Opportunities and problems in determining proton and light nuclear radii

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Abstract. We briefly review the so-called 'proton puzzle', i.e., the disagreement of the newly extracted value of the proton charge radius r_p from muonic hydrogen spectroscopy with other extractions, its possible significance and related problems. After describing the conventional theory to extract the proton radius from atomic spectroscopy we focus on a novel consistent approach based on the Breit equation. With this new tool, we confirm that the radius has indeed become smaller compared to the value extracted from scattering experiments, but the existence of different theoretical approaches casts some doubt on the accuracy of the new value. Precision measurements in atomic physics do provide the opportunity to extract light nuclear radii but the accuracy is limited by the methods of incorporating the nuclear structure effects.

1. Introduction

Scientific discoveries can take and have taken in the past different paths and routes: serendipity, advancement in technology (e.g., by increasing the energy), and precision measurements, to mention a few. As it appears now, new breakthroughs in physics to which we got so used to in the twentieth century are hard to get and a permanent increase in energy (e.g., with particle accelerators) has its physical and financial limits. A seemingly not understood anomaly of something which we thought to have mastered a long time ago is then a welcome opportunity especially when high precision is involved. Indeed, the ubiquitous hydrogen atom has turned into a precision instrument in the last 15 or so years. The transition energies for example, are being measured with unprecedented precision surpassing the theoretical predictions and opening the door for speculations (of missing contributions or new physics). A recent example is the precision measurement of the $1S$–$2S$ frequency in hydrogen which was measured [1] to be $f_{1S-2S} = 2466061413187018(11)$ with a relative uncertainty of 4.5×10^{-15}. More to

the point of the present work, the Lamb shift in muonic hydrogen has been recently measured with a high precision allowing to relate it to contributions such as the finite size corrections (FSC) due to the structure of the proton [2]. These corrections can often be parametrized in terms of the mean radius of the proton, r_p, defined by the expectation value $r_p^2 = \int \rho_C(\mathbf{r}) r^2 d\mathbf{r}$, where ρ_C is the charge density of the proton. The charge density of the proton, $\rho_C(\mathbf{r})$, can be determined from experiments as the Fourier transform of the measured Sachs form factor $G_E(\mathbf{q}^2)$. Such a relation is however valid only in the Breit frame (see §2 and 3 for a more detailed discussion). The extraction of the proton radius from the Lamb shift performed by Pohl *et al* [3] led to the surprising finding that the extracted value of $r_p = 0.84184(67)$ fm was much smaller than the world average CODATA value of 0.8768(69) fm [4], relying on scattering experiments and, partly, also on measurements in the standard hydrogen atom. There are some extractions from hydrogen spectroscopy [5] whose central value is smaller than the average CODATA value to which they contribute. However, the hydrogenic spectroscopy and *ep* scattering yield results for r_p that have commensurate error bars. It seems then that it is the high precision of the 'new' proton radius that makes it incompatible with the CODATA value and most of the radii extracted from scattering experiments. In this context, it is worth noticing that to be able to extract the radius from a measured number, one indeed needs to provide a theoretical input. We have some confidence that quantum electrodynamics (QED) and, in general, the electroweak part of the Standard Model are capable of delivering precise numbers. However, in nuclear physics the model dependence of a calculation weighs more than in the electroweak sector and this uncertainty could make the error of the 'new' proton radius larger than previously thought. We do not know whether the 'shrunk proton' is a harbinger of new physics or not, but it does make sense to test the extracted value against any model dependence. This is the motivation of our work here. We present another consistent theoretical approach (for the more commonly used approaches, see [6]) to extract the proton radius from atomic spectroscopy. The calculation confirms qualitatively that the 'proton has shrunk', but, of course, does not give the same numbers which demonstrates the model dependence.

The shrunk proton gave rise to numerous proposals involving even exotic options such as quantum gravity, large extra dimensions and others such as non-identical protons [7] in order to fix the problem. In the present work we do not attempt to review all these works but rather concentrate on the methods of extracting the proton radius. Before proceeding, we would like to remind the reader that we had and still have a similar kind of problem at hand. The anomalous magnetic moment of the muon has been measured with a very high accuracy. This measured value does not seem to be ompatible with the theory. The theoretical uncertainty lies again in the hadronic sector [8].

