PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 83, No. 4
October 2014
pp. 481–491

Biased trapping issue on weighted hierarchical networks

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MS received 24 September 2013; revised 18 February 2014; accepted 25 February 2014
DOI: 10.1007/s12043-014-0814-6; ***e*Publication:** 20 August 2014

Abstract. In this paper, we present trapping issues of weight-dependent walks on weighted hierarchical networks which are based on the classic scale-free hierarchical networks. Assuming that edge's weight is used as local information by a random walker, we introduce a biased walk. The biased walk is that a walker, at each step, chooses one of its neighbours with a probability proportional to the weight of the edge. We focus on a particular case with the immobile trap positioned at the hub node which has the largest degree in the weighted hierarchical networks. Using a method based on generating functions, we determine explicitly the mean first-passage time (MFPT) for the trapping issue. Let parameter a $(0 < a < 1)$ be the weight factor. We show that the efficiency of the trapping process depends on the parameter a; the smaller the value of a, the more efficient is the trapping process.

Keywords. Weighted hierarchical networks; weight-dependent walks; mean first passage time.

PACS Nos **05.40.Fb; 89.75.–K; 89.75.Fb**

1. Introduction

As a powerful and versatile mathematical tool, complex networks have represented and modelled the structure of complex systems [1,2]. Much attention has been paid to the study of complex networks and empirical evidence has shown that many transportation systems can be better described by complex networks [3,4]. Complex networks attracted a lot of research in the past decade because of their applications in different areas of science [5,6].

The first-passage time (FPT), denoted by $T_i(g)$, is the expected time for a particle, originating at node i, to first reach the trap on the gth generation network. This quantity is fundamental in the study of transport-limited reactions, because it gives the reaction time in the limit of perfect reaction and it is also useful in target search problems and other physical systems [7,8]. We are interested in the mean first-passage time (MFPT), which is the average of the node-to-trap FPT over the entire network. MFPT is important because of its role in real situations such as transport in disordered media, neuron firing, spread of diseases and target search processes. Moreover, MFPT is a fundamental quantity pertaining to trapping.

The modularity plays an important role in shaping up scale-free networks [9]. The trap located at the node with the highest degree can simultaneously capture scale-free behaviour and modular structure. The deterministic nature of the hierarchical networks makes it possible to investigate analytically, the trapping process defined on them. On the other hand, authors in refs [10,11] compared the behaviour of trapping on the hierarchical networks with those of other networks, and showed that the hierarchical networks can be helpful for enhancing the efficiency of the trapping process. However, the trapping process on weighted hierarchical networks with modular structure remains less understood, in spite of the facts that modularity plays an important role in shaping up scale-free networks, and that taking into account the modular structure of scale-free networks leads to a better understanding of how the underlying systems work.

Weighted networks are realistic forms of networks, where a weight is attached to each link. This may denote a physical property of interest, e.g., the length of a road. Similarly, a strength is attached to each node. This may represent, for instance, the computational capacity of an internet router. Weighted networks carry more information and they are more realistic as far as the representation of real systems are concerned. In addition, individual links (and nodes) on weighted networks are essentially different. Recently, attention has been given to the MFPT [12,13] based on weight-dependent walk in weighted communication networks. For weight-dependent walk, a walker chooses one of its neighbours with a probability proportional to the weight of the edge linking them. The probability of trapping based on weighted-dependent walks are extremely different from the one on unweighted networks. That is why we add weight to the edge on hierarchical networks in this paper, i.e., weighted hierarchical networks, which is different from the classic hierarchical scale-free networks.

In the present paper, we discuss a type of trapping problem on weighted hierarchical networks. The trapping problem, first introduced in [14], is in fact a random-walk issue, where a trap is positioned at a given location, which absorbs all particles visiting it. The primarily interesting quantity closely related to the trapping problem is the MFPT, which is useful in the study of transport-limited reactions [15,16], target search [17,18] and other physical problems. MFPT is the mean of the node-to-trap (a single immobile trap positioned at a given site) first-passage time, over the entire weighted hierarchical network. Here, we focus on a particular case with the trap located at the node with the highest degree. We derive rigorous solution to the MFPT that characterizes the trapping process. Moreover, we compare the behaviour of the trapping on the weighted hierarchical networks and the behaviour on the unweighted networks, and find that the weighted hierarchical networks are more useful for improving the efficiency of the trapping process.

