PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 83, No. 6
December 2014
pp. 885–906

Analytical solutions of time–space fractional, advection–dispersion and Whitham–Broer–Kaup equations

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MS received 14 January 2014; revised 23 February 2014; accepted 4 March 2014
DOI: 10.1007/s12043-014-0821-7; ***e*Publication:** 5 September 2014

Abstract. In this article, we study time–space fractional advection–dispersion (FADE) equation and time–space fractional Whitham–Broer–Kaup (FWBK) equation that have significant roles in hydrology. We introduce suitable transformations to convert fractional-order derivatives to integer-order derivatives and as a result these equations transform into partial differential equations (PDEs). Then the Lie symmetries and the corresponding optimal systems of the resulting PDEs are derived. The symmetry reductions and exact independent solutions based on optimal system are investigated which constitute the exact solutions of original fractional differential equations.

Keywords. Modified Riemann–Liouville fractional derivative; Lie symmetries, optimal system; invariant solutions.

PACS Nos 02; 02.20.Qs; 02.20.Sv

1. Introduction

Recent advances in the fields of science and economics acknowledge the significance of fractional differential equations that were in the realm of theory a few decades ago. They are indeed powerful and more accurate and preferred over integer-order differential equations that used to model those phenomenons where it was necessary to keep the description of memory and hereditary properties of various materials and processes. It is because fractional differential operator has a non-local property, i.e. it is defined as a definite integral over some interval, rather than on a small neighbourhood of a point. However, the definition of a fractional differential operator is not unique as it appears in different forms as defined by Riemann–Liouville, Caputo, Weyl and many others. In [1], Jumarie defined a modified version of Riemann–Liouville operator which resembles the ordinary differential operator.

The applications of fractional differential equations can be found in fractals, acoustics, control theory, continuous time random walk, signal processing and many other problems [2,3]. Specifically, it arises in hydrology to describe the anomalous transport of fluids. Within the era of last two decades, several methods such as Laplace transform method, Green's function method [2], variational iteration method [4], Adomian decomposition method [5], homotopy perturbation method [6] and symmetries method [7] have been proposed to obtain numerical and analytical solutions of these equation.

In this monograph, we consider fractional advection–dispersion equation (FADE)

$$u_t^{\alpha} = -\nu u_x^{\beta} + k u_{xx}^{2\beta}, \qquad 0 < \alpha, \qquad \beta < 1, \qquad t > 0, \qquad x > 0, \tag{1}$$

where u_t^{α}, u_x^{β} and u_{xx}^{β} are fractional derivatives of u with respect to t and x of order α and β respectively used in modified Riemann–Liouville sense. In eq. (1), $u = u(t, x)$ describes the concentration of solute at time t along the longitudinal direction at position x, positive constants ν and k represent the medium flow velocity and the dispersion coefficient respectively. The transport of dissolved solutes in soil and aquifers plays an important role in various fields including absorption of nutrients by plant roots, remediation of contaminated soils and aquifers and leaching of agrochemicals to groundwater. In nature, solutes are transported in fluids due to advection and dispersion. If dispersion is smooth, then this can be modelled by utilizing simple Brownian motion. However, in most of the real applications, it is not an appropriate tool for dispersion of particles because of the anomaly where the mean square displacement of a particle does not increase linearly with time. The anomalous dispersion is categorized as subdispersion and superdispersion. Subdispersion is often caused by memory effects and Lèvy-type statistics, due to 'traps' that have infinite mean-waiting time as it happens in subsurface hydrology due to heterogeneity of natural geological media at various scales. The anomalous transport of solute does not obey Brownian motion, and therefore a random process is required to describe non-Brownian motion. Berkowitz and Scher [8] introduced continuous time random walk (CTRW) approach to simulate solute transport in geological media by considering the movement as a series of transitions. Later on Berkowitz *et al* [9] characterized solute transport by using joint probability distribution. FADE is a special case of CTRW where solute has a considerable probability to move long distances and its distribution follows a power law. Cushman [10] derived FADE from a general stochastic continuum model in which hydraulic properties of heterogeneous medium is treated as stochastic processes. Over the last decade FADE has proven to be an effective tool to simulate solute transport in both surface and subsurface [11–15]. Initially, FADE was defined in terms of Riemann–Liouville fractional derivative. However, due to lack of physical results for solute movement in soil columns, Zhang *et al* [16] redefined FADE in terms of Caputo fractional derivative. This modification was necessary to overcome the drawbacks of the Riemann–Liouville fractional derivative that includes hypersingular improper integral and that the fractional derivative of a constant is not zero. Due to the increasing water and air pollution FADE has drawn serious attention of scientists, engineers and mathematicians. Many researchers have discussed FADE and various solutions of FADE have been found using variational iteration method [17], Adomian decomposition method [18] and homotopy perturbation method [19]. We consider FADE with modified Riemann–Liouville fractional derivatives which not only handles these issues but also has a similarity with classical derivative.

We also study the time–space fractional Whitham–Broer–Kaup (FWBK) equations which are used to describe the anomalous propagation of long waves in shallow water. The FWBK equations are expressed as

$$u_t^\alpha + uu_x^\alpha + v_x^\alpha + \beta u_{xx}^{2\alpha} = 0,$$

$$v_t^\alpha + vu_x^\alpha + uv_x^\alpha - \beta v_{xx}^{2\alpha} + \gamma u_{xxx}^{3\alpha} = 0, \qquad t > 0, \qquad x \in [a,b] \subset \mathbb{R}, \quad (2)$$

where $u = u(t,x)$ is the velocity potential of waves moving in x-direction, $v = v(t,x)$ is the vertical displacement from equilibrium position of the fluid, β and γ are real constants that describe different diffusion powers and α is the order of the fractional derivative. These equations are generalizations of classical Whitham–Broer–Kaup equations [20–22] which are obtained by using the Boussinesq approximation. If $\beta = 0$, $\gamma = 1$, eqs (2) reduce to fractional long wave equations, whereas the chice of $\beta = 0$ and $\gamma = 1$ gives fractional modified Boussinesq equations. These equations are generalizations of classical long wave equations [23] and modified Boussinesq equations [24] respectively. The exact solutions of FWBK equations were constructed in [25], where the authors have used travelling wave transformation to reduce (2) to a nonlinear system of third-order fractional ODE. The generalized exp-function method was utilized to derive exact solutions of the reduced system which in terms of original variables formed the solutions of FWBK equations. In [26], Atangana and Baleanu studied FWBK using the Sumudu transform homotopy method.

The Lie theory of symmetry group, to ordinary or partial differential equations, is the most powerful tool to obtain invariant solutions that are usually of closed form and often describe the asymptotic behaviour of general types of solutions. These invariant solutions depend upon their invariant transformations obtained from the symmetry group admitted by the differential equation. However, a Lie group or Lie algebra contains infinitely many subgroups or subalgebras of the same dimensions and it is impossible to use all of them to construct invariant solutions. A well-known method was proposed by Ovsiannikov [27] to classify all subalgebras to equivalence classes of conjugate subalgebras by using its adjoint representation group. A set consisting of one representative from each equivalence class is known as an optimal system which provides all invariant solutions. In this work, we first convert the FADE and FWBK equations into partial differential equations (PDEs) by using suitable transformations. Then we compute Lie symmetries of the resulting PDEs and find their optimal systems to construct invariant solutions. To the best of our knowledge the solutions obtained for FADE and FWBK equations are new and not reported elsewhere. We have used a systematic way to construct transformations for reduced FADE and FWBK equations which are more general and the transformations considered in other articles are retrieved here.

We have organized this paper as follows. In §2, we give some basic definitions about fractional differential operators along with their properties. Section 3 is devoted to FADE which is reduced to PDE and we compute Lie symmetries and optimal system based on the adjoint representations of vector field. Exact solutions of FADE for all cases are also presented in §3. In §4, the exact solutions of FWBK equations are derived using the same procedure as in §3. The concluding remarks are summarized in §5.