2. Structure and form factors

The electromagnetic probe of a nucleon is given by its electromagnetic current density which couples to the photon, i.e.,

$$\langle p'|j_\mu(0)|p\rangle = e\bar{u}(p')\Gamma_\mu(q)u(p), \tag{1}$$

where the most general vertex Γ_μ is known to contain, in general, four form factors depending on the square of the four-momentum transfer q^2 [9]. However, two of these

form factors violate discrete symmetries and can be neglected in the first approximation. The contribution of the electric dipole form factor to any process is certainly small as it violates the time-reversal symmetry. The contribution of the parity violating anapole form factor to hydrogen spectroscopy might however be an interesting undertaking. The vertex in terms of the F_1 and F_2 form factors can be written as follows:

$$\Gamma_\mu = F_1(q^2)\gamma_\mu + \frac{\sigma_{\mu\nu}q^\nu}{2m_N}F_2(q^2).$$ (2)

The form factors of the nucleons are subject to parametrizations from experiments. The earliest such experiments used elastic electron–proton scattering. The differential cross-section in the one-photon-exchange approximation for *e–p* scattering is given in the laboratory frame as [2]

$$\frac{d\sigma}{d\Omega_e} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \frac{E_e}{E_{beam}} \Big[F_1^2(q^2)$$
$$+\tau\left(F_2^2(q^2) + 2(F_1(q^2) + F_2(q^2))^2 \tan^2(\theta_e/2)\right)\Big],$$ (3)

where $\tau = q^2/4m_p^2$ and $E_e/E_{beam} = (1+\tau)^{-1}$ is the recoil factor. F_1 and F_2 are related to the so-called Sachs form factors by $G_E = F_1 - \tau F_2$ and $G_M = F_1 + F_2$. To extract the neutron form factors one usually uses experiments with deuterium relying then on theoretical skills to separate the neutron part. The Sachs form factors have a significance which is directly associated with the charge density of the nucleon and hence with the definition of the charge radius of the proton. It can be shown that in the Breit frame, G_E is the Fourier transform of the charge density ρ_C (putting $q_0 = 0$), whereas the Fourier transform of G_M gives the magnetization density ρ_M. However, in the literature there is no general agreement regarding how one should relate the densities with the form factors (and hence, eventually with the radius). Friar *et al* [10] advocate $\tilde{G}_E^2(q^2) = G_E^2(q^2)(1+\tau)^{-1}$ as the correct Fourier transform of the charge density and in [11], the author suggests the use of $\tilde{G}_E^2(q^2) = G_E^2(q^2)(1+\tau)^\lambda$ such that λ eventually appears as a parameter in the definition of the radius. The author obtains the root mean square (rms) radius, $\langle r_p^2 \rangle_\lambda^{1/2} = [\xi^2 - (3\lambda/2m_p^2)]^{1/2}$, where $\xi^2 = -6(d\ln G_E(q^2)/dq^2)_{q^2\to 0}$. One should bear in mind such uncertainties when comparing the results of nuclear radii emerging from scattering data and atomic spectroscopy. The radius from scattering experiments will always be calculated using the charge density. A second fact worth mentioning here is the existence of two form factors or two densities (charge and magnetic) and not one. Characterizing the finite size of the proton by only one number, the proton charge radius, must be a simplification or an artifact of the parametrization of the form factors. Indeed, if in parametrizations of both form factors only one parameter occurs, then this parameter can be related to r_p. However, in general, all formalisms using only one density (mostly the charge density) are necessarily incomplete. This happens often in the analysis of FSC in atomic spectroscopy. To assume that the effects of the second form factor (G_M) are negligibly small is permissible in an order-of-magnitude estimate, but should not be an *a priori* assumption in claims of high precision. What is called for is a formalism for the FSC which does not have this deficit and includes both form factors with their effects over the full range of *q* or *r*. Such a formalism [12,13] is presented in §5. Nevertheless, a simple dipole parametrization with only one parameter

is used in order to be comparable with other spectroscopic results in the literature which contain only powers of r_p for the FSC. The dipole parametrization reads: $G_D(Q^2) = 1/(1 + Q^2/m^2)^2 \simeq G_E^p(Q^2) \simeq G_M^p(Q^2)/\mu_p$, where $(1 + \kappa_p) = \mu_p = 2.793$ is the proton's magnetic moment, Q^2 is the three-momentum transfer squared and m is the dipole parameter (connected to the radius r_p). The Breit potential (eq. (14)) which appears in §5 is derived using the standard non-relativistic approximation of $q^2 = -\mathbf{q}^2$ ($|\mathbf{q}| = Q$ above).

