{\text{TP}}$ [19–21]. As a concrete example, let us assume that in the fixed frame the momentum-spin states in eq. (1) are of the form

$$|\Psi\rangle = \int\int \mathrm{d}^3p\, \mathrm{d}^3q\, [a_{++}(\mathbf{p}, \mathbf{q}, +, +)|\mathbf{p}, \mathbf{q}, +, +\rangle + a_{--}(\mathbf{p}, \mathbf{q}, -, -)|\mathbf{p}, \mathbf{q}, -, -\rangle], \tag{8}$$

whose spin state, for equal amplitudes, is a Bell one with maximal spin–spin entanglement [29,31]. To pursue an analytic expression, we have assumed that both momenta are along the y-axis, normal to the direction of the boost; the x-axis (the normalization of eq. (7) is based on this assumption). For the state of eq. (8), the partially transposed density operator is easily calculated, giving

$$(\rho^{\Lambda}_{\text{spin}})^{\text{TP}} = F_1|+,+\rangle\langle+,+| + F_2|+,-\rangle\langle-,+| + F_2^*|-,+\rangle\langle+,-| + (1-F_1)|-,-\rangle\langle-,-|, \tag{9}$$

where the transformed probability amplitudes are

$$F_1 = A\left(\frac{p_0^{\Lambda} q_0^{\Lambda}}{p_0 q_0}\right)\int\int \mathrm{d}^3p\, \mathrm{d}^3q\, |a_{++}(\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda})|^2 f_1(\mathbf{p}, \mathbf{q}, \zeta) \tag{10}$$

and

$$F_2 = A\left(\frac{p_0^{\Lambda} q_0^{\Lambda}}{p_0 q_0}\right)\int\int \mathrm{d}^3p\, \mathrm{d}^3q\, a_{++}(\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda}) a^*_{--}(\mathbf{p}^{\Lambda}, \mathbf{q}^{\Lambda}) f_2(\mathbf{p}, \mathbf{q}, \zeta), \tag{11}$$

with $A = [(p_0 + mc)(p_0^{\Lambda} + mc)(q_0 + mc)(q_0^{\Lambda} + mc)]^{-1}$. In eqs (10) and (11), the momentum distributions have been modified, under the transformation (boost and rotation), by the factors $f_1(\mathbf{p}, \mathbf{q}, \zeta)$ and $f_2(\mathbf{p}, \mathbf{q}, \zeta)$ defined by

$$f_1(\mathbf{p}, \mathbf{q}, \zeta) = \prod_{\mathbf{P}=\mathbf{p},\mathbf{q}} [(P_0 + mc)^2 \cosh^2(\zeta/2) + \mathbf{P}^2 \sinh^2(\zeta/2)] \tag{12}$$

and

$$f_2(\mathbf{p}, \mathbf{q}, \zeta) = \prod_{\mathbf{P}=\mathbf{p},\mathbf{q}} [(P_0 + mc)\cosh(\zeta/2) + i\mathbf{P}\sinh(\zeta/2)]^2. \tag{13}$$

Diagonalizing the matrix representation of eq. (9), the eigenvalues of the partially transposed spin density operator are readily obtained as

$$\lambda_1 = F_1, \qquad \lambda_2 = 1 - F_1, \qquad \lambda_{3,4} = \pm|F_2|. \tag{14}$$

From eqs (10) and (12), it is evident that F_1 is positive definite and less than one (due to normalization), so that λ_1 and λ_2, as well as $\lambda_3 = |F_2|$, do not contribute to the negativity. As we shall see, in the next section, the above statement also holds true for sharp probability distributions. The behaviour of spin–spin entanglement is then solely determined by eqs (11) and (13). In the next section, we employ eqs (11) and (13) to investigate the effect of the momentum probability distributions, on the spin–spin entanglement, as viewed by the fixed and moving observers.

3. Gaussian momentum distributions

In this section, the result of the previous section is examined for Gaussian distributions. Suppose, in the fixed frame

$$a_{\pm,\pm}(\mathbf{p},\mathbf{q}) = (2\pi w_{1(2)}^2)^{-1/4} \exp\left(-\frac{p^2+q^2}{2w_{1(2)}^2}\right) \tag{15}$$

gives the probability distributions for the occurrence of $|\mathbf{p},\mathbf{q},+,+\rangle$ and $|\mathbf{p},\mathbf{q},-,-\rangle$, respectively [30,31]. Substituting eq. (15) into eq. (11) and using eqs (3) and (4), one easily finds

