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Dynamics of coupled field solitons: A collective coordinate approach

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Abstract. In this paper we consider a class of systems of two coupled real scalar fields in bidimensional space-time, with the main motivation of studying classical stability of soliton solutions using collective coordinate approach. First, we present the class of systems of the collective coordinate equations which are derived using the presented method. After that, we follow the dynamics of the coupled fields with local inhomogeneity like a delta function potential wall as well as a delta function potential well. The results of the investigation of the two coupled fields are compared to each other and the differences are discussed. The method can predict most of the characters of the interaction.

Keywords. Solitons; collective coordinate; coupled fields.

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1. Introduction

The notion we want to explore in this paper originates from the investigations already presented in some previous works [1,2] which where motivated by investigations introduced in [3,4], in which the scattering of solitons of integrable systems from the potentials have been studied. Similarly, solitary behaviour of coupled fields in the presence of an inhomogeneity is a thought-provoking phenomenon which will be considered in this paper.

Coupled systems of scalar fields with different types of potentials have already been studied by many scholars; they proved to have practical application of the supersymmetry (SUSY) in nonrelativistic quantum mechanics [5]. The presence of topological defects inside the domain walls in a specific system of coupled scalar fields has already been stuied [6]. The collision of coupled field solitons leads to resonance structure depending on the energy exchange between solitons and the internal vibrational modes [7]. Soliton solutions of such systems are of great interest in several branches of physics. However, there are no general rules for investigating the presence of soliton solutions of coupled field theories. Some methods are presented for investigating the presence of soliton solutions in relativistic systems of coupled scalar fields [8]. Nevertheless, these methods are not applicable for all kinds of coupled field systems. In this paper, we do not aim to show how the solutions of a considered Lagrangian are obtained. On the other hand, we are interested in the scattering of the solutions in a defective medium.

Solitons appear in a nonlinear medium with a fine tuning between nonlinear and dispersive effects. This means that they may disappear in the absence of this precise balance in the medium. It is clear that a real medium contains disorders and impurities. Therefore, stability and propagation of solitons in such media are of great interest because of their applications and theoretical interests. In order to understand the behaviour of nonlinear excitations in a disordered system, it is important to investigate the interaction of solitons with impurities. The methods for adding such impurities as potential barrier and potential well to the coupled fields will also be discussed.

External potentials can be added to the equation of motion using different methods. One way to do is to add an external potential to the equation of motion as perturbative terms [1,9]. These effects can also be taken into account by making some parameters of the equation of motion to become a function of space or time [10,11]. Another way is to add an external potential to the field through the metric of background space-time [3,12,13]. This method can be used for models in which the Lagrangians are Lorentz invariant, such as sine-Gordon model, ϕ^4 theory, CP^{N} model, NKG models etc. In this paper we shall focus on the behaviour of solitons of the coupled fields and try to investigate the interaction of their solitary solutions with defects using the first method mentioned above.

Bazeia *et al* [14] considered a system of two coupled real scalar fields with a particular self-interaction potential in a way that the static solutions are derivable from the first-order coupled differential equations. Riazi *et al* [15] employed the same method to investigate the stability of the single-soliton solutions of a particular system of this type. Inspired by the coupled system introduced in [16], we propose a new coupled system of two real scalar fields which shows interesting types of solitons with well-defined topological charges and rest energies (masses). The collective coordinate approach helps us to find analytical equations for the evolution of localized solutions, if such suitable variables can be constructed.

Therefore, a coupled field system and its soliton solutions are introduced in §2. The presented model is introduced and discussed in §3. The results of the model are compared for potential barrier and potential well systems for both fields and the outcomes are also discussed in the final section.

2. The coupled field soliton solutions

The Lagrangian density used to describe the system of the two coupled real scalar fields is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - U(\phi_1, \phi_2), \tag{1}$$

where the potential $U(\phi_1, \phi_2)$ is a scalar function of the two fields ϕ_1 and ϕ_2, bounded from below. It is well known that a potential having disconnected vacua leads to the appearance of the so-called topological solitons in (1+1) space-time dimensions. We assume the potential to be similar to the one in [16],

$$U(\phi_1, \phi_2) = \lambda(x)\left(\frac{1}{4}\left(\phi_1^2 - 1\right)^2 + \frac{1}{2}m^2\phi_2^2 + \frac{1}{4}\gamma\phi_2^4 + \frac{1}{2}d\phi_2^2\left(\phi_1^2 - 1\right)\right), \tag{2}$$