3. Scattering vs. spectroscopy

The size of the proton can be determined by two completely different means: precision spectroscopy of atomic hydrogen and elastic electron–proton scattering. Thus, the definition of the radius of the proton depends on the methods used for extracting it. In the former approach, the measured difference between energy levels of the hydrogen atom is compared with the theoretical one, leaving the proton radius as an adjustable parameter. The Dirac eigenvalue for an electron in a Coulomb field is modified by several corrections such as relativistic recoil, vacuum polarization, two- and three-photon corrections and the fact that the proton is not point-like, among others. The proton radius thus enters the theoretical expression via the finite-size corrections to the energy levels (which, as we shall see, themselves depend on the theoretical approach used to evaluate them). For example, the proton radius in [3] was determined from an accurate measurement of the transition energy, $\Delta(=E_{2P_{3/2}}^{f=2} - E_{2S_{1/2}}^{f=1}) = 206.2949(32)$ meV in muonic hydrogen and its comparison with the expression, $\Delta = 209.9779(49) - 5.2262 r_p^2 + 0.0347 r_p^3$ meV, derived after including several quantum electrodynamic (QED) as well as finite-size corrections.

Determination of the proton radius from elastic scattering experiments basically relies on a good determination of the nucleon form factors from the measured cross-sections. The main limitation here arises from the errors involved in the determination of the form factors. Having determined the so-called Sachs form factors from e–p cross-sections, one can relate the electric Sachs form factor $G_E^p(Q^2)$ to the Fourier transform of the charge density distribution of the proton as explained in the previous section. For a proton charge density of $\rho_C(r)$ related to $G_E^p(Q^2)$ by $G_E^p(Q^2) = (4\pi/Q) \int r\rho_C(r) \sin(Qr) \mathrm{d}r$, on expanding $\sin(Qr)$ in the integrand it is easy to show that

$$\left. \frac{\mathrm{d}G_E^p(Q^2)}{\mathrm{d}Q^2} \right|_{Q^2=0} = -\frac{\langle r_p^2 \rangle}{6}. \tag{4}$$

Knowing the charge distribution $\rho_C(Q^2)$, determined from the measured electric form factor $G_E^p(Q^2)$ one could also evaluate the root mean square (rms) radius, r_p, from the proton charge distribution as

$$r_p^2 = \langle r_p^2 \rangle = \int_0^\infty r^2 \, \rho_C(r) \, 4\pi r^2 \, \mathrm{d}r. \tag{5}$$

The rms radius thus determined, however, depends on the density distribution at large r, where it is poorly known. The difficulties arising due to the upper cut-off radius in
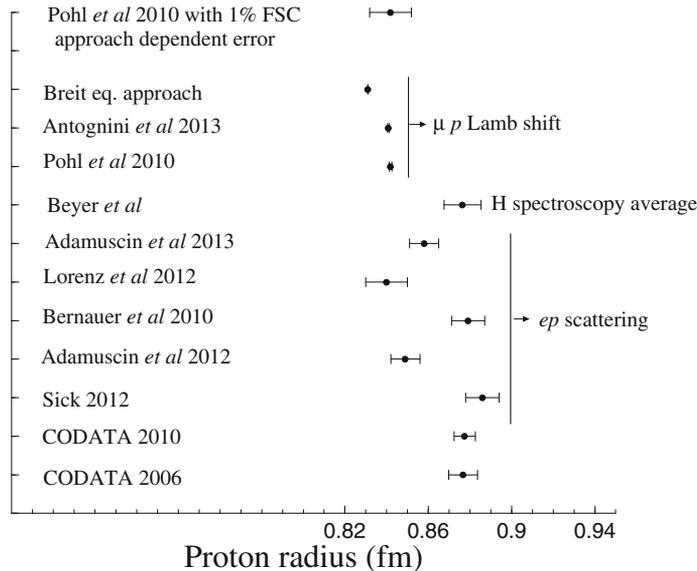
Figure 1. Comparison of the proton radius values extracted from the Lamb shift [3,15], CODATA values [4,16] and some recent analyses [14,17] of the *ep* scattering data. The hydrogen spectroscopy average is from [5]. The top-most point displays the result of [3] with a possible 1% error due to dependence on the FSC approach used [13].

the above integral are discussed in [14]. Figure 1 presents a comparison of the proton radius values extracted from *e–p* scattering and hydrogen spectroscopy. The point labelled Breit eq. extracts the proton radius in the same manner as in Pohl *et al* [3] from the muonic Lamb shift and using the same QED corrections. The difference lies only in the approach used for the proton FSC. One can see that the difference in approach can introduce a sizable error.