$$F_1 = \frac{D_1[2(p_0+mc)^2\cosh^2(\zeta/2) + w_1^2\sinh^2(\zeta/2)]^2}{\sum_{i=1,2} D_i[2(p_0+mc)^2\cosh^2(\zeta/2) + w_i^2\sinh^2(\zeta/2)]^2} \tag{16}$$

and

$$F_2 = \frac{8\sqrt{D_1D_2}\frac{w_1w_2}{w_1^2+w_2^2}[(p_0+mc)^2\cosh^2(\zeta/2) - \frac{w_1^2w_2^2}{w_1^2+w_2^2}\sinh^2(\zeta/2)]^2}{\sum_{i=1,2} D_i[2(p_0+mc)^2\cosh^2(\zeta/2) + w_i^2\sinh^2(\zeta/2)]^2}, \tag{17}$$

where

$$D_j = \exp\left[-\frac{2p_0^2\sinh^2(\zeta)}{w_j^2}\right], \qquad j = 1, 2.$$

It is observed that the expression for F_2, which in turn determines the degree of spin–spin entanglement (through eq. (14)) strongly depends upon the distribution widths and the rapidity, ζ. Moreover, as any of the widths approaches zero (sharp distribution), F_2 and thus the negativity, vanish in all frames; spin states disentangle. This, of course, is due to the fact that for sharp distributions eq. (16) yields $F_1 = 0$ or 1, and consequently, only one of the spin states, either $|+,+\rangle$ or $|-,-\rangle$ (see eq. (9)), is present and is not affected by a Wigner rotation about the z-axis. It is then clear that the degree of entanglement is solely determined by the widths of the Gaussian distributions in the fixed frame and can be varied accordingly. In the limit $\zeta \to \infty$, F_2 approaches zero, consequently, and regardless of the (unequal) widths, spin–spin entanglement diminishes as seen by an observer moving close to c [30,31]. To support these conclusions, the behaviour of negativity as functions of rapidity for different width ratios are illustrated in figure 1. Moreover, the graphs in figure 1 particularly show that as the ratio of the widths increases, spin–spin states become less entangled for every rapidity. The reason behind this behaviour is the fact that for larger widths ratio, more momentum states are present in the system, so that less information about spins is available.

As for Gaussian distributions of equal widths, $w_1 = w_2 = w$, one easily finds that $F_2 = \frac{1}{2}$ (see eq. (17)), with maximal entanglement, in the fixed frame $\zeta = 0$. However, in view of the moving observer,

$$F_2 = \left[\frac{1 - \frac{w^2\tanh^2(\zeta/2)}{2(p_0+mc)^2}}{1 + \frac{w^2\tanh^2(\zeta/2)}{2(p_0+mc)^2}}\right]^2 \leq 1. \tag{18}$$

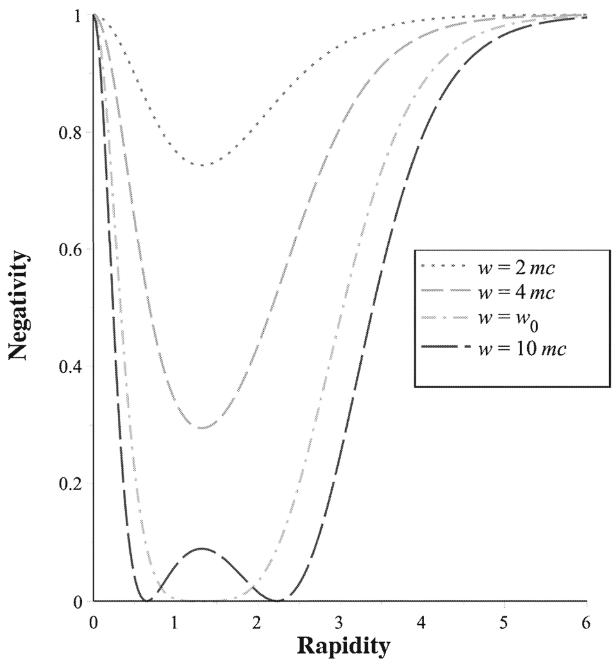

Figure 1. Negativity vs. the rapidity for equal distribution widths. w_0 indicates the critical width.