in which $\lambda(x) = 1 + V(x)$, m^2 and d are real and positive constants and $\gamma = d(d - 2m^2)/(1 - 2m^2)$. This potential has already been used in different contexts in condensed matter physics [17]. It was also proposed in the context of quantizing charged solitons [18,19]. $V(x)$ is a potential parameter and carries the effects of the external potential. Potential $V(x)$ is a localized function which is nonzero only in a certain region of space. The equations of motion obtained from the Lagrangian density (2) in bidimensional space-time are

$$\partial_\mu\partial^\mu\phi_1 + \lambda(x)(\phi_1(\phi_1^2 - 1)^2 + d\phi_1\phi_2^2) = 0 \tag{3}$$

and

$$\partial_\mu\partial^\mu\phi_2 + \lambda(x)\left(m^2\phi_2 + \gamma\phi_2^3 + d\phi_2\left(\phi_1^2 - 1\right)\right) = 0. \tag{4}$$

These equations cannot be solved analytically because the potential has a spatial dependence. However, they are solvable if we take $V(x) = 0$. Rajaraman has found the following localized solutions for the above equations in the absence of an external potential. Soliton solutions for this equations are [16]:

$$\phi_1(x,t) = \tanh\left(m\frac{x - X_1(t)}{\sqrt{1 - \dot{X}_1^2}}\right) \tag{5}$$

and

$$\phi_2(x,t) = \pm\sqrt{\frac{1 - 2m^2}{d}}\,\mathrm{sech}\left(m\frac{x - X_2(t)}{\sqrt{1 - \dot{X}_2^2}}\right), \tag{6}$$

where $X_1(t) = x_0 - \dot{X}_1 t$ and $X_2(t) = x_0 - \dot{X}_2 t$. Here, x_0 is the initial position of solitons, $\dot{X}_1$ and $\dot{X}_2$ are their velocities. The signs + and − in eq. (6) denote the kink and antikink solutions respectively. Figure 1 presents the above solutions with $\dot{X}_1 = \dot{X}_2 = 0.5$, $m = 1/\sqrt{3}$, $x_0 = 0$ and $\gamma = d = 1$ at $t = 0$. Clearly, there are other solutions for the above equations: ϕ_1 being topological and ϕ_2 being zero are refered to as type 1 solitons. We have also named solutions (5) and (6) as type 2 solitons. The energy of type 2 soliton is less than that of type 1. This suggests that type 1 soliton is unstable towards decay into type 2 soliton for $m^2 < 1/2$. In this paper we shall investigate the behaviour of type 2 soliton during the interaction with an external potential using the collective coordinate technique. The results are valid for other coupled field theories, as well.

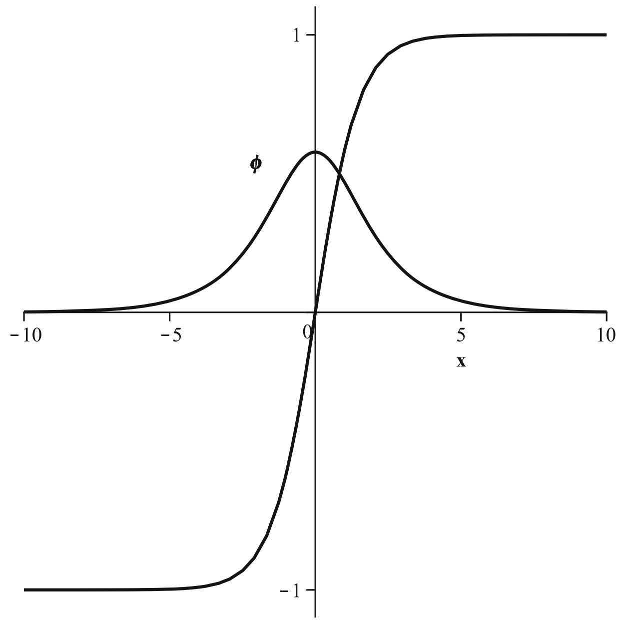

Figure 1. Topological and non-topological soliton solutions of the coupled field system described by Lagrangian (1).