2. Weighted hierarchical modular networks

In this section, we aim at constructing the weighted hierarchical modular networks. Inspired by hierarchical scale-free networks [9], we built weighted hierarchical networks in an iterative way. Let us introduce the model for the weighted hierarchical networks controlled by a positive integer M and a real number a with a modular structure. We denote the network model by G_g after g ($g \geq 1$) iterations.

(1) Initially, when $g = 1$, the network G_1 consists of a central node, called the hub (root) node A, and peripheral (external) nodes M with $M \geq 2$. All these initial $M + 1$ nodes are fully connected to each other, forming a complete graph. Each edge carries a standard initial weight $w = 1$.

(2) At the second generation ($g = 2$), we generate M copies of G_1, whose weighted edges have been scaled, respectively, by a factor a. The M copies are labelled as $G_1^{(1)}, G_1^{(2)}, \ldots, G_1^{(M)}$, and we connect M external nodes of each replica $G_1^{(i)} (i = 1, 2, \ldots, M)$ to the root of the original G_1 through edges of unitary weight. The original G_1 in G_2 has label $G_1^{(0)}$. The hub of the original $G_1^{(0)}$ and M^2 peripheral nodes in the replicas $G_1^{(i)} (i = 1, 2, \ldots, M)$ become the hub and peripheral nodes of G_2, respectively. The hub node in G_2 also has label A. The process is described in figure 1.

(3) Suppose one has G_{g-1}, the next generation network G_g can be obtained from G_{g-1} by adding M replicas of G_{g-1}, whose weighted edges have been scaled, respectively, by a factor a. The M copies are labelled as $G_{g-1}^{(1)}, G_{g-1}^{(2)}, \ldots, G_{g-1}^{(M)}$, and their external nodes are linked to the hub of the original G_{g-1} through edges of unitary weight. The original G_{g-1} in G_g has label $G_{g-1}^{(0)}$. In G_g, its hub is the hub of the $G_{g-1}^{(0)}$, and its external nodes are composed of all the peripheral nodes of $G_{g-1}^{(i)} (i = 1, 2, \ldots, M)$. The hub node in G_g always has label A.

(4) Repeating the replication and connection steps, we obtain the weighted hierarchical networks.

According to the weighted network construction, we can see that G_g is characterized by two parameters g and M where g stands for the number of generations and M represents the replication factor. In G_g, the number of nodes, often called order of the network denoted as N_g, is $N_g = (M + 1)^g$. All these nodes can be classified into the following

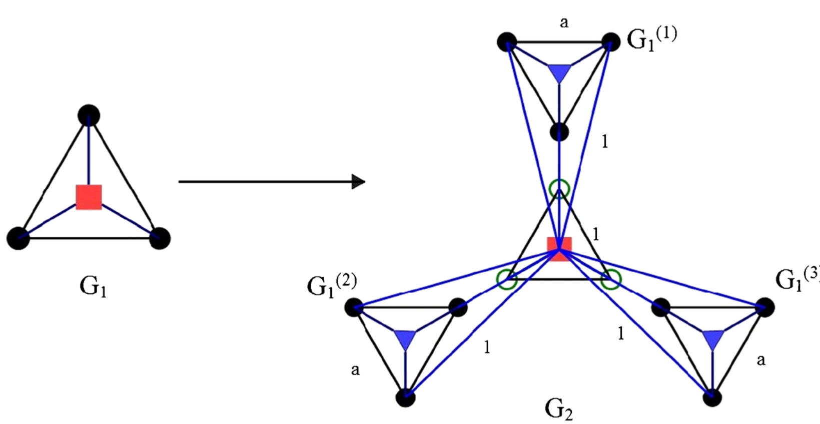

Figure 1. The iterative construction process from G_1 to G_2 for $M = 3$.