4. Conventional approaches

The fact that the proton inside the hydrogen atom is not point-like can be incorporated using different approaches ranging from rather simple ones to those based on quantum field theory. In a very simple approach one can replace the proton classically by a spherical charge distribution of radius R and write the Coulomb potential $V_C(r)$ as, $V_C(r) = -e^2(1/2R)[3 - (r/R)^2]$ for $r < R$ and $-e^2/r$ for $r > R$. However, with the electron wave function being everywhere, a proper calculation must be performed quantum mechanically. The correction to the energy level can be evaluated using the correction to the point Coulomb potential as

$$\delta E = e \int d\mathbf{r} \, |\Psi(\mathbf{r})|^2 \left[V(r) + \frac{e}{4\pi r} \right], \tag{6}$$

where the quantity in square brackets is the correction, $\delta V(r) = V(r) - (-e/r)$. Approximating the wave function by its value at the origin and after some manipulations and using the Poisson's equation, $\nabla^2(\delta V(r)) = -\rho_C(r)$, one gets [18]

$$\Delta E \simeq \frac{e^2}{6} |\Psi(0)|^2 \langle r_p^2 \rangle \tag{7}$$

i.e., a finite-size correction which depends on the mean square radius of the proton.

The theoretical approach used to calculate the proton FSC in [3] is based on the work by Friar [19]. The FSC are derived to depend on the mean square radius of the proton. This, however, is made possible by approximating the hydrogen atom wave function by its value at the origin. Besides, it is only the charge density distribution (and not the magnetization density distribution) which appears in the formalism. Friar [19] begins by splitting the Coulomb potential into a point part and a finite size part ΔV_C given by

$$\Delta V_C(r) = -Z\alpha \int d^3 s \, \rho_C(s) \left(\frac{1}{|\mathbf{r} - \mathbf{s}|} - \frac{1}{r} \right). \tag{8}$$

Using perturbation theory and assuming a constant atomic wave function, Φ_n, the energy correction is found to be

$$\Delta E \simeq \frac{2\pi Z\alpha}{3} |\Phi_n(0)|^2 \left(\langle r^2 \rangle - \frac{Z\alpha\mu}{2} \langle r^3 \rangle_{(2)} + \cdots \right). \tag{9}$$

The second term is the third moment of the convoluted proton charge density and is defined as: $\langle r^3 \rangle_{(2)} = \int d^3 r \, r^3 \rho_{(2)}(r)$, where the convoluted charge density $\rho_{(2)}(r)$ is given by $\rho_{(2)}(r) = \int d^3 z \, \rho_C(|\mathbf{z} - \mathbf{r}|) \, \rho_C(r)$. The third moment can be related to the mean square radius as [20]

$$\langle r^3 \rangle_{(2)} = \frac{48}{\pi} \int_0^\infty \frac{dQ}{Q^4} \left(G_E^2(Q^2) - 1 + \frac{Q^2}{3} \langle r^2 \rangle \right). \tag{10}$$

The FSC correction used in [3] which gives a Lamb shift of the form $\Delta E_{LS} = A + B\langle r^2 \rangle + C\langle r^2 \rangle^{3/2}$ is obtained by using a dipole form factor to connect the third moment with the mean square radius. The use of a dipole form factor implies the utilization of a certain input radius to fit the radius appearing in the expression of the Lamb shift. Though the coefficient appearing before $\langle r^2 \rangle^{3/2}$ is small, it is not so clear as to how much error would be introduced on changing the parametrization of the form factors.

In [15] a better method of extracting the radius from Lamb shift (LS) was presented. Antognini *et al* [15] measured two 2S–2P transitions ($2S_{1/2}^{f=0} - 2P_{3/2}^{f=1}$ and $2S_{1/2}^{f=1} - 2P_{3/2}^{f=2}$) and independently deduced the experimental Lamb shift $\Delta E_{LS} = \Delta E_{2P_{1/2} - 2S_{1/2}}$ and the 2S hyperfine splitting from their linear combinations. The theoretical LS was written as $\Delta E_{LS}^{th} = 206.0336(15) - 5.2275(10)r_p^2 + \Delta E_{TPE}$ with the two-photon exchange (TPE) effect given by $\Delta E_{TPE} = 0.0332(20)$ meV. A comparison of the theoretical and experimental Lamb shift led to the proton radius, $r_p = 0.84087(39)$.