The derivation of the collective action for the motion of the vortex centres starts with the elegant idea of Manton [20]. A collective action can be constructed by substituting the collective vortex ansatz for the field configuration with vortices at $X_i(t), i = 1, \ldots, N$, into the effective field theory action and reduce the action to a function of the collective coordinates, $L[X_i(t)] = \int \mathcal{L}(\psi(x, t, X_i(t)))$ [21]. It is clear that (5) and (6) are not exact solutions for eqs (3) and (4). But they are approximate solutions if $V(x)$ is a local weak perturbation [22].

By inserting solutions (5) and (6) in the Lagrangian (1) and using adiabatic approximation [1] we have

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{6}\left(\dot{X}_1^2 - 1\right)\operatorname{sech}^4\left(\frac{x - X_1}{\sqrt{3}}\right) + \frac{1}{18}\left(\dot{X}_2^2 - 1\right) \\ & \times \tanh^2\left(\frac{x - X_2}{\sqrt{3}}\right)\operatorname{sech}^2\left(\frac{x - X_2}{\sqrt{3}}\right) \\ & - \lambda(x)\left(\frac{1}{12}\operatorname{sech}^4\left(\frac{x - X_1}{\sqrt{3}}\right) + \frac{1}{18}\operatorname{sech}^2\left(\frac{x - X_2}{\sqrt{3}}\right)\right. \\ & \left. + \frac{1}{36}\operatorname{sech}^4\left(\frac{x - X_2}{\sqrt{3}}\right)\right). \end{aligned} \tag{7}$$

In the adiabatic approximation one would suppose that the soliton velocity changes in an adiabatic process. Therefore, the soliton speed changes very slowly. Moreover, we have considered solitons with a small and slowly varying velocity.

3. Collective coordinate for the coupled fields

The Lagrangian density of the solitons is described by (1) for the coupled field system. The internal structure of the solitons can be omitted by integrating the Lagrangian density (or Hamiltonian density) with respect to the variable x. The integrated Lagrangian is called the collective Lagrangian. After the integration, the solitons appear as point-like particles. However, the effect of their extended nature still reflects in the kinetic and also potential parts of the collective Lagrangian. The dynamics of the point-like particles can be described by equations which are derived from collective Lagrangian. It is interesting to compare the results of the collective equations with those of direct numerical simulation of the main Lagrangian density. Let us derive the collective Lagrangian and the point-like particle equations of motion in the presented coupled field model.

By integrating Lagrangian (7) over the variable x, $X_1(t)$ and $X_2(t)$ remain as collective coordinates. If we take the potential $V(x) = \epsilon\delta(x)$, collective Lagrangian is derived from (7) as

$$L = \frac{2\sqrt{3}}{9}\dot{X}_1^2 + \frac{\sqrt{3}}{27}\dot{X}_2^2 - \epsilon\left(\frac{1}{12}\mathrm{sech}^4\left(\frac{X_1}{\sqrt{3}}\right) + \frac{1}{18}\mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right)\right.$$
$$\left. + \frac{1}{36}\mathrm{sech}^4\left(\frac{X_2}{\sqrt{3}}\right)\right) - \frac{14\sqrt{3}}{27}, \qquad (8)$$

where $M_1 = 4\sqrt{3}/9$ and $M_2 = 2\sqrt{3}/27$ are the rest mass of ϕ_1 and ϕ_2 solitons respectively. Equations of motion for the variables $X_1(t)$ and $X_2(t)$ are derived from (8) as

$$\frac{4\sqrt{3}}{9}\ddot{X}_1 - \frac{\epsilon}{3\sqrt{3}}\tanh\left(\frac{X_1}{\sqrt{3}}\right)\mathrm{sech}^4\left(\frac{X_1}{\sqrt{3}}\right) = 0 \qquad (9)$$

and

$$\frac{2\sqrt{3}}{27}\ddot{X}_2 - \frac{\epsilon}{9\sqrt{3}}\tanh\left(\frac{X_2}{\sqrt{3}}\right)\mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right)\left[1 + \mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right)\right] = 0. \qquad (10)$$

We can define collective forces on solitons if we look at the above equations as $F_1 = M_1\ddot{X}_1$ and $F_2 = M_2\ddot{X}_2$, where M_1 and M_2 are the rest masses of the solitons. Therefore, we have

$$F_1 = \frac{\epsilon}{3\sqrt{3}}\tanh\left(\frac{X_1}{\sqrt{3}}\right)\mathrm{sech}^4\left(\frac{X_1}{\sqrt{3}}\right) \qquad (11)$$

and

$$F_2 = \frac{\epsilon}{9\sqrt{3}}\tanh\left(\frac{X_2}{\sqrt{3}}\right)\mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right)\left[1 + \mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right)\right]. \qquad (12)$$

The above equations show that the peak of each soliton moves under the influence of complicated forces which are functions of external potentials and the positions of solitons. Suppose that a soliton moves toward a potential barrier. Its velocity will reduce due to the effect of a repulsive force. While the soliton is moving away, its velocity increases.