As $\zeta \to \infty$, it is evident that $F_2 \to \frac{1}{2}$, which again gives a maximal spin–spin entanglement. To explore the behaviour of F_2 and thus the negativity between these two limits, we rewrite eq. (17) in the form

$$F_2 = \frac{1}{2}\left(\frac{1-\frac{1}{2}(\frac{w}{mc})^2 g(\zeta)}{1+\frac{1}{2}(\frac{w}{mc})^2 g(\zeta)}\right)^2,$$

with obvious definition for $g(\zeta)$, and take its derivative. One finds

$$\frac{\mathrm{d}F_2}{\mathrm{d}\zeta} = \frac{-2(\frac{w}{mc})^2 \frac{\mathrm{d}g(\zeta)}{\mathrm{d}\zeta}(1-\frac{1}{4}(\frac{w}{mc})^4 g^2(\zeta))}{(1+\frac{1}{2}(\frac{w}{mc})^2 g(\zeta))^4}, \tag{19}$$

which vanishes for

$$\frac{\mathrm{d}g(\zeta)}{\mathrm{d}\zeta} = 0 \quad \text{and} \quad \left(1-\frac{1}{4}\left(\frac{w}{mc}\right)^4 g^2(\zeta)\right) = 0.$$

The first condition gives $\zeta = \cosh^{-1}(1) = 0$, $\zeta = \cosh^{-1}(2)$ and $\zeta = \infty$, regardless of the width. As a result, the spin–spin entanglement exhibits at least two maxima and a minimum for every width. The number of roots of the second condition, however, indeed depends upon the distribution widths. A simple calculation shows that for widths less than a critical one, $w_0 = \sqrt{54}mc$ it does not vanish at all. For widths greater than (or

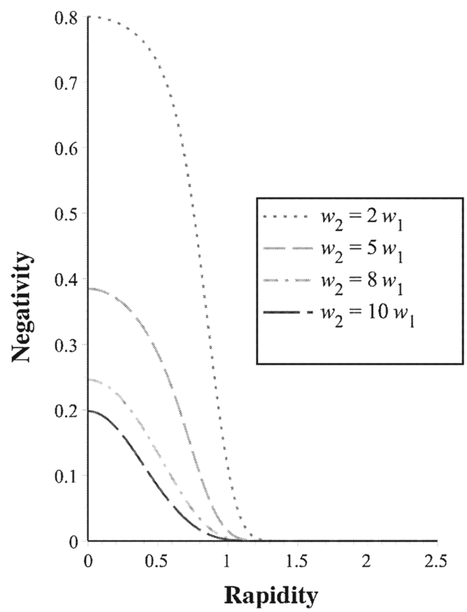

Figure 2. Negativity vs. the rapidity for unequal distribution widths.

equal to) w_0, on the other hand, two (one) distinct nonnegative roots do (does) exist. This fact indicates that for $w \geq w_0$, at the most two vanishing minima and a maximum in the entanglement occur. The behaviour of entanglement as functions of rapidity and for unequal distribution widths is depicted in figure 2, showing a complete agreement with the forgoing conclusions. It is worth mentioning that, to some extent, this behaviour in the entanglement has also been observed in ref. [30] by considering the concurrence.

4. Conclusion

In the present article, we have systematically investigated the properties of negativity, as a measure of spin–spin entanglement, relative to moving observers, for a system of two massive spin-$\frac{1}{2}$ particles. We have assumed that the spin–spin states of the system occur with Gaussian probability distributions. Although the behaviour of entanglement is thoroughly discussed in the preceding section, in what follows, we outline the more important aspects of this report.

(1) The negativity, which indicates the degree of entanglement between the two spin-$\frac{1}{2}$ particles, similar to concurrence, is a frame-dependent quantity.
(2) The spin–spin entanglement, as a function of rapidity, starts from a maximum in the fixed frame and diminishes, approaching zero, at high rapidities. The maximum value of entanglement, as well as its slope, decreases as the distribution's widths ratio increases.
(3) For probability distributions of equal widths, the entanglement is again maximal in the fixed frame, decreases to a minimum at a certain rapidity and returns to its

maximal value at extremely high rapidity. This behaviour is under the restriction that the width is smaller than a critical value, w_0. The value of minimum entanglement can be adjusted by varying the width. These particular conclusions are indeed in accordance with the concurrence as reported in ref. [29].

(4) For widths greater than w_0, one finds two vanishing minima and one maximum in the entanglement, in addition to the two extreme limiting cases. Here again, one can adjust the maximum value by setting the widths. This point in the behaviour of entanglement, however, has been overlooked in ref. [30].

In short, the data presented in this article, support the general belief [14,30] that degree of entanglement determined by the negativity (or concurrence [30]), is frame-dependent. It is then evident that a systematic definition of reduced spin density matrix for massive particles, consistent with principles of special relativity, is deemed necessary [35].

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