5. Breit equation for hydrogenic atoms

The Breit equation basically involves a method for deriving coordinate space potentials from quantum field theory (QFT) [21,22]. This method goes back to a relation between

the non-relativistic scattering amplitude and the potential which is obviously the Fourier transform. More explicitly, this can be done in the following steps:

(1) Start with a one-photon exchange diagram for two spin-1/2 particles.
(2) Write down the elastic scattering amplitude expanded in powers of $1/c^2$ and depending on the three momentum transferred $\mathbf{q}$. The amplitude in terms of two-component spinors $w_{e,p}$ is typically of the form, $M_{fi} = -2m_e \cdot 2m_p(w_e'^* w_p'^*)\hat{U}(\mathbf{p}_e, \mathbf{p}_p, \mathbf{q})(w_e w_p)$.
(3) The Fourier transform of $\hat{U}(\mathbf{p}_e, \mathbf{p}_p, \mathbf{q})$ is the potential $\hat{U}(\hat{\mathbf{p}}_e, \hat{\mathbf{p}}_p, \mathbf{r})$:

$$\hat{U}(\hat{\mathbf{p}}_e, \hat{\mathbf{p}}_p, \mathbf{r}) = \int e^{i\mathbf{q}\cdot\mathbf{r}} \, \hat{U}(\mathbf{p}_e, \mathbf{p}_p, \mathbf{q}) \frac{\mathrm{d}^3 q}{(2\pi)^3}, \tag{11}$$

where in the centre of mass system we can identify $\hat{\mathbf{p}}_e = -\hat{\mathbf{p}}_p = -i\nabla$.

Since the potential in coordinate space is a Fourier transform of the scattering amplitude over the whole q-space, the external particles (proton and electron) in the Feynman diagram are not necessarily on shell (as should be the case in bound states like the hydrogen atom). The standard result for the potential in momentum space can be found in previous works [21]. The various terms in the potential correspond to the Coulomb, Darwin, fine, hyperfine structure terms and retarded potentials. In [12] a mild extension of the above formalism to include the electromagnetic form factors at the proton vertex in the e–p scattering amplitude was presented. Everyone of the above-mentioned terms (fine, hyperfine etc.) thus receives its FSC in a natural way depending on the form factors. The resulting e–p interaction potential along with the first-order time-independent perturbation theory was used to evaluate the corrections to the energy levels in the hydrogen atom [13]. Applying these corrections to the energy levels measured in [3], a new expression for the difference in energies, $\Delta = E_{2P_{3/2}}^{f=2} - E_{2S_{1/2}}^{f=1}$, depending on the proton radius r_p was obtained. The form factors were assumed to be of a dipole form to facilitate analytic calculations which would provide a better insight into the problem. The expression thus derived,

$$\Delta = 209.16073 + 0.11388r_p - 4.3029r_p^2 + 0.020585r_p^3 \, \text{meV} \tag{12}$$

had an additional term dependent on r_p and different coefficients of the r_p^2 and r_p^3 terms as compared to the result in [3], namely

$$\Delta^{\text{Pohl}\,et\,al} = 209.9779(49) - 5.2262r_p^2 + 0.0347r_p^3 \, \text{meV}. \tag{13}$$

Details regarding the reasons behind the differences in the two results can be found in [13]. However, the main difference becomes clear if we look at the Breit potential with form factors, namely [13]

$$\hat{U}(\mathbf{p}_X, \mathbf{p}_p, \mathbf{q}) = 4\pi e^2 \left[F_1^X F_1^p \left(-\frac{1}{\mathbf{q}^2} + \frac{1}{8m_X^2 c^2} + \frac{1}{8m_p^2 c^2} \right. \right.$$

$$\left. + \frac{i\boldsymbol{\sigma}_p \cdot (\mathbf{q} \times \mathbf{p}_p)}{4m_p^2 c^2 \mathbf{q}^2} - \frac{i\boldsymbol{\sigma}_X \cdot (\mathbf{q} \times \mathbf{p}_X)}{4m_X^2 c^2 \mathbf{q}^2} + \frac{\mathbf{p}_X \cdot \mathbf{p}_p}{m_X m_p c^2 \mathbf{q}^2} \right.$$