It is interesting to compare the effective potentials of the solitons by integrating collective forces of eqs (11) and (12) with respect to the collective variables. By following this rule, it is not difficult to derive potentials as follows:

$$V_1(X_1) = \int -F_1 \mathrm{d}X_1 = \frac{\epsilon}{4}\mathrm{sech}^4\left(\frac{X_1}{\sqrt{3}}\right), \tag{13}$$

for ϕ_1 soliton and

$$V_2(X_2) = \int -F_2 \mathrm{d}X_2 = \frac{\epsilon}{9}\left[\frac{5}{4}\mathrm{sech}^4\left(\frac{X_2}{\sqrt{3}}\right) + \frac{1}{2}\mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right)\right], \tag{14}$$

for ϕ_2 soliton respectively. Figure 2 shows the effective potentials as a function of collective variable for the coupled solitons. It can be concluded that the effective potential of ϕ_1 soliton (dash line) is stronger than the effective potential of ϕ_2 soliton (solid line). These differences act on the dynamics of solitary waves in opposite ways. The effects of these differences are studied in the next section.

Interestingly, eqs (9) and (10) have exact solutions for $\dot{X}_1$ and $\dot{X}_2$ as follows:

$$\dot{X}_1^2 - \dot{X}_{01}^{\,2} = -\frac{\sqrt{3}\epsilon}{8}\left[\mathrm{sech}^4\left(\frac{X_1}{\sqrt{3}}\right) - \mathrm{sech}^4\left(\frac{X_{01}}{\sqrt{3}}\right)\right] \tag{15}$$

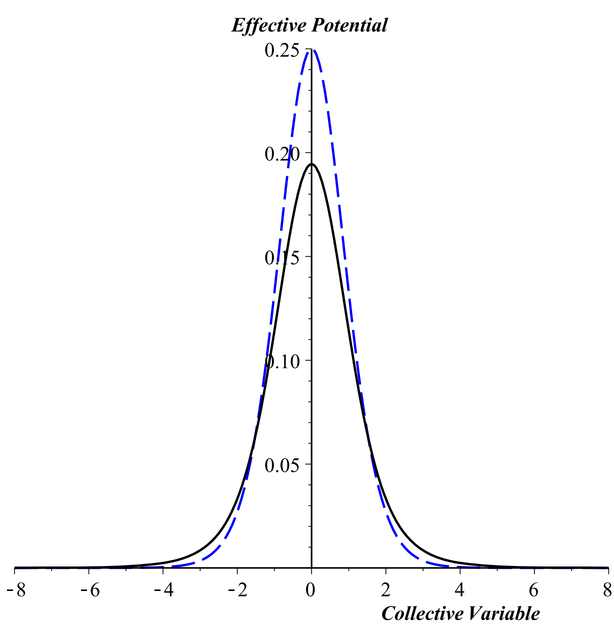

Figure 2. Effective potential as a function of collective variable for the coupled fields.

and

$$\dot{X}_2^2 - \dot{X}_{02}^{\,2} = -\frac{\sqrt{3}\epsilon}{2}\left[\frac{1}{2}\mathrm{sech}^4\left(\frac{X_2}{\sqrt{3}}\right) + \mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right) - \frac{1}{2}\mathrm{sech}^4\left(\frac{X_{02}}{\sqrt{3}}\right) - \mathrm{sech}^2\left(\frac{X_{02}}{\sqrt{3}}\right)\right], \tag{16}$$

where X_{01} and X_{02} are the initial positions of the soliton and $\dot{X}_{01}$ and $\dot{X}_{02}$ are their initial velocities. Some of the physical features of a soliton-potential system can be discovered using the above equations. The collective energy can be obtained from Lagrangian (8) as follows:

$$E = \frac{2\sqrt{3}}{9}\dot{X}_1^2 + \frac{\sqrt{3}}{27}\dot{X}_2^2 + \epsilon\left(\frac{1}{12}\mathrm{sech}^4\left(\frac{X_1}{\sqrt{3}}\right) + \frac{1}{18}\mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right) + \frac{1}{36}\mathrm{sech}^4\left(\frac{X_2}{\sqrt{3}}\right)\right) + \frac{14\sqrt{3}}{27}. \tag{17}$$