2. Preliminaries

Fractional calculus is a generalization of integer-order integrals and derivative to an arbitrary real order. There are several definitions of fractional derivatives which are useful for applications. The most famous among those is the Riemann–Liouville definition considered as the classic which has the property of a non-zero derivative of a constant function. Caputo [28,29] redeveloped this definition to contend this issue. It also utilizes integer-order initial conditions to solve fractional differential equations. The definitions by Weyl [30] and Riesz [31] are also noteworthy. Jumarie [1] proposed a modified version of Riemann–Liouville fractional derivative that uses weak conditions on the derivative and has some features coincide with those in classical calculus. This section deals with the fundamental definitions and properties of Riemann–Liouville and Jumarie's modified Riemann–Liouville fractional derivatives. For details, the interested reader is referred to [1–3].

2.1 *Definition*

A real valued function $f(x) \in \mathcal{C}_\infty$, $x > 0$, $\alpha \in \mathcal{R}$ if there exists a real number $\beta > \alpha$ such that $f(x) = x^\beta g(x)$, where $g(x) \in \mathcal{C}[0, \infty)$ and it is said to be in space $\mathcal{C}_\alpha^n$ if and only if $f^{(n)} \in \mathcal{C}_\alpha$; $n \in \mathcal{N}$.

2.2 *Riemann–Liouville fractional derivative*

The fractional integral of order α is defined as

$$D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) \mathrm{d}\xi, \tag{3}$$

where $\mathrm{Re}(\alpha) > 0$. An alternative notation used is $I^\alpha f(x) = D_x^{-\alpha} f(x)$. The operator I^α holds the following fundamental relations:

$$I^\alpha (I^\beta f(x)) = I^{\alpha+\beta} f(x) \tag{4}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} I^{\alpha+1} f(x) = I^\alpha f(x). \tag{5}$$

One can define fractional-order derivative of f as

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} f = \frac{\mathrm{d}^n}{\mathrm{d}x^n} I^{n-\alpha} f, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}. \tag{6}$$

Using the above relation, we can define Riemann–Liouville fractional derivatives

$$D_x^\alpha f(x) = \begin{cases} (\mathrm{d}^n/\mathrm{d}x^n) I^{n-\alpha} f(x), & \alpha > 0, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N} \\ f(x) & \alpha = 0. \end{cases} \tag{7}$$

More precisely

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x f(\xi)(x-\xi)^{n-\alpha-1} \mathrm{d}\xi, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}, \tag{8}$$

where $f \in \mathcal{C}_\alpha^n$. The Liebnitz rule can be generalized for fractional derivative:

$$D_x^\alpha(f(x)g(x)) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_x^{\alpha-n}(f(x)) D_x^n(g(x)). \tag{9}$$

If f is a function of two independent variables t and x, then Riemann–Liouville fractional derivative of f with respect to t is

$$D_t^\alpha f(t,x) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_0^t f(\xi,x)(t-\xi)^{n-\alpha-1} \mathrm{d}\xi, \quad n-1 < \alpha \le n, \quad \in \mathbb{N}. \tag{10}$$

The fractional derivative with respect to x can be defined in a similar fashion.

2.3 *Modified Riemann–Liouville fractional derivative*

Guy Jumarie [1] proposed some modifications in Riemann–Liouville fracional derivative and derived fractional Taylor series of nondifferentiable functions. The new modified fractional derivative has some features similar to the classical derivative. The definition and properties of modified fractional derivatives are defined below.

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (f - f(0))(x-\xi)^{-\alpha-1} \mathrm{d}\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^x (f(\xi) - f(0))(x-\xi)^{-\alpha} \mathrm{d}\xi, & 0 < \alpha \le 1, \\ (f^{(n-1)}(x))^{(\alpha-n+1)}, & n-1 < \alpha \le n, \quad n \ge 2. \end{cases} \tag{11}$$

The modified Riemann–Liouville fractional derivative bears some interesting properties:

$$D_x^\alpha x^\mu = \frac{\Gamma(1+\alpha)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}, \qquad \mu > 0, \tag{12}$$

$$D_x^\alpha(f(x)g(x)) = g(x) D_x^\alpha f(x) + f(x) D_x^\alpha g(x), \tag{13}$$

$$D_x^\alpha f[g(x)] = \frac{\mathrm{d}f[g(x)]}{\mathrm{d}g(x)} D_x^\alpha g(x). \tag{14}$$

3. Fractional advection–dispersion equation

Consider the time–space fractional advection–dispersion equation (FADE)

$$u_t^\alpha = -\nu u_x^\beta + k u_{xx}^{2\beta}, \qquad 0 < \alpha, \quad \beta \le 1, \qquad \nu, \ k > 0, \qquad t, x > 0. \tag{15}$$

We introduce the transformations

$$X = \frac{p x^\beta}{\Gamma(1+\beta)}, \qquad T = \frac{q t^\alpha}{\Gamma(1+\alpha)}, \qquad W(T,X) = u(t,x), \qquad p, \ q \ne 0. \tag{16}$$

Equation (15) with the help of (12), (14) and (16) transforms to second-order partial differential equation

$$q W_T + \nu p W_X - k p^2 W_{XX} = 0. \tag{17}$$

Computing the Lie symmetries manually is a tedious task. However, many computer software packages have been designed to compute the Lie symmetries for partial differential equations. Using Maple, the Lie symmetries of eq. (17) are

$$\begin{aligned} V_1 &= \partial_X, \qquad V_2 = \partial_T, \qquad V_3 = W\partial_W, \\ V_4 &= 2qT\partial_T + (qX + \nu pT)\partial_X, \\ V_5 &= (qXW - p\nu TW)\partial_W - 2p^2kT\partial_X, \\ V_6 &= -4p^2qkT^2\partial_T - 4p^2qkXT\partial_X \\ &\quad +(-2pq\nu TX + 2p^2qkT + p^2\nu^2T^2 + q^2X^2)W\partial_W \end{aligned}$$

and

$$V_\infty = F_1(T, X)\partial_W, \tag{18}$$

where $F_1(T, X)$ satisfies

$$kp^2\frac{\partial^2 F_1}{\partial X^2} - q\frac{\partial F_1}{\partial T} - p\nu\frac{\partial F_1}{\partial X} = 0. \tag{19}$$

3.1 *Adjoint representation and optimal system*

In this section, we shall compute the adjoint representations and optimal system based on the vector fields. The optimal system gives the minimal list of one-dimensional subalgebras of the Lie algebra $\mathfrak{G}$, each of which is used to construct a set of invariant solutions such that if there is any other solution, then there exists a further symmetry which transforms that solution to a solution in the set. To construct an optimal system of one-dimensional subalgebra of the Lie algebra of eq. (17), we proceed in the following manner as given in [32].

Without loss of generality we assume $p = q = \nu = k = 1$ to make our calculations as simple as possible. The first step in this regard is to calculate the commutators $[V_\alpha, V_\beta]$, which is defined as

$$[V_\alpha, V_\beta] = V_\alpha V_\beta - V_\beta V_\alpha, \qquad \alpha,\ \beta = 1, 2, ..., 6. \tag{20}$$

The Lie algebra of the infinitesimal symmetries of eq. (17) is depicted by commutator table 1.

The second step is to compute the adjoint representations generated by the basis symmetries given in (18). These adjoint representations will help in sorting similar one-dimensional subalgebras. The adjoint representation is given by

$$\mathrm{Ad}(\exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{1}{2}\epsilon^2[V_i, [V_i, V_j]] - \cdots. \tag{21}$$

A complete adjoint representation is presented in table 2.

Let

$$V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + a_6V_6, \qquad a_i \in \mathbb{R}, \qquad i = 1, 2, ..., 6 \tag{22}$$

Table 1. Commutator table.