four sets: external node set, **E**, locally external node set, $\mathbf{E}_n (1 \le n < g)$, set **H** only consisting of the hub node of G_g and the local hub set, $\mathbf{H}_n (1 \le n < g)$ (see figure 2). The cardinalities, defined as the number of nodes in a set,

$$|\mathbf{E}| = M^g$$

and

$$|\mathbf{H}| = 1.$$

3. Mean first passage time for biased random walk

In this section, we express the trapping issue on the hierarchical weighted networks G_g. We next focus on the trapping problem on G_g with the trap fixed on the hub node, i.e., node A. The hub node in G_g is labelled 1, the other $N_g - 1$ nodes in G_g are labelled as $2, 3$ and N_g. The particular choice for the trap position allows us to compute the MFPT analytically, which will be discussed in detail in the following section. During the trapping process, the particle, starting from any node except the trap A, jumps to any of its nearest neighbours independently with a probability proportional to its weight. In the weighted networks, a weight ω_{ij} is assigned to the edge connecting the nodes i and j, and the strength of the node i can be defined as

$$s_i = \sum_{j \in v(i)} w_{ij},$$

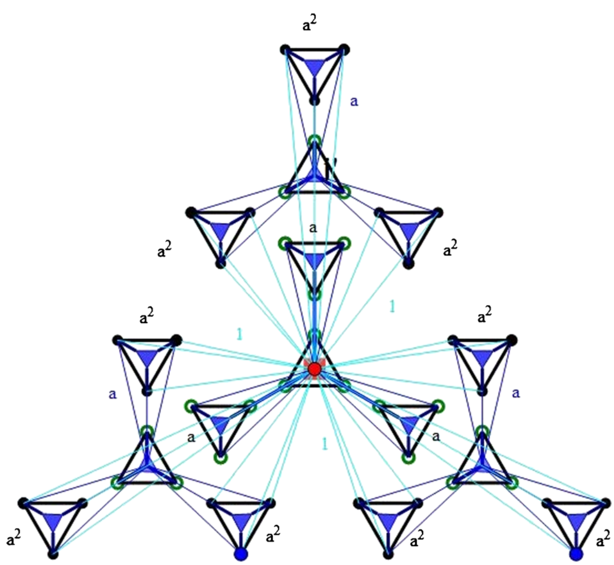

Figure 2. Classification of nodes in network G_3 for $M = 3$. The filled circles, open circles, full square and triangles represent external nodes, locally external nodes, hub nodes and locally hub nodes, respectively.

where the sum index j runs over the set $v(i)$ of neighbours of i [19]. The strength of a node integrates the information about its connectivity and the weights of its links.

For weight-dependent walk, a walker chooses one of its neighbours with a probability proportional to the weight of edge linking to them. The transition probability from node i to its neighbour j is

$$p^{w}_{i\to j} = \frac{w_{ij}}{s_i} = \frac{w_{ij}}{\sum_{j\in v(i)} w_{ij}}.$$

For the trapping issue on G_g, in order to get the general formula for MFPT, $\langle T\rangle_g$, first we define some related intermediate quantities. We denote λ_g to represent the set of nodes in G_g and separate them into two subsets: one subset is λ_{g-1} made up of nodes in the original G_{g-1} and the other subset, denoted by $\bar{\lambda}_g$, is the set of nodes of the M copies of G_{g-1}. Let $T_i(g)$ denote the trapping time for a walker originating at node i on the gth generation network to first reach the trap node. Obviously, for all $g \geq 0$, $T_1(g) = 0$. Then, by definition, the MFPT $\langle T\rangle_g$ can be expressed as

$$\langle T\rangle_g = \frac{1}{N_g - 1}\sum_{i=2}^{N_g} T_i(g),$$

where the summation term $\sum_{i=2}^{N_g} T_i(g)$ can be rewritten as

$$\begin{aligned}\sum_{i=2}^{N_g} T_i(g) &= \sum_{i=2}^{N_{g-1}} T_i(g) + \sum_{i\in\bar{\lambda}_g} T_i(g)\\ &= \sum_{i=2}^{N_{g-1}} T_i(g-1) + \sum_{i\in\bar{\lambda}_g} T_i(g).\end{aligned}$$

Thus, we have

$$\langle T\rangle_g = \frac{N_{g-1}-1}{N_g - 1}\langle T\rangle_{g-1} + \frac{1}{N_g-1}\sum_{i\in\bar{\lambda}_g} T_i(g). \quad (1)$$