It is the energy of two particles with the velocities of $\dot{X}_1$ and $\dot{X}_2$ which move under the influence of an external effective potential. By substituting $\dot{X}_1$ from (15) and $\dot{X}_2$ from (16) in eq. (17), it is shown that the energy of the system is a function of the initial conditions of the soliton and therefore it is conserved.

4. The dynamics of solitons

Critical velocity is the minimum velocity for a soliton to pass over the barrier. A soliton with a velocity smaller than the critical velocity reflects back from the impurity. However, if it has enough enrgy to overcome the barrier, it can go through it. Finding such a velocity with numerical simulation is straightforward and simple. Let us consider how we can find this velocity using a collective coordinate approach. As mentioned before, we have no dissipation in the total energy of the system. So, critical velocities can be obtained for both coupled field solitons without any numerical simulations. Equation (17) is reduced to

$$E = \frac{2\sqrt{3}}{9}\dot{X}_{01}^{\,2} + \frac{\sqrt{3}}{27}\dot{X}_{02}^{\,2} + \frac{14\sqrt{3}}{27}$$

when the solitons are far from the centre of the delta-like potential which is located at the origin. It is the energy of two particles with the masses $M_1 = 4\sqrt{3}/9$ and $M_2 = 2\sqrt{3}/27$ and velocities $\dot{X}_{01}$ and $\dot{X}_{02}$ respectively. The energy of the system when ϕ_1 soliton is located in the origin ($X_1 = 0$) with zero velocity, comes from (17) as follows:

$$E(X_1 = 0) = \frac{\sqrt{3}}{27}\dot{X}_2^2 + \epsilon\left(\frac{1}{12} + \frac{1}{18}\mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right) + \frac{1}{36}\mathrm{sech}^4\left(\frac{X_2}{\sqrt{3}}\right)\right) + \frac{14\sqrt{3}}{27}. \tag{18}$$

On the other hand, solitons which come from infinity with initial velocities v_{1c} and v_{2c} have the energy

$$E = \frac{2\sqrt{3}}{9}v_{1c}^2 + \frac{\sqrt{3}}{27}v_{2c}^2 + \frac{14\sqrt{3}}{27}.$$

So, it is easy to calculate the critical velocity of ϕ_1 soliton by comparing the energy of the system when ϕ_1 soliton is located at the origin to its energy at infinity. The critical velocity is calculated as

$$v_{1\mathrm{c}} = \sqrt{\frac{3\sqrt{3}}{2}\epsilon\left(\frac{1}{12} + \frac{1}{18}\mathrm{sech}^2\left(\frac{X_2}{\sqrt{3}}\right) + \frac{1}{36}\mathrm{sech}^4\left(\frac{X_2}{\sqrt{3}}\right)\right)}. \tag{19}$$

The above equation shows that the critical velocity of ϕ_1 soliton depends on the position of ϕ_2 soliton. If ϕ_2 soliton is located at infinity, the calculated critical velocity is reduced to $v_{1\mathrm{c}} = \sqrt{\sqrt{3}\epsilon/8}$. Critical velocity of ϕ_1 soliton has maximum value when ϕ_2 soliton is located at the origin. In this situation, the energy of the system is made up of 'soliton+potential'. Therefore, ϕ_1 soliton needs higher energy to pass over the barrier. The calculated critical velocity in this situation is $v_{1\mathrm{c}} = \sqrt{\sqrt{3}\epsilon/4}$.

With the same analysis one can find that the critical velocity of ϕ_2 soliton is derived as

$$v_{2\mathrm{c}} = \sqrt{\frac{3\sqrt{3}}{4}\epsilon\left(1 + \mathrm{sech}^4\left(\frac{X_1}{\sqrt{3}}\right)\right)}. \tag{20}$$

The above critical velocity is reduced to $v_{2\mathrm{c}} = \sqrt{3\sqrt{3}\epsilon/4}$ when ϕ_1 soliton is located at infinity. Moreover, the same result is derived by substituting $\dot{X}_2 = 0$, $\dot{X}_{02} = v_{2\mathrm{c}}$, $X_{02} = \infty$ and $X_2 = 0$ in (16).