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	0	0	V_1	V_3	$2V_5$
V_2	0	0	0	V_1+2V_2	$-2V_1-V_3$	$2V_3-4V_4-2V_5$
V_3	0	0	0	0	0	0
V_4	$-V_1$	$-V_1-2V_2$	0	0	V_5	$2V_6$
V_5	$-V_3$	$2V_1+V_3$	0	$-V_5$	0	0
V_6	$-2V_5$	$-2V_3+4V_4+2V_5$	0	$-2V_6$	0	0

be a general vector of the Lie algebra $\mathfrak{G}$. Then, every element of $\mathfrak{G}$ can be expressed as a vector $V=(a_1,a_2,a_3,a_4,a_5,a_6)$ for some $a_i\in\mathbb{R}$. Now we compute a real function $\eta(V)$ termed as invariant. This can be done by using the following symmetries:

$$\Delta_i = c_{ij}^k a^j \frac{\partial}{\partial a^k}, \qquad i=1,2,\ldots,6,$$

where c_{ij}^k are the structure constants in table 1 (see [33], vol. 2). Thus we have

$$\begin{aligned}
\Delta_1 &= a_4\frac{\partial}{\partial a_1}+a_5\frac{\partial}{\partial a_3}+2a_6\frac{\partial}{\partial a_5},\\
\Delta_2 &= a_4\left(\frac{\partial}{\partial a_1}+2\frac{\partial}{\partial a_2}\right)+a_5\left(-2\frac{\partial}{\partial a_1}-\frac{\partial}{\partial a_3}\right)+a_6\left(2\frac{\partial}{\partial a_3}-4\frac{\partial}{\partial a_4}-2\frac{\partial}{\partial a_5}\right),\\
\Delta_3 &= 0,\quad \Delta_4=-a_1\frac{\partial}{\partial a_1}+a_2\left(-\frac{\partial}{\partial a_1}-2\frac{\partial}{\partial a_2}\right)+a_5\frac{\partial}{\partial a_5}+2a_6\frac{\partial}{\partial a_6},\\
\Delta_5 &= -a_1\frac{\partial}{\partial a_3}+a_2\left(2\frac{\partial}{\partial a_1}+\frac{\partial}{\partial a_3}\right)-a_4\frac{\partial}{\partial a_5},\\
\Delta_6 &= -2a_1\frac{\partial}{\partial a_5}+a_2\left(-2\frac{\partial}{\partial a_3}+4\frac{\partial}{\partial a_4}+2\frac{\partial}{\partial a_5}\right)-2a_4\frac{\partial}{\partial a_6}\,.
\end{aligned}\tag{23}$$

Table 2. Adjoint representation.

$\mathrm{Ad}(\exp(\epsilon i)j)$	V_1	V_2	V_3	V_4	V_5	V_6
V_1	V_1	V_2	V_3	$-\epsilon V_1+V_4$	$-\epsilon V_3+V_5$	$\epsilon^2-2\epsilon V_5+V_6$
V_2	V_1	V_2	V_3	$-\epsilon V_1-2\epsilon V_2$ $+V_4$	$2\epsilon V_1+\epsilon V_3$ $+V_5$	$-4\epsilon^2V_2+4\epsilon V_4$ $(-2\epsilon+\epsilon^2)V_3$ $+2\epsilon V_5+V_6$
V_3	V_1	V_2	V_3	V_4	V_5	V_6
V_4	$\mathrm{e}^{\epsilon}\mathrm{V}_1$	$(\mathrm{e}^{2\epsilon}-\mathrm{e}^{\epsilon})\mathrm{V}_1$ $+\mathrm{e}^{2\epsilon}V_2$	V_3	V_4	$\mathrm{e}^{-\epsilon}\mathrm{V}_5$	$\mathrm{e}^{-2\epsilon}\mathrm{V}_6$
V_5	$V_1+\epsilon V_3$	$-2\epsilon V_1+V_2$ $(-\epsilon^2-\epsilon)V_3$	V_3	$V_4+\epsilon V_5$	V_5	V_6
V_6	$V_1+2\epsilon V_5$	$V_2+2\epsilon V_3$ $-4\epsilon V_4-2\epsilon V_5$ $-4\epsilon^2V_6$	V_3	$V_4+2\epsilon V_6$	V_5	V_6

Now $\psi = \psi(a_1, a_2, a_3, a_4, a_5, a_6)$ is an invariant of the full adjoint action if it satisfies

$$\Delta_1(\psi) = 0, \qquad \Delta_2(\psi) = 0, \qquad \Delta_3(\psi) = 0, \qquad \Delta_4(\psi) = 0,$$
$$\Delta_5(\psi) = 0, \qquad \Delta_6(\psi) = 0. \tag{24}$$

The solution of the above system is

$$\psi = \psi(\eta_1(V), \eta_2(V)) = \psi\Bigg(a_4^2 + 4a_2a_6 - \frac{1}{2(a_4^2+4a_2a_6)}(4a_2a_4a_6 + a_4^3 + 2a_2^2a_6$$
$$-4a_1a_2a_6 + 8a_2a_3a_6 + 2a_2a_4a_5 - 2a_2a_5^2 + 2a_1^2a_6 + 2a_3a_4^2 - 2a_1a_4a_5)\Bigg). \tag{25}$$

Precisely

$$\eta_1(V) = a_4^2 + 4a_2a_6 \tag{26}$$

and

$$\eta_2(V) = -\frac{1}{\eta_1(V)}(4a_2a_4a_6 + a_4^3 + 2a_2^2a_6 - 4a_1a_2a_6$$
$$+8a_2a_3a_6 + 2a_2a_4a_5 - 2a_2a_5^2 + 2a_1^2a_6 + 2a_3a_4^2 - 2a_1a_4a_5). \tag{27}$$

To begin the classification process, we concentrate on a_2, a_4 and a_6 coefficients of V as defined in (22). Now

$$\tilde{V} = \sum_{i=1}^{6} \tilde{a}_i V_i = \mathrm{Ad}(\exp(\alpha V_6)) \circ \mathrm{Ad}(\exp(\beta V_2))V \tag{28}$$

has coefficients

$$\tilde{a}_2 = a_2 - 2\beta a_4 - 4\beta^2 a_6,$$
$$\tilde{a}_4 = -4\alpha(a_2 - 2\beta a_4 - 4\beta^2 a_6) + a_4 + 4\beta a_6,$$
$$\tilde{a}_6 = -4\alpha^2(a_2 - 2\beta a_4 - 4\beta^2 a_6) + 2\alpha(a_4 + 4\beta a_6) + a_6. \tag{29}$$

The following three cases need to be considered.

Case 1. $\eta_1(V) > 0$.

- If $a_6 \neq 0$, choose

  $$\beta = \frac{-a_4 + \sqrt{\eta_1(V)}}{4a_6} \quad \text{and} \quad \alpha = \frac{a_6}{-2\sqrt{\eta_1}}$$

  then $\tilde{a}_2 = \tilde{a}_6 = 0$ while $\tilde{a}_4 = \sqrt{\eta_1(V)}$. So V is equivalent to a multiple of

  $$\tilde{V} = V_4 + \tilde{a}_1 V_1 + \tilde{a}_3 V_3 + \tilde{a}_5 V_5. \tag{30}$$

Acting further by adjoint maps generated respectively by V_5 and V_1, i.e.,

$$\begin{aligned}\tilde{\tilde{V}} &= \mathrm{Ad}(\exp(\theta V_1)) \circ \mathrm{Ad}(\exp(\phi V_5))\tilde{V} \\ &= V_4 + (\tilde{a}_1 - \theta)V_1 + (\tilde{a}_1\phi + \tilde{a}_3 - \theta\phi - \theta\tilde{a}_5)V_3 + (\phi + \tilde{a}_5). \end{aligned} \tag{31}$$

We can make the coefficients of V_1 and V_5 equal to zero by setting $\theta = \tilde{a}_1$ and $\phi = -\tilde{a}_5$. Hence

$$\tilde{\tilde{V}} = V_4 + aV_3. \tag{32}$$

- If $a_6 = 0$, choose $\beta = a_2/2a_4$ and $\alpha = 0$ and proceeding in the same manner as above we obtain

$$\tilde{\tilde{V}} = V_4 + aV_3. \tag{33}$$

- If $a_6 = a_4 = 0$ then $\eta_1(V) = 0$ which is not the case.
  Therefore every element of $\eta_1(V) > 0$ is equivalent to $V_4 + aV_3$ where $a \in \mathbb{R}$.

Case 2. $\eta_1(V) < 0$.