$$-\frac{(\mathbf{p}_X \cdot \mathbf{q})(\mathbf{p}_\mathrm{p} \cdot \mathbf{q})}{m_X m_\mathrm{p} c^2 \mathbf{q}^4} - \frac{i\boldsymbol{\sigma}_\mathrm{p} \cdot (\mathbf{q} \times \mathbf{p}_X)}{2m_X m_\mathrm{p} c^2 \mathbf{q}^2} + \frac{i\boldsymbol{\sigma}_X \cdot (\mathbf{q} \times \mathbf{p}_\mathrm{p})}{2m_X m_\mathrm{p} c^2 \mathbf{q}^2}$$

$$+ \frac{\boldsymbol{\sigma}_X \cdot \boldsymbol{\sigma}_\mathrm{p}}{4m_X m_\mathrm{p} c^2} - \frac{(\boldsymbol{\sigma}_X \cdot \mathbf{q})(\boldsymbol{\sigma}_\mathrm{p} \cdot \mathbf{q})}{4m_X m_\mathrm{p} c^2 \mathbf{q}^2} \Bigg) + F_1^X F_2^\mathrm{p} \left(\frac{1}{4m_\mathrm{p}^2 c^2} + \frac{i\boldsymbol{\sigma}_\mathrm{p} \cdot (\mathbf{q} \times \mathbf{p}_\mathrm{p})}{2m_\mathrm{p}^2 c^2 \mathbf{q}^2} \right.$$

$$\left. - \frac{i\boldsymbol{\sigma}_\mathrm{p} \cdot (\mathbf{q} \times \mathbf{p}_X)}{2m_X m_\mathrm{p} c^2 \mathbf{q}^2} - \frac{(\boldsymbol{\sigma}_X \cdot \mathbf{q})(\boldsymbol{\sigma}_\mathrm{p} \cdot \mathbf{q})}{4m_X m_\mathrm{p} c^2 \mathbf{q}^2} + \frac{\boldsymbol{\sigma}_X \cdot \boldsymbol{\sigma}_\mathrm{p}}{4m_X m_\mathrm{p} c^2} \right)$$

$$+ F_2^X F_1^\mathrm{p} \left(\frac{1}{4m_X^2 c^2} - \frac{i\boldsymbol{\sigma}_X \cdot (\mathbf{q} \times \mathbf{p}_X)}{2m_X^2 c^2 \mathbf{q}^2} + \frac{i\boldsymbol{\sigma}_X \cdot (\mathbf{q} \times \mathbf{p}_\mathrm{p})}{2m_X m_\mathrm{p} c^2 \mathbf{q}^2} - \frac{(\boldsymbol{\sigma}_X \cdot \mathbf{q})(\boldsymbol{\sigma}_\mathrm{p} \cdot \mathbf{q})}{4m_X m_\mathrm{p} c^2 \mathbf{q}^2} \right.$$

$$\left. + \frac{\boldsymbol{\sigma}_X \cdot \boldsymbol{\sigma}_\mathrm{p}}{4m_X m_\mathrm{p} c^2} \right) + F_2^X F_2^\mathrm{p} \left(\frac{\boldsymbol{\sigma}_X \cdot \boldsymbol{\sigma}_\mathrm{p}}{4m_X m_\mathrm{p} c^2} - \frac{(\boldsymbol{\sigma}_X \cdot \mathbf{q})(\boldsymbol{\sigma}_\mathrm{p} \cdot \mathbf{q})}{4m_X m_\mathrm{p} c^2 \mathbf{q}^2} \right) \Bigg], \tag{14}$$

where $X = e$ or μ. The Breit potential is a true two-body interaction potential where the masses, momenta and spins of the two particles enter in a symmetric way. The Fourier transform of the first term, $-4\pi e^2 F_1^\mathrm{p} F_1^X / \mathbf{q}^2$, namely the Coulomb potential, $V_\mathrm{C}(r)$, with form factors is given as $V_\mathrm{C}(r) = V_\mathrm{C} + \Delta V_\mathrm{C}$, i.e.,

$$V_\mathrm{C}(r) = -\frac{e^2}{r} \left[1 - \left(1 + \frac{\kappa_\mathrm{p}}{(1-k^2)^2} \right) \mathrm{e}^{-mr} \right.$$

$$\left. - \left(1 + \frac{\kappa_\mathrm{p}}{(1-k^2)} \right) \frac{m}{2} r \mathrm{e}^{-mr} + \frac{\kappa_\mathrm{p}}{(1-k^2)^2} \mathrm{e}^{-mkr} \right], \tag{15}$$

where $k = 2m_\mathrm{p}/m$ with m_p being the mass of the proton and m the parameter entering the dipole form factor. If, instead of taking just the Coulomb potential in momentum space, we decide to club one of the Darwin terms with it, we find

$$\tilde{V}_\mathrm{C}(\mathbf{q}) = 4\pi e^2 \left[F_1^X F_1^\mathrm{p} \left(-\frac{1}{\mathbf{q}^2} \right) + F_1^X F_2^\mathrm{p} \left(\frac{1}{4m_\mathrm{p}^2 c^2} \right) \right]$$

$$= -4\pi\alpha \left[\frac{G_\mathrm{E}^p(\mathbf{q}^2)}{\mathbf{q}^2} \right]. \tag{16}$$