Note that the critical velocities of the solitons depend on their initial positions as well as their initial velocities. For solitons which are located at some positions like X_{01} and X_{02} (which are not necessarily infinity), the critical velocities will not be the ones calculated in eqs (19) and (20). So a soliton which comes from an initial position with an initial velocity will have the critical initial velocity considering that it has a velocity of zero at the top of the barrier. Consider ϕ_1 soliton with initial conditions of X_{01} and $\dot{X}_{01}$. If we set $X_1 = 0$ and $\dot{X}_1 = 0$ in eq. (15) then $v_{1\mathrm{c}} = \dot{X}_{01}$. Thus, for such a soliton we have

$$v_{1\mathrm{c}} = \sqrt{\frac{\sqrt{3}}{8}\epsilon\left(1 - \mathrm{sech}^4\left(\frac{X_{01}}{\sqrt{3}}\right)\right)}. \tag{21}$$

But the critical velocity of ϕ_2 soliton becomes

$$v_{2\mathrm{c}} = \sqrt{\frac{\sqrt{3}}{2}\epsilon\left(\frac{3}{2} - \mathrm{sech}^2\left(\frac{X_{02}}{\sqrt{3}}\right) - \frac{1}{2}\mathrm{sech}^4\left(\frac{X_{02}}{\sqrt{3}}\right)\right)}. \tag{22}$$

Figure 3 shows the critical velocities of the two solitons as a function of the potential strength for $X_0 = -2$. This figure shows that ϕ_1 soliton results in a smaller critical velocity compared to ϕ_2 soliton because of the differences between their rest masses. The rest mass of the ϕ_1 soliton is $M_1 = 4\sqrt{3}/9$ and the rest mass of ϕ_2 soliton is $M_2 = 2\sqrt{3}/27$. Obviously, a soliton with a large rest mass needs a smaller velocity to reach

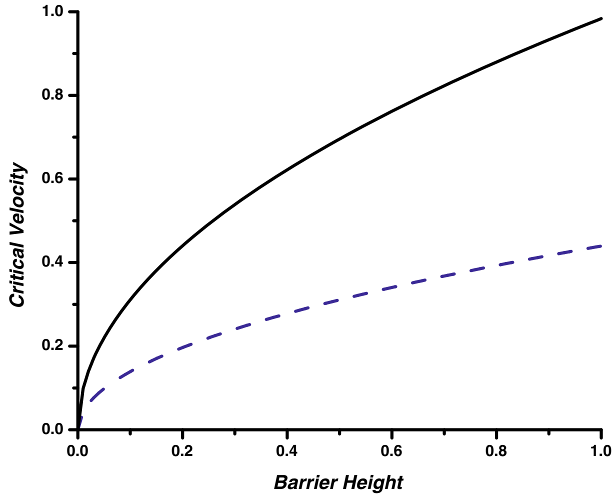

Figure 3. The calculated critical velocities as a function of barrier height for ϕ_1 soliton (dash line) and ϕ_2 soliton (solid line) in initial position $X_0 = -2$.

the potential peak. It is interesting to depict the critical velocities as a function of the initial position of solitons. The critical velocities are demonstrated as a function of the initial position in figure 4 for the two solitons with $\epsilon = 0.2$. This figure shows a considerable disagreement between the two solitons. For solitons at infinity, the plots demonstrate that the critical velocities are not dependent on the initial positions of solitons. However, for a soliton which moves at an initial position close to the centre of the potential, different results are obtained. It is clear from this figure that a soliton needs a lower initial velocity to pass over the barrier if it is closer to the centre of the potential.