Set $\alpha = a_4/4a_2$ and $\beta = 0$ where $a_2 \neq 0$ as $\eta_1(V) < 0$. For these α and β we can make $\tilde{a}_4 = 0$. Now we have

$$\begin{aligned}\tilde{V} &= \tilde{a}_1V_1 + \tilde{a}_2V_2 + \tilde{a}_3V_3 + \tilde{a}_4V_4 + \tilde{a}_5V_5 + \tilde{a}_6V_6 \\ &= a_1V_1 + a_2V_2 + \tilde{a}_3V_3 + \tilde{a}_5V_5 + \frac{\eta_1(V)}{4a_2}V_6. \end{aligned} \tag{34}$$

As a_2 and a_6 are not zero because of the restriction on $\eta_1(V)$, dividing $\tilde{V}$ in (34) by a_2 to make the coefficient of V_2 equals 1 yield

$$\tilde{V} = \frac{a_1}{a_2}V_1 + V_2 + \frac{(a_4 + 2a_3)}{2a_2}V_3 + \frac{(a_1a_4 - a_4a_2 + 2a_2a_5)}{2a_2^2}V_5 + \frac{\eta_1(V)}{4a_2^2}V_6. \tag{35}$$

Acting further by adjoint maps generated respectively by V_5 and V_1, we have

$$\tilde{\tilde{V}} = \tilde{\tilde{a}}_1V_1 + V_2 + \tilde{\tilde{a}}_3V_3 + \tilde{\tilde{a}}_5V_5 + \tilde{a}_6V_6 \tag{36}$$

where

$$\begin{aligned}\tilde{\tilde{a}}_1 &= \frac{a_1}{a_2} - 2\phi, \\ \tilde{\tilde{a}}_3 &= \left(\frac{a_1}{a_2} - 1\right)\phi - \phi^2 + \frac{(a_4 + 2a_3)}{2a_2} - \frac{(a_1a_4 - a_4a_2 + 2a_2a_5)}{2a_2^2}\theta + \frac{\eta_1(V)}{4a_2^2}\theta^2, \\ \tilde{\tilde{a}}_5 &= \frac{(a_1a_4 - a_4a_2 + 2a_2a_5)}{2a_2^2} - \frac{\eta_1(V)}{2a_2^2}\theta. \end{aligned} \tag{37}$$

- Setting

$$\phi = \frac{a_1}{2a_2}$$

and

$$\theta = \frac{a_4a_1 - a_4a_2 + 2a_2a_5}{\eta_1(V)}$$

in (37) show that $\tilde{\tilde{V}}$ is equivalent to $V_2 + aV_3 + bV_6$, $a, b \in \mathbb{R}$, $b \neq 0$. No further simplification is possible.

- If we set

$$\phi = \frac{a_1}{2a_2} \quad \text{and} \quad \theta = \frac{1}{\eta_1(V)}[a_1a_4 - a_2a_4 + 2a_2a_5 + (a_2^2\eta_1(V) - 2a_2\eta_2(V))^{1/2}],$$

$$(a_2^2\eta_1(V) - 2a_2\eta_2(V)))^{1/2} > 0$$

with $\eta_1(V) < 0$ in (37) we see that $\tilde{\tilde{V}}$ is equivalent to $V_2 + aV_5 + bV_6$, $a, b \in \mathbb{R}$, $b \neq 0$ which cannot be simplified further.

- If

$$\theta = \frac{a_1a_4 - a_2a_4 + 2a_2a_5}{\eta_1(V)},$$

$$\phi = \frac{1}{2a_2\eta_1(V)}[(a_1 - a_2)\eta_1(V) + (2a_2\eta_1(V)\eta_2(V))^{1/2}],$$

$$2a_2\eta_1(V)\eta_2(V) > 0$$

then (36) reduces to $aV_1 + V_2 + bV_6$, $a, b \in \mathbb{R}$, $b \neq 0$ which is in its simplest form.

Case 3. $\eta_1(V) = 0$.

- Suppose $a_2, a_4, a_6 \neq 0$. In this case, we cannot make two of the coefficients in (29) equal to zero simultaneously. However, if we choose $\alpha = a_4/4a_2$ and $\beta = 0$ in (29) then $\tilde{V} = \tilde{a}_1V_1 + \tilde{a}_2V_2 + \tilde{a}_3V_3 + \tilde{a}_5V_5$. Now by the action of V_1 and V_5 respectively we can either simultaneously make the coefficient of V_1 and V_3 equals zero or V_1 and V_5 equals zero, resulting in either $V_2 + aV_5$ or $V_2 + aV_3$ respectively.
- If $a_2 = a_4 = 0$, $a_6 \neq 0$ then $V = a_1V_1 + a_3V_3 + a_5V_5 + a_6V_6$. Under the action of the adjoint maps generated by V_1 and V_5 respectively, V reduces to $aV_1 + V_6$ when $a_1 \neq 0$. For $a_1 = 0$, we obtain either $aV_3 + V_6$ or $aV_5 + V_6$.
- If $a_4 = a_6 = 0$, $a_2 \neq 0$ then V reduces to $a_1V_1 + a_2V_2 + a_3V_3 + a_5V_5$. Such a V reduces to either $V_2 + aV_5$ or $V_2 + aV_3$ or $aV_1 + V_2$ if we use the group generated by V_2 and V_5.
- If $a_2 = a_4 = a_6 = 0$, then we have $V = a_1V_1 + a_3V_3 + a_5V_5$ which by the action of adjoint maps generated by V_1 and V_2 reduces to V_1 or V_3.

The optimal system obtained is summarized in table 3.

Remark

- Note that $V_2 + aV_3 + bV_6 \sim V_2 + aV_5 + bV_6 \sim aV_1 + V_2 + bV_6$ using $\text{Adj}(\exp(a/2b)V_1)$ and $\text{Adj}(\exp(a/2)V_5)$ respectively.

Table 3. Optimal system of subalgebra admitted by fractional advection dispersion equation.

1	$V_4 + aV_3$	$\eta_1(V) > 0$	$a_6 \neq 0$ or $a_4 \neq 0$	$a \in \mathbb{R}$
2	$V_2 + aV_3 + bV_6$	$\eta_1(V) < 0$	$a_2 \neq 0$ and $a_6 \neq 0$	$a \in \mathbb{R},\quad b \neq 0$
3	$V_2 + aV_5$	$\eta_1(V) = 0$	$a_2\, a_4\, a_6 \neq 0$	$a \in \mathbb{R}$
4	$V_2 + aV_3$	$\eta_1(V) = 0$	$a_2\, a_4\, a_6 \neq 0$	$a \in \mathbb{R}$
5	$aV_3 + V_6$	$\eta_1(V) = 0$	$a_2 = a_4 = 0,\ a_6 \neq 0$	$a \in \mathbb{R}$
6	V_1	$\eta_1(V) = 0$	$a_2 = a_4 = a_6 = 0$	
7	V_3	$\eta_1(V) = 0$	$a_2 = a_4 = a_6 = 0$	

- It can be seen that $V_2 + aV_3$ transforms to $aV_1 + V_2$ for adjoint map $\mathrm{Adj}(\exp(a/2)V_5)$.
- One can find equivalence between $aV_3 + V_6$ and $aV_5 + V_6$ using $\mathrm{Adj}(\exp(a/2)V_2)$.

3.2 *Reductions and exact solutions of eq. (15)*

In this section, we find the similarity reduction and classify the group invariant solutions of fractional advection dispersion equation (FADE). The symmetry reduction method to find invariant solution is well known in literature.