Note that this potential now depends only on the electric form factor and is hence connected to the charge density distribution. Taking the Fourier transform of this potential and using first-order perturbation theory leads to a result (for the r_p^2 term) similar to that used in [3]. However, taking into account the other Darwin terms in the above potential changes the coefficients appearing in the final energy difference expression [13]. The term in r_p^3 using the Breit potential method cannot be directly compared with that in [3]. The r_p^3 term in [3] arises from the third moment of the convoluted proton charge density. Including such a term with the proton form factor squared in the Breit potential would be possible with a two-photon exchange calculation which would however lead to a correction one order higher in α in this approach and hence would be quite small. The Breit potential has been evaluated with only one photon exchange diagrams. Before summarizing the discussions, let us briefly examine the root of the difference between the hyperfine correction using the approach in [13] and the one used in [3].

6. The hyperfine difference

In [12] a discrepancy between the finite-size correction to the hyperfine structure as eval-uated in [12] and using the standard Zemach procedure [23] was noticed. In this section we analyse the similarities and the differences between the two procedures and mention *a priori* that the difference lies in the way of using time-independent perturbation theory. The finite-size correction to the hyperfine structure (hfs) was evaluated in [12,13] using the appropriate part of eq. (14). For the case $l = 0$ and noting that the Sachs form factor $G_{\mathrm{M}}(q^2) = F_1(q^2) + F_2(q^2)$,

$$\hat{U}(\mathbf{q})_{\mathrm{hfs}} = \pi\alpha \left[\frac{(\boldsymbol{\sigma}_X \cdot \boldsymbol{\sigma}_{\mathrm{p}})}{m_X\, m_{\mathrm{p}}c^2} - \frac{(\boldsymbol{\sigma}_X \cdot \mathbf{q}) \cdot (\boldsymbol{\sigma}_{\mathrm{p}} \cdot \mathbf{q})}{m_X\, m_{\mathrm{p}}c^2\mathbf{q}^2} \right] (1 + \kappa_\mu)\, G_{\mathrm{M}}^{\mathrm{p}}(\mathbf{q}^2), \qquad (17)$$

where $X = e$ or μ. A Fourier transform of this potential $\hat{V}_{\mathrm{hfs}}$ is then used in the evalu-ation of the correction to the hyperfine energy levels using first-order time-independent perturbation theory. Thus the correction to the hyperfine level is

$$\Delta E = \int |\Phi_{\mathrm{C}}(\mathbf{r})|^2\, \hat{V}_{\mathrm{hfs}}(\mathbf{r})\, \mathrm{d}\mathbf{r}, \qquad (18)$$

where $\Phi_{\mathrm{C}}(\mathbf{r})$ is the unperturbed hydrogen atom wave function. The spin operators are included in the definition of $\hat{V}_{\mathrm{hfs}}$.

Let us now examine eq. (2.3) of [23], namely,

$$\Delta E = -\frac{2}{3}\mu_1\, \mu_2\, \langle \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \rangle \int |\Phi(\mathbf{r})|^2\, f_m(\mathbf{r})\, \mathrm{d}\mathbf{r}, \qquad (19)$$