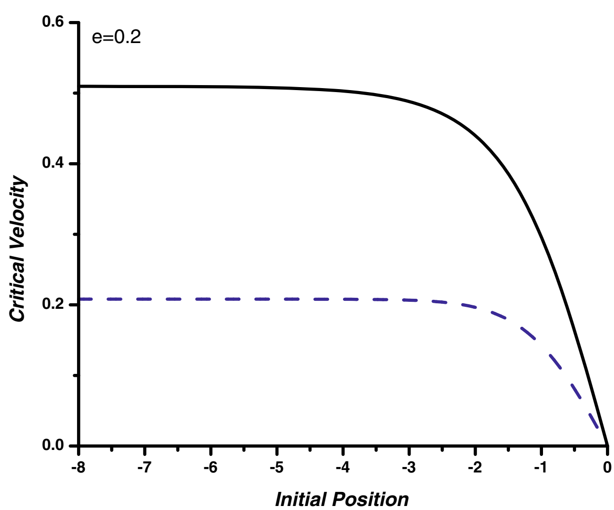

Figure 4. Critical velocities of the solitons as a function of initial position for $\epsilon = 0.2$.

In the framework of classical mechanics, a moving object will never turn back until it reaches a turning point, at which the velocity vanishes. At the microscopic scale, where the wave character of particles becomes important, quantum mechanics allows the reflection of a particle in a classically permitted region even when there is no classical turning point. Quantum reflection can occur above a repulsive potential barrier or an attractive potential well, and may take place in an attractive potential tail [23] or at a potential step [24]. The above-mentioned points inspire one to expect that a classical object will never reflect from an attractive impurity. But the classical reflection of solitons from a potential well has been shown by some authors [1,13,25]. It is demonstrated that at some velocities smaller than the critical velocity, the soliton may reflect back over the potential although it is expected that the soliton should be trapped in the potential well. This quantum-like behaviour has been found in some very narrow windows of initial velocities.

Let us consider whether we can find the classical reflection by the collective coordinate method. This situation is worth investigating because of some differences between a point particle and a soliton in the potential well.

If we choose $\epsilon < 0$ in eq. (15), then the soliton moves toward the potential well. The solution for the system of ϕ_1 soliton is

$$\dot{X}_1^2 - \dot{X}_{01}^2 = \frac{\sqrt{3}\epsilon}{8}\left[\operatorname{sech}^4\left(\frac{X_1}{\sqrt{3}}\right) - \operatorname{sech}^4\left(\frac{X_{01}}{\sqrt{3}}\right)\right]. \tag{23}$$

Similary, for ϕ_2 soliton we have

$$\dot{X}_2^2 - \dot{X}_{02}^2 = \frac{\sqrt{3}\epsilon}{2}\left[\frac{1}{2}\operatorname{sech}^4\frac{X_2}{\sqrt{3}} + \operatorname{sech}^2\frac{X_2}{\sqrt{3}} - \frac{1}{2}\operatorname{sech}^4\frac{X_{02}}{\sqrt{3}} - \operatorname{sech}^2\frac{X_{02}}{\sqrt{3}}\right]. \tag{24}$$

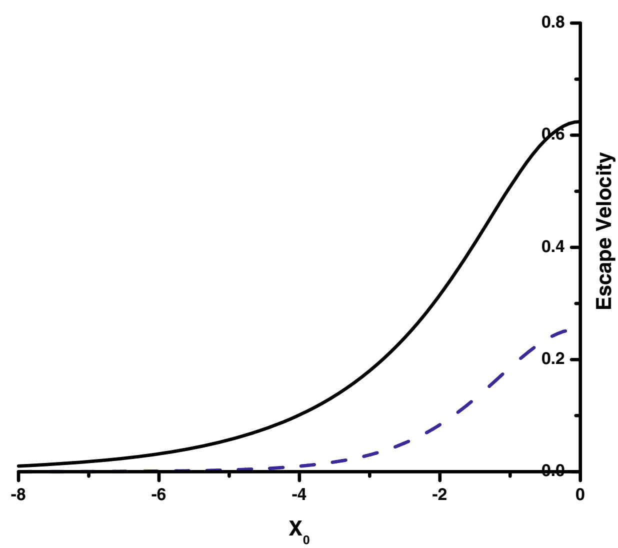

Figure 5. Escape velocity as a function of initial position for $\epsilon = 0.3$.