1. *Subalgebra $V = V_4 + aV_3$, $a \in \mathbb{R}$*

   In this case the symmetry V is

$$V = 2qT\partial_T + (qX + pvT)\partial_X + aW\partial_W, \tag{38}$$

   which can be written in characteristic form as

$$\frac{\mathrm{d}T}{2qT} = \frac{\mathrm{d}X}{(qX + pvT)} = \frac{\mathrm{d}W}{aW}. \tag{39}$$

   The solution of (39) gives the similarity variables

$$r = XT^{-1/2} - \frac{pv}{q}T^{1/2}, \qquad B(r) = WT^{-a/2q}, \tag{40}$$

   which reduces (17) to

$$2kp^2 B_{rr} + qrB_r - aB = 0. \tag{41}$$

   We introduce

$$R = \frac{qr^2}{4kp^2}, \quad B(r) = r\exp\left(-\frac{qr^2}{4kp^2}\right)Z(R). \tag{42}$$

   With this substitution (41) transforms to

$$RZ_{RR} + \left(\frac{3}{2} - R\right)Z_R - \frac{a+2q}{2q}Z = 0, \tag{43}$$

   which is a Kummer differential equation. Equation (43) can be rewritten as

$$zy'' + (\nu - z)y' - \mu y = 0, \tag{44}$$

where

$$R = z, \qquad Z = y, \qquad \nu = \frac{3}{2}, \qquad \mu = \frac{a+2q}{2q}. \tag{45}$$

The solution of (44) yields

$$y(z) = C_1 \text{ KummerM}(\mu, \nu, z) + C_2 \text{ KummerU}(\mu, \nu, z), \tag{46}$$

where special functions KummerM and KummerU are defined in appendix. Equation (46) with the help of (42) and (45) implies

$$B(r) = r \exp\left(-\frac{qr^2}{4kp^2}\right)(C_1 \text{ KummerM}(\mu, \nu, z) + C_2 \text{ KummerU}(\mu, \nu, z)). \tag{47}$$

After substituting $B(r)$ from (47) in (40) and then writing in terms of original variables (16), we obtain solution of eq. (15) as

$$u(t,x) = -\frac{1}{\sqrt{\frac{qt^\alpha}{\Gamma(1+\alpha)}}}\left(\frac{qt^\alpha}{\Gamma(1+\alpha)}\right)^{1/2\,\frac{a}{q}}\left(\frac{pvqt^\alpha}{\Gamma(1+\alpha)} - \frac{pqx^\beta}{\Gamma(1+\beta)}\right)\mathrm{e}^{-1/4\,\frac{\left(vt^\alpha\Gamma(1+\beta)-x^\beta\Gamma(1+\alpha)\right)^2}{\Gamma(1+\alpha)(\Gamma(1+\beta))^2kt^\alpha}}$$
$$\cdot\left\{C_1 \text{ KummerM}\left(1/2\,\frac{2\,q+a}{q},\ 3/2,\ 1/4\,\frac{\left(vt^\alpha\Gamma(1+\beta) - x^\beta\Gamma(1+\alpha)\right)^2}{\Gamma(1+\alpha)(\Gamma(1+\beta))^2\,kt^\alpha}\right)\right.$$
$$\left.+C_2 \text{ KummerU}\left(1/2\frac{2\,q+a}{q},\ 3/2,\ 1/4\frac{\left(vt^\alpha\Gamma(1+\beta)-x^\beta\Gamma(1+\alpha)\right)^2}{\Gamma(1+\alpha)(\Gamma(1+\beta))^2\,kt^\alpha}\right)\right\}. \tag{48}$$

2. *Subalgebra* $V = V_2 + aV_3 + bV_6$, $a, b \in \mathbb{R}$, $\ b \neq 0$

In this case

$$V = (1 - 4bp^2qkT^2)\partial_T - 4bp^2qkXT\partial_X$$
$$+(a - 2bpqvXT + 2bp^2qkT + bp^2v^2T^2 + bq^2X^2)W\partial_W. \tag{49}$$

The change of variables is

$$r = \frac{X}{\sqrt{4bkp^2qT^2 - 1}},$$

$$B(r) = W(4bkp^2qT^2 - 1)^{1/4}.\exp\left\{-\left(\frac{-8q^3bkp\sqrt{bqk}X^2T}{4bkp^2qT^2-1} - pv^2\sqrt{bqk}T\right.\right.$$
$$+4vq\sqrt{bqk}X + 4aqk\,\operatorname{arctanh}\left(\frac{2bkpqT}{\sqrt{bqk}}\right)$$
$$\left.\left.+v^2\,\operatorname{arctanh}(2\sqrt{bqk}pT)\right)/8qkp\sqrt{bqk}\right\}, \tag{50}$$

which transforms eq. (17) to

$$k^2 p^2 B_{rr} + \left[qk(a - bq^2 r^2) + \frac{1}{4}v^2\right] B = 0. \tag{51}$$

Introducing

$$B(r) = \frac{1}{\sqrt{r}} M(R), \qquad R = \frac{q^{3/2} b^{1/2} r^2}{p k^{1/2}}, \tag{52}$$

eq. (51) is transformed to

$$M_{RR} + \left(-\frac{1}{4} + \frac{4akq + v^2}{16 p k^{3/2} q^{3/2} b^{1/2} R} + \frac{3}{16 R^2}\right) M = 0, \tag{53}$$

which is a Whittaker equation. The solution of Whittaker equation (53) is

$$M(R) = C_1 \,\mathrm{WhittakerM}\,(\kappa, \mu, R) + C_2 \,\mathrm{WhittakerW}\,(\kappa, \mu, R)\,, \tag{54}$$

where WhittakerM and WhittakerW are Whittaker's functions of first and second type respectively which are special solutions of Whittaker equation. The Whittaker equation is a modified form of confluent hypergeometric equation. The parameters κ and μ used in (54) are

$$\kappa = \frac{4akq + v^2}{16 p k^{3/2} q^{3/2} b^{1/2}}, \qquad \mu = \frac{1}{4}. \tag{55}$$

In order to solve (51) two cases should be considered, namely, $b > 0$ and $b < 0$. For $b > 0$, we have the following invariant solution of (15):

$$\begin{aligned}
u(t,x) &= \sqrt{\frac{\Gamma(1+\beta)}{p x^{\beta}}} \\
&\times\Bigg[C_1 \,\mathrm{WhittakerM}\left(\frac{4aqk+v^2}{16pk^{3/2}q^{3/2}b^{1/2}}, \frac{1}{4}, \frac{pq^{3/2}b^{1/2}x^{2\beta}\Gamma(1+\alpha)^2}{k^{1/2}\Gamma(1+\beta)^2(4bp^2q^3kt^{2\alpha}-\Gamma(1+\alpha)^2)}\right) \\
&+C_2 \,\mathrm{WhittakerW}\left(\frac{4aqk+v^2}{16pk^{3/2}q^{3/2}b^{1/2}}, \frac{1}{4}, \frac{pq^{3/2}b^{1/2}x^{2\beta}\Gamma(1+\alpha)^2}{k^{1/2}\Gamma(1+\beta)^2(4bp^2q^3kt^{2\alpha}-\Gamma(1+\alpha)^2)}\right)\Bigg] \\
&\times\exp\Bigg[\Bigg[\Gamma(1+\alpha)\Gamma(1+\beta)^2\left(aqk+\frac{1}{4}v^2\right)\mathrm{arctanh}\left(\frac{2pq^{3/2}b^{1/2}k^{1/2}t^{\alpha}}{\Gamma(1+\alpha)}\right)\left(-\frac{1}{4}\Gamma(1+\alpha)^2\right. \\
&\left.+bp^2q^3kt^{2\alpha}\right) - \frac{1}{2}p\left\{p^2t^{\alpha}k^{3/2}q^{9/2}b^{3/2}\left(vt^{\alpha}\Gamma(1+\beta) - x^{\beta}\Gamma(1+\alpha)\right)^2\right. \\
&\left.-\frac{1}{4}vq^{3/2}b^{1/2}k^{1/2}\Gamma(1+\beta)\Gamma(1+\alpha)^2 \left(-2x^{\beta}\Gamma(1+\alpha) + vt^{\alpha}\Gamma(1+\beta)\right)\right\}\Bigg] \\
&/\left[2pq^{3/2}k^{3/2}b^{1/2}\Gamma(1+\alpha)\Gamma(1+\beta)^2\left(-\frac{1}{4}p\Gamma(1+\alpha)^2 + bp^2q^3kt^{2\alpha}\right)\right]\Bigg].
\end{aligned} \tag{56}$$