(where $f_m(\mathbf{r})$ is the Fourier transform of $G_{\mathrm{M}}(\mathbf{q}^2)$) which leads to the Zemach cor-rection, $\Delta E = \Delta E_0(1 - 2m_1\alpha \int f_e(\mathbf{u})|\mathbf{u} - \mathbf{r}|f_m(\mathbf{r})\mathrm{d}\mathbf{r})$. Whereas $\Phi_{\mathrm{C}}(\mathbf{r})$ in (18) is a solution of the $1/r$ Coulomb potential, $\Phi(\mathbf{r})$ is the solution of the potential which includes finite-size corrections to the Coulomb potential and is given in [23] as, $\Phi(\mathbf{r}) = \Phi_{\mathrm{C}}(\mathbf{r}) + m_1\alpha\Phi_{\mathrm{C}}(0) \int f_e(\mathbf{u})|\mathbf{u} - \mathbf{r}|\mathrm{d}\mathbf{u}$. After some algebraic manipulations, it is straight-forward to see that eqs (18) and (19) give the same result provided we replace Φ_{C} by Φ in (18). Thus, the difference in the hyperfine corrections obtained from (18) and (19) lies in the different usage of the unperturbed wave function in the energy correction. Had the authors in [12,13] used $\Phi(\mathbf{r})$ as their unperturbed wave function, they would have arrived at exactly the same hyperfine correction as that obtained using the Zemach radius. In other words, in [13], the total Hamiltonian is taken as $H = H_0 + H_{\mathrm{C}}^{\mathrm{FF}} + H_{\mathrm{hfs}}^{\mathrm{FF}}$ with the unper-turbed H_0 containing the $1/r$ Coulomb potential, $H_{\mathrm{C}}^{\mathrm{FF}}$, the FSC correction to the Coulomb potential and $H_{\mathrm{hfs}}^{\mathrm{FF}}$ the hyperfine interaction with form factors leading to the energy correc-tion in first-order perturbation theory as $\Delta E = \langle \Phi_{\mathrm{C}}|H_{\mathrm{C}}^{\mathrm{FF}}|\Phi_{\mathrm{C}}\rangle + \langle \Phi_{\mathrm{C}}|H_{\mathrm{hfs}}^{\mathrm{FF}}|\Phi_{\mathrm{C}}\rangle$. In [23] however, one finds $H = \tilde{H}_0 + H_{\mathrm{hfs}}^{\mathrm{FF}}$, with $\tilde{H}_0$ which includes FSC to the Coulomb potential taken as the unperturbed Hamiltonian. In a calculation which involves finite-size correc-tions to the Coulomb as well as hyperfine structure taken separately (as in [13]) it seems reasonable to use the former prescription with $\Delta E = \langle \Phi_{\mathrm{C}}|H_{\mathrm{C}}^{\mathrm{FF}}|\Phi_{\mathrm{C}}\rangle + \langle \Phi_{\mathrm{C}}|H_{\mathrm{hfs}}^{\mathrm{FF}}|\Phi_{\mathrm{C}}\rangle$ in order to avoid double counting of the FSC to the Coulomb term. The r_{p}^2 and r_{p}^3 terms in eqs (12), (13) for example appear after the explicit inclusion of the FSC to the $(1/r)$ Coulomb potential. We notice from the above discussion that as far as the hyperfine FSC alone is concerned, the Breit equation and the Zemach method lead to

the same correction if the time-independent perturbation theory is handled in the same way.

Finally, the point to note here is that using the result in (12) the proton radius further reduced to 0.831 fm. The QED corrections in [13] were included in exactly the same manner as in [3]. This meant that the difference in the approach used for the inclusion of the finite-size effects had introduced 1.3% uncertainty on the result. A FSC approach dependent (see [12] for a discussion on some other approaches) uncertainty would reduce the claimed accuracy of the proton radius and increase the possibility of reducing the disagreement with other data (see figure 1).

7. Summary and outlook

To summarize the discussion of the 'proton radius puzzle' we can say that advances in technology and precision as the one involved in the measurement of the muonic hydrogen Lamb shift have allowed a more precise extraction of the proton radius from spectroscopy as compared to its electronic (standard hydrogen spectroscopy) counterpart. The nuclear structure effects continue to mask the precision though. The uncertainties apart from the right use of the form factors [11] can also arise from the approach used to include the finite-size effects of the protons. As the discussion of the deviation of the radius extracted from muonic hydrogen spectroscopy as compared to extractions from electron–proton elastic scattering continues, experiments on muonic deuterium and helium are planned [24] for the future. The formalism of Breit-like equations can be extended to consider the FSC due to the light nuclei in the μ-d and μ-helium atoms. Indeed, a similar extension for the case of pionic atoms and the exotic beryllium nucleus was performed in the past [25]. The case of the deuteron would however not be trivial due to its being a spin-1 nucleus. We close this discussion with the mention of a proposed simultaneous measurement of $\mu^+ p$ and $e^+ p$ elastic scattering as well as $\mu^- p$ and $e^- p$ elastic scattering by the MUSE Collaboration [26]. It is hoped that these measurements will provide a consistency check on the likeness of the μp and ep interactions.

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