The soliton will run away from the potential and will go to infinity if the initial velocity of the soliton is more than the escape velocity. The escape velocity is the minimum velocity for a soliton with which it can go through a well. A soliton in an initial position X_0 reaches infinity with a zero final velocity if its initial velocity is

$$\dot{X}_{\text{escape1}} = \sqrt{\frac{\sqrt{3}}{8}\epsilon\,\text{sech}^4\left(\frac{X_{01}}{\sqrt{3}}\right)} \tag{25}$$

and

$$\dot{X}_{\text{escape2}} = \sqrt{\frac{\sqrt{3}}{2}\epsilon\left(\frac{1}{2}\text{sech}^4\left(\frac{X_{02}}{\sqrt{3}}\right) + \text{sech}^2\left(\frac{X_{02}}{\sqrt{3}}\right)\right)}, \tag{26}$$

which are calculated using the presented model for ϕ_1 and ϕ_2 solitons respectively. In other words, a soliton which is located in the initial position X_0 can escape to infinity if its initial velocity $\dot{X}_0$ is more than the escape velocity $\dot{X}_{\text{escape}}$. Figure 5 shows the escape velocity from a potential well as a function of initial position of the solitons for the two solitons. Owing to its bigger rest mass, the ϕ_1 soliton needs lower escape velocity in comparison with the ϕ_2 soliton.

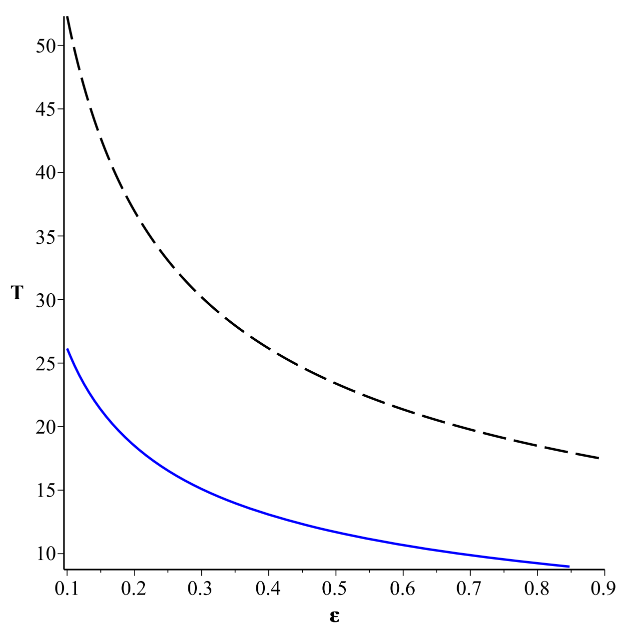

Figure 6. Period of oscillation of solitons in the potential well as a function of ϵ for ϕ_1 soliton (dash line) and ϕ_2 soliton (solid line).

Lastly, let us consider a situation in which the initial velocity of soliton is not big enough to pass through the potential. The soliton reaches a maximum distance from the centre of the potential with zero velocity and then comes back toward the centre of the potential well. Therefore, the soliton oscillates in the well with a specific period. The period of oscillation of both solitons can be calculated analytically using eqs (9) and (10). For small amplitudes, these equations are reduced to

$$\frac{4\sqrt{3}}{9}\ddot{X}_1 - \frac{\epsilon}{9}X_1 = 0, \qquad \frac{2\sqrt{3}}{27}\ddot{X}_2 - \frac{2\epsilon}{27}X_2 = 0. \tag{27}$$

These are equations of motion of two particles with the periods of oscillation $T_1 = 2\pi\sqrt{4\sqrt{3}/\epsilon}$ and $T_2 = 2\pi\sqrt{\sqrt{3}/\epsilon}$ respectively. In figure 6 we see the period as a function of ϵ. It shows that the results of the two solitons have some discrepancies that come from the different masses of the solitons.

Finally, we had a desire to guess the soliton reflection from the potential well by the collective coordinate model presented here. But, the method fails to predict the narrow windows of soliton reflection.

In summary, an analytical model for the interaction of the coupled field solitons with delta function potential has been presented. The critical velocity was obtained for each soliton separately in the soliton-barrier system by using the proposed model. The critical velocity is a function of initial conditions and also the potential identities. The dynamics of each soliton is related to another one which affects their characteristics. We have defined escape velocity for a soliton-well system as a substitute for the critical velocity. The model can explain most of the features of the system analytically. It is better to test the validity of our model by numerical simulations. This would be our future work. An interesting question about the possibility of quantum reflection of coupled field solitons from attractive potential cannot be predicted by the analytical model presented here. So, it is expected to find a more powerful model with a suitable collective coordinate method to explain this behaviour.

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