If $b < 0$, then eq. (54) yields the following solution of (15):

$$u(t,x) = \sqrt{\frac{\Gamma(1+\beta)}{px^{\beta}}}$$

$$\times\Bigg[C_1 \text{ WhittakerM}\left(\frac{(4aqk+v^2)\,i}{16pk^{3/2}q^{3/2}b^{1/2}}, \frac{1}{4}, \frac{pq^{3/2}b^{1/2}x^{2\beta}\Gamma(1+\alpha)^2\,i}{k^{1/2}\Gamma(1+\beta)^2(4bp^2q^3kt^{2\alpha}+\Gamma(1+\alpha)^2)}\right)$$

$$+C_2 \text{ WhittakerW}\left(\frac{(4aqk+v^2)\,i}{16pk^{3/2}q^{3/2}b^{1/2}}, \frac{1}{4}, \frac{pq^{3/2}b^{1/2}x^{2\beta}\Gamma(1+\alpha)^2\,i}{k^{1/2}\Gamma(1+\beta)^2(4bp^2q^3kt^{2\alpha}+\Gamma(1+\alpha)^2)}\right)\Bigg]$$

$$\times\exp\Bigg[\Bigg[\Gamma(1+\alpha)\Gamma(1+\beta)^2\left(aqk+\frac{1}{4}v^2\right)\ \arctan\left(\frac{2pq^{3/2}b^{1/2}k^{1/2}t^{\alpha}}{\Gamma(1+\alpha)}\right)\left(\frac{1}{4}\Gamma(1+\alpha)^2\right.$$

$$\left.+bp^2q^3kt^{2\alpha}\right) - \frac{1}{2}p\Big\{p^2t^{\alpha}k^{3/2}q^{9/2}b^{3/2}\left(vt^{\alpha}\Gamma(1+\beta)-x^{\beta}\Gamma(1+\alpha)\right)^2$$

$$+\frac{1}{4}vq^{3/2}b^{1/2}k^{1/2}\Gamma(1+\beta)\Gamma(1+\alpha)^2\left(-2x^{\beta}\Gamma(1+\alpha)+vt^{\alpha}\Gamma(1+\beta)\right)\Big\}\Bigg]$$

$$\Big/\left[2pq^{3/2}k^{3/2}b^{1/2}\Gamma(1+\alpha)\Gamma(1+\beta)^2\left(\frac{1}{4}p\Gamma(1+\alpha)^2+bp^2q^3kt^{2\alpha}\right)\right]\Bigg]. \tag{57}$$

Note: The solutions obtained in (56) and (57) can also be expressed in terms of Kummer functions using

$$\text{WhittakerM}(\kappa,\mu,z) = \exp(-z/2)z^{\mu+\frac{1}{2}}\text{KummerM}\left(\mu-\kappa+\frac{1}{2}, 1+2\mu, z\right), \tag{58}$$

$$\text{WhittakerW}(\kappa,\mu,z) = \exp(-z/2)z^{\mu+\frac{1}{2}}\text{KummerU}\left(\mu-\kappa+\frac{1}{2}, 1+2\mu, z\right). \tag{59}$$

3. *Subalgebra* $V = V_2 + aV_5$, $a \in \mathbb{R}$

The similarity variables

$$r = X - ap^2kT^2, \qquad B(r) = W\exp\left(aqXT - \frac{2}{3}a^2p^2qkT^3 - \frac{1}{2}apvT^2\right), \tag{60}$$

transforms (17) to

$$kp^2B_{rr} - pvB_r + aq^2rB = 0, \tag{61}$$

which finally results in

$$u(t,x) = [C_1 \text{ AiryAi}(\varphi) + C_2 \text{ AiryBi}(\varphi)]\mathrm{e}^{1/2\frac{vx^{\beta}}{\Gamma(1+\beta)k}+2/3\frac{a^2q^4p^2k(t^{\alpha})^3}{(\Gamma(1+\alpha))^3}-\frac{aq^2t^{\alpha}px^{\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)}}, \tag{62}$$

where

$$\varphi=\frac{v^2\Gamma(1+\beta)\,(\Gamma(1+\alpha))^2-4aq^2px^\beta k(\Gamma(1+\alpha))^2+4\,a^2q^4k^2p^2t^{2\alpha}\Gamma(1+\beta)}{4\,(\Gamma\,(1+\alpha))^2\,\Gamma\,(1+\beta)\,k^{4/3}p^{2/3}a^{2/3}q^{4/3}}.$$

4. *Subalgebra* $V=V_2+aV_3$, $a\in\mathbb{R}$

The symmetry V is expressed as

$$V=\partial_T+aW\partial_W \tag{63}$$

which gives the following change of variables:

$$r=X,\qquad B(r)=W\exp(-aT). \tag{64}$$

The reduced equation is

$$kp^2B_{rr}-pvB_r-aqB=0. \tag{65}$$

The solution of (15) obtained in this case is

$$u(t,x)=C_1\mathrm{e}^{\frac{\left(v+\sqrt{4qka+v^2}\right)x^\beta}{2\Gamma(1+\beta)k}+\frac{aqt^\alpha}{\Gamma(1+\alpha)}}+C_2\mathrm{e}^{\frac{\left(v-\sqrt{4qka+v^2}\right)x^\beta}{2\Gamma(1+\beta)k}+\frac{aqt^\alpha}{\Gamma(1+\alpha)}}. \tag{66}$$

5. *Subalgebra* $V=aV_3+V_6$, $a\in\mathbb{R}$

For linear combination aV_3+V_6, we have

$$r=\frac{X}{T},\qquad B(r)=WT^{1/2}\exp\left(\frac{p^2v^2T^2+q^2X^2-a-2pqvXT}{4p^2qkT}\right). \tag{67}$$

Using similarity variables (67), eq. (17) reduces to

$$4p^4k^2B_{rr}+aB=0. \tag{68}$$

In order to solve eq. (68) three cases need to be considered.

If $a>0$ then we obtain the following solution:

$$\begin{aligned}u(t,x)=&\frac{1}{\sqrt{\frac{qt^\alpha}{\Gamma(1+\alpha)}}}\left(C_1\sin\left(1/2\,\frac{\sqrt{a}x^\beta\Gamma\,(1+\alpha)}{p\Gamma\,(1+\beta)\,kqt^\alpha}\right)\right.\\&\left.+C_2\cos\left(1/2\,\frac{\sqrt{a}x^\beta\Gamma\,(1+\alpha)}{p\Gamma\,(1+\beta)\,kqt^\alpha}\right)\right)\\&\times\mathrm{e}^{1/4\left(-\frac{p^2v^2q^2(t^\alpha)^2}{(\Gamma(1+\alpha))^2}-\frac{p^2q^2\left(x^\beta\right)^2}{(\Gamma(1+\beta))^2}+a+2\,\frac{p^2q^2vx^\beta t^\alpha}{\Gamma(1+\beta)\Gamma(1+\alpha)}\right)\Gamma(1+\alpha)p^{-2}q^{-2}k^{-1}(t^\alpha)^{-1}}.\end{aligned} \tag{69}$$

For $a<0$ we find

$$\begin{aligned}u(t,x)=&\frac{1}{\sqrt{\frac{qt^\alpha}{\Gamma(1+\alpha)}}}\left(C_1\sinh\left(1/2\,\frac{\sqrt{-a}x^\beta\Gamma\,(1+\alpha)}{p\Gamma\,(1+\beta)\,kqt^\alpha}\right)\right.\\&\left.+C_2\cosh\left(1/2\,\frac{\sqrt{-a}x^\beta\Gamma\,(1+\alpha)}{p\Gamma\,(1+\beta)\,kqt^\alpha}\right)\right)\\&\times\mathrm{e}^{1/4\left(-\frac{p^2v^2q^2(t^\alpha)^2}{(\Gamma(1+\alpha))^2}-\frac{p^2q^2\left(x^\beta\right)^2}{(\Gamma(1+\beta))^2}+a+2\,\frac{p^2q^2vx^\beta t^\alpha}{\Gamma(1+\beta)\Gamma(1+\alpha)}\right)\Gamma(1+\alpha)p^{-2}q^{-2}k^{-1}(t^\alpha)^{-1}}.\end{aligned} \tag{70}$$

When $a = 0$, then eq. (68) together with (67) present

$$u(t,x) = \left(\frac{C_1 p x^{\beta} \Gamma(\alpha) \alpha + C_2 q t^{\alpha} \Gamma(\beta) \beta}{q \Gamma(1+\beta) \sqrt{\frac{q t^{\alpha}}{\Gamma(\alpha)\alpha}} t^{\alpha}} \right) \times \mathrm{e}^{1/4 \, \frac{\Gamma(1+\alpha) t^{-\alpha} \left(-pvT^2 (\Gamma(\beta))^2 \beta^2 + 2 x^{\beta} pvT q p \Gamma(\beta) \beta - x^{2\beta} p^2 q^2 \right)}{p^2 q^2 k (\Gamma(1+\beta))^2}}, \tag{71}$$

which forms a solution of (15).

6. *Subalgebra* $V = V_1 = \partial_X$

The similarity variables are

$$r = T, \qquad B(r) = W. \tag{72}$$

Using similarity variables in eq. (17) and after simplification we arrived at

$$u(t,x) = C, \tag{73}$$

where C is constant.

7. *Subalgebra* $V = V_3 = W \partial_W$

No invariant solution is obtained in this case.

4. Fractional Whitham–Broer–Kaup equations

The fractional Whitham–Broer–Kaup (FWBK) equations are [25]

$$\begin{aligned} & u_t^{\alpha} + u u_x^{\alpha} + v_x^{\alpha} + \beta u_{xx}^{2\alpha} = 0, \\ & v_t^{\alpha} + v u_x^{\alpha} + u v_x^{\alpha} - \beta v_{xx}^{2\alpha} + \gamma u_{xxx}^{3\alpha} = 0, \\ & \quad 0 < \alpha \leq 1, \quad \beta,\ \gamma \in \mathbb{R}, \quad t > 0, \quad x \in [a,b] \subset \mathbb{R}. \end{aligned} \tag{74}$$

We introduce $\frac{1}{2}\ell$ the transformations

$$\begin{aligned} & X = \frac{p x^{\alpha}}{\Gamma(1+\alpha)}, \qquad T = \frac{q t^{\alpha}}{\Gamma(1+\alpha)}, \qquad U(T,X) = u(t,x), \\ & V(T,X) = v(t,x), \qquad p, q \neq 0. \end{aligned} \tag{75}$$

Substitution of (75) in (74) results in

$$\begin{aligned} & qU_T + pUU_X + pV_X + \beta p^2 U_{XX} = 0, \\ & qV_T + pU_X V + pUV_X - \beta p^2 V_{XX} + \gamma p^3 U_{XXX} = 0. \end{aligned} \tag{76}$$

The system (76) resembles the classical Whitham–Broer–Kaup (WBK) equations which were considered by many researchers (see e.g [34–36]). The system (76) admits the following Lie symmetries for $\beta \neq 0$ and $\gamma \neq 0$:

$$\begin{aligned} \Gamma_1 &= \partial_T, \qquad \Gamma_2 = \partial_X, \\ \Gamma_3 &= pT\partial_X + q\partial_U, \\ \Gamma_4 &= T\partial_T + \frac{1}{2} X \partial_X - \frac{1}{2} U \partial_U - V \partial_V. \end{aligned} \tag{77}$$

In order to compute the independent invariant solutions of (76) we need to determine the optimal system. The optimal system of classical WBK equations based on vector field (77) given below was found in [34].

$$\Gamma_4, \qquad \Gamma_3+\Gamma_1, \qquad \Gamma_3, \qquad \Gamma_2, \qquad \Gamma_1. \tag{78}$$

1. *Subalgebra* $\Gamma = \Gamma_4 = T\partial_T + \frac{1}{2}X\partial_X - \frac{1}{2}U\partial_U - V\partial_V$

   Similarity variables

$$r = \frac{X}{\sqrt{T}}, \qquad A(r) = VT, \qquad B(r) = U\sqrt{T}, \tag{79}$$

   reduce PDE system (76) to system of third-order ODE

$$\beta p^2 B_{rr} + pBB_r + pA_r - \frac{1}{2}qrB_r - \frac{1}{2}qB = 0, \tag{80a}$$

$$\gamma p^3 B_{rrr} - \beta p^2 A_{rr} + pBA_r + pAB_r - \frac{1}{2}qrA_r - qA = 0. \tag{80b}$$

   Integrating (80a) with respect to r gives rise to

$$A(r) = \frac{1}{p}\left(\frac{1}{2}qrB - \frac{1}{2}pB^2 - \beta p^2 B_r - C_1\right) \tag{81}$$

   which on substituting in (80b) and integrating gives rise to the following nonlinear ODE:

$$4\,p^4\left(\beta^2+\gamma\right)B_{rrr} + \left(-q^2r^2 + 6qrpB - 6p^2B^2 - 4pC_1\right)B_r$$
$$-3q^2rB + 4qpB^2 + 4qC_1 = 0. \tag{82}$$

   Equation (82) does not admit any symmetry, hence it cannot be reduced further. However, it admits the polynomial solution

$$B(r) = \frac{qr}{p}. \tag{83}$$

   Substituting (83) in (81) we obtain

$$A(r) = \frac{-\beta pq + C_1}{p}. \tag{84}$$

   In terms of original variables, eqs (83) and (84) can be written as

$$u(t,x) = \frac{x^\alpha}{t^\alpha}, \qquad v(t,x) = \frac{\Gamma(1+\alpha)(-\beta pq + C_1)}{pqt^\alpha}, \tag{85}$$

   which satisfies system (74).
2. *Subalgebra* $\Gamma = \Gamma_2 + \Gamma_4$

   In this case

$$\Gamma = \partial_T + pT\partial_x + q\partial_U. \tag{86}$$

Similarity variables of Γ are

$$r = -\frac{1}{2}pT^2 + X, \qquad A(r) = V, \qquad B(r) = -qT + U. \tag{87}$$

System (76) in terms of similarity variables (87) reduces to

$$\begin{aligned} &q^2 + pBB_r + pA_r + \beta p^2 B_{rr} = 0, \\ &p(AB)_r - \beta p^2 A_{rr} + \gamma p^3 B_{rrr} = 0. \end{aligned} \tag{88}$$

The solution of (88) is

$$\begin{aligned} A(r) &= \frac{1}{p}\left(-q^2 r - \frac{p}{2}f(r)^2 - \beta p^2 \frac{d}{dr}f(r) + c_1\right), \\ B(r) &= f(r), \end{aligned} \tag{89}$$

where $f(r)$ satisfies

$$p^3(\beta^2+\gamma)\frac{\mathrm{d}^2}{\mathrm{d}r^2}f(r) + \frac{1}{2}\left(-pf(r)^3 - 2q^2 r f(r) + 2C_1 f(r) + 2C_2 p^3 \beta^2 + 2C_2 p^3 \gamma\right) = 0. \tag{90}$$

One can write the solution of (74) as

$$\begin{aligned} u(t,x) &= f(r) + \frac{q^2 t^\alpha}{\Gamma(1+\alpha)}, \\ v(t,x) &= \frac{1}{p}\left(-q^2 r - \frac{p}{2}f(r)^2 - bp^2 \frac{\mathrm{d}}{\mathrm{d}r}f(r) + C_1\right), \end{aligned} \tag{91}$$

where

$$r = -1/2\,\frac{pq^2\,(t^\alpha)^2}{(\Gamma\,(1+\alpha))^2} + \frac{px^\alpha}{\Gamma\,(1+\alpha)}$$

and $f(r)$ satisfies

$$p^3(\beta^2+\gamma)\frac{\mathrm{d}^2}{\mathrm{d}r^2}f(r) + \frac{1}{2}(-pf(r)^3 - 2q^2 r f(r) + 2C_1 f(r) + 2C_2 p^3 \beta^2 + 2C_2 p^3 \gamma) = 0. \tag{92}$$

3. *Subalgebra* $\Gamma = \Gamma_2$

The similarity variables of

$$\Gamma = pT\partial_X + q\partial_U \tag{93}$$

are

$$r = T, \qquad A(r) = V, \qquad B(r) = -\frac{qX}{pT} + U, \tag{94}$$

which transforms (76) to

$$rB_r + B = 0,$$
$$rA_r + A = 0. \tag{95}$$

Solving system (95) and then with the help of (94) and (75) we obtain

$$u(t,x) = \frac{[qx^\alpha + C_1\Gamma(1+\alpha)]}{qt^\alpha}, \qquad v(t,x) = \frac{C_2\Gamma(1+\alpha)}{qt^\alpha}, \tag{96}$$

which comprise the solution of (74).

4. *Subalgebra* $\Gamma = \Gamma_1 + \alpha\Gamma_2 = \partial_T + \alpha\partial_X$

In this case the similarity variables are

$$r = \alpha T - X, \qquad A(r) = U, \qquad B(r) = V. \tag{97}$$

Using change of variables (97), system (76) transforms to

$$\alpha q A_r - pAA_r - pB_r + \beta p^2 A_{rr} = 0,$$
$$\alpha q B_r - pA_r B - pAB_r - \beta p^2 B_{rr} - \gamma p^3 A_{rrr} = 0. \tag{98}$$

The following closed form solutions are obtained:

$$u_1(t,x) = \frac{1}{p}\left[\alpha q + \sqrt{2pC_1 + \alpha^2 q^2}\right.$$
$$\left.\times \tanh\left(\frac{1}{2}\frac{\sqrt{2pC_1+\alpha^2q^2}\left(\frac{\alpha q t^\alpha}{\Gamma(1+\alpha)} - \frac{px^\alpha}{\Gamma(1+\alpha)} + c_3\right)}{p^2\sqrt{\beta^2+\gamma}}\right)\right],$$

$$v_1(t,x) = -\frac{1}{p^2\sqrt{\beta^2+\gamma}}\left[(\sqrt{\beta^2+\gamma}+\beta)\left(pC_1 + \frac{1}{2}\alpha^2q^2\right)\right.$$
$$\left.\times\left\{\tanh^2\left(\frac{1}{2}\frac{\sqrt{2pC_1+\alpha^2q^2}\left(\frac{\alpha q t^\alpha}{\Gamma(1+\alpha)} - \frac{px^\alpha}{\Gamma(1+\alpha)} + c_3\right)}{p^2\sqrt{\beta^2+\gamma}}\right) - 1\right\}\right],$$
$$\beta^2 + \gamma > 0, \tag{99}$$

$$u_2(t,x) = \frac{1}{p}\left[\alpha q - \sqrt{2pC_1 + \alpha^2 q^2}\right.$$
$$\left.\times \tanh\left(\frac{1}{2}\frac{\sqrt{2pC_1+\alpha^2q^2}\left(\frac{\alpha q t^\alpha}{\Gamma(1+\alpha)} - \frac{px^\alpha}{\Gamma(1+\alpha)} + c_3\right)}{p^2\sqrt{\beta^2+\gamma}}\right)\right],$$

$$v_2(t,x) = \frac{1}{p^2\sqrt{\beta^2+\gamma}}\left[(-\sqrt{\beta^2+\gamma}+\beta)\left(pC_1 + \frac{1}{2}\alpha^2q^2\right)\right.$$
$$\left.\times\left\{\tanh^2\left(\frac{1}{2}\frac{\sqrt{2pC_1+\alpha^2q^2}\left(\frac{\alpha q t^\alpha}{\Gamma(1+\alpha)} - \frac{px^\alpha}{\Gamma(1+\alpha)} + c_3\right)}{p^2\sqrt{\beta^2+\gamma}}\right) - 1\right\}\right],$$
$$\beta^2 + \gamma > 0, \tag{100}$$

$$u_3(t,x)=\frac{1}{p}\left[\alpha q-i\sqrt{2pC_1+\alpha^2q^2}\right.$$
$$\left.\times\tanh\left(\frac{1}{2}\frac{\sqrt{2\,pC_1+\alpha^2q^2}\left(\frac{\alpha\,qt^\alpha}{\Gamma(1+\alpha)}-\frac{px^\alpha}{\Gamma(1+\alpha)}+c_3\right)}{p^2\sqrt{-(\beta^2+\gamma)}}\right)\right],$$

$$v_3(t,x)=-\frac{1}{p^2\sqrt{-(\beta^2+\gamma)}}\left[(i\beta-\sqrt{-(\beta^2+\gamma)})\left(pC_1+\frac{1}{2}\alpha^2q^2\right)\right.$$
$$\left.\times\left\{\tanh^2\left(\frac{1}{2}\frac{\sqrt{2\,pC_1+\alpha^2q^2}\left(\frac{\alpha\,qt^\alpha}{\Gamma(1+\alpha)}-\frac{px^\alpha}{\Gamma(1+\alpha)}+c_3\right)}{p^2\sqrt{-(\beta^2+\gamma)}}\right)+1\right\}\right],$$
$$\beta^2+\gamma<0, \tag{101}$$

$$u_4(t,x)=\frac{1}{p}\left[\alpha q+i\sqrt{2pC_1+\alpha^2q^2}\right.$$
$$\left.\times\tanh\left(\frac{1}{2}\frac{\sqrt{2\,pC_1+\alpha^2q^2}\left(\frac{\alpha\,qt^\alpha}{\Gamma(1+\alpha)}-\frac{px^\alpha}{\Gamma(1+\alpha)}+c_3\right)}{p^2\sqrt{-(\beta^2+\gamma)}}\right)\right],$$

$$v_4(t,x)=\frac{1}{p^2\sqrt{-(\beta^2+\gamma)}}\left[(i\beta+\sqrt{-(\beta^2+\gamma)})\left(pC_1+\frac{1}{2}\alpha^2q^2\right)\right.$$
$$\left.\times\left\{\tanh^2\left(\frac{1}{2}\frac{\sqrt{2\,pC_1+\alpha^2q^2}\left(\frac{\alpha\,qt^\alpha}{\Gamma(1+\alpha)}-\frac{px^\alpha}{\Gamma(1+\alpha)}+c_3\right)}{p^2\sqrt{-(\beta^2+\gamma)}}\right)+1\right\}\right],$$
$$\beta^2+\gamma<0. \tag{102}$$

One can easily verify that the solutions obtained above satisfy FWBK equations (74).

5. Conclusion

It is a tedious task to solve fractional differential equations due to limited number of methods available in the literature. One can use appropriate transformations to convert fractional differential equations to ordinary/partial differential equations and these can be solved by various methods that are only compatible with ordinary/partial differential equations. The inverse transformations then yield the exact solutions of original fractional differential equations. This technique is successfully applied in this paper, to derive analytical solutions of FADE and FWBK equations.

Using transformations (6), the fractional advection dispersion equation (FADE) was transformed to the second-order PDE (7). The Lie symmetries and their adjoint representation groups of transformed FADE (7) were constructed and then utilized to find optimal system of one-dimensional subalgebras. The invariant solutions using subalgebras for each case of (7) were derived which in terms of original variables constituted the solutions of FADE. A similar analysis was carried out to obtain exact solutions of FWBK equations. The similarity transformations to solve reduced FADE and FWKB equations were obtained in a systematic way using the symmetry methods which were more general and yielded independent solutions, whereas the other methods developed to find exact solutions for fractional differential equations were restricted to certain class of equations. This study points out new ways of finding analytical solutions of fractional differential equations.

Appendix

Lie symmetries

A computer package SADE [37] which works in MAPLE environment is used to compute Lie symmetries of partial differential equation (17). In MAPLE we use the SADE routines

```
with(SADE):
eq:=diff(W(T,X),T)+v*p*diff(W(T,X),X)-k*p^2*diff(W(T,X),X,X)=0;
liesymmetries(eq,[W(T,X)]);
```

The above routines generate Lie symmetries of the partial differential equation (17) labelled in (18).

Kummer function

KummerM is a Kummer function of the first kind which is a generalized hypergeometric function

$$\mathrm{KummerM}(\mu,\nu,z)=\sum_{n=0}^{\infty}\frac{\mu^{(n)}z^n}{\nu^{(n)}n!}={}_1F_1(\mu;\nu;z), \tag{103}$$

where

$$\begin{aligned}\mu^{(0)}&=1,\\ \mu^{(n)}&=\mu(\mu+1)(\mu+2)\cdots(\mu+n-1).\end{aligned} \tag{104}$$

$\nu^{(n)}$ in (103) can be defined in a similar fashion. The Kummer function of the second type KummerU can be defined in terms of KummerM as

$$\begin{aligned}\mathrm{KummerU}(\mu,\nu,z)=&\frac{\Gamma(1-\nu)}{\Gamma(\mu-\nu+1)}\,\mathrm{KummerM}(\mu,\nu,z)\\ &+\frac{\Gamma(\nu-1)}{\Gamma(\mu)}z^{1-\nu}\,\mathrm{KummerM}(\mu,\nu,z).\end{aligned} \tag{105}$$

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