Next, let $E_g(t)$ stand for the probability that the particle starting from any external node in **E** first arrives at the hub after t jumps at the generation g; and let $H_g(t)$ represent the probability that the walker originating from the hub to first reach any node belonging to **E** after t steps. At first, we bring in the Kronecker delta function, $\delta_{t,1}$, which is defined as follows: $\delta_{t,1} = 1$ if t is equal to 1, and $\delta_{t,1} = 0$ otherwise. Then, the fundamental relation $E_g(t)$ can be established:

$$\begin{aligned}E_g(t) = &\frac{1}{Ma^{g-1} + a^{g-2} + \cdots + a + 1}\delta_{t,1}\\ &+ \frac{a^{g-1}}{Ma^{g-1} + a^{g-2} + \cdots + a + 1}(M-1)E_g(t-1)\\ &+ \sum_{n=1}^{g-1}\sum_{i=1}^{t-1}\frac{a^{g-n}}{Ma^{g-1} + a^{g-2} + \cdots + a + 1}H_n(i)E_g(t-1-i), \quad (2)\end{aligned}$$

The three parts on the right side of eq. (2) can be be understood based on the following three processes: the first part stands for the probability that the walker takes only one time step to first reach the hub; the second part on the right-hand side accounts for the case that the particle gets first to one of its $M-1$ neighbours belonging to $\mathbf{E}$ in one time step, and then it takes $t-1$ steps more to first arrive at the target node; the last part on the right hand side explains the probability of the process, where the walker first makes a jump to a local hub node belonging to $\mathbf{E}_n$, then it takes i time steps, starting from the local hub nodes, to reach one of the nodes in $\mathbf{E}$, and continues to jump $t-1-i$ steps more to first reach the hub.

The latter quantity $H_g(t)$ obeys the following recursive relation:

$$\begin{aligned} H_g(t) &= \frac{M^g}{M^g+M^{g-1}+\cdots+M}\delta_{t,1} \\ &\quad+\sum_{n=1}^{g-1}\sum_{i=1}^{t-1}\frac{M^n}{M^g+M^{g-1}+\cdots+M}\delta_{t,1}E_n(i)H_g(t-1-i). \end{aligned} \tag{3}$$

Analogously, the two parts of the terms of eq. (3) can be elaborated as follows: the first part represents the occurring probability of the process that the walker, starting from the hub, only needs one time step to reach an external node in $\mathbf{E}$. The second part explains the probability that the particle, originating from the hub, first makes one jump to a local external node in $\mathbf{E}_n$, then makes i jumps to the hub, and continues to the destination (any node belonging to $\mathbf{E}$), taking $t-1-i$ steps more.

Note that in eqs (2) and (3), the equivalence of nodes in the same set is used. Equations (2) and (3) provide two basic intermediate relations governing the trapping problem performing on G_g, from which almost all subsequent results are derived. The obtained relations $E_g(t)$ and $H_g(t)$ are very useful quantities for the following calculations.

Based on the above two intermediate relations, we define two generating functions, $\tilde{E}_g(x)$ and $\tilde{H}_g(x)$, that represent the probability distribution of first-passage time described in eqs (2) and (3), and they can be denoted as follows:

$$\begin{aligned} \tilde{E}_g(x) &:= \sum_{t=0}^{\infty}E_g(t)x^t = \frac{x}{Ma^{g-1}+a^{g-2}+\cdots+a+1} \\ &\quad+\frac{(M-1)a^{g-1}}{Ma^{g-1}+a^{g-2}+\cdots+a+1}x\tilde{E}_g(x) \\ &\quad+\frac{x}{Ma^{g-1}+a^{g-2}+\cdots+a+1}\sum_{n=1}^{g-1}a^{g-n}\tilde{H}_n(x)\tilde{E}_g(x) \end{aligned} \tag{4}$$

and

$$\begin{aligned} \tilde{H}_g(x) &:= \sum_{t=0}^{\infty}H_g(t)x^t \\ &= \frac{M^g x}{M^g+M^{g-1}+\cdots+M}+\frac{\tilde{H}_g(x)}{M^g+M^{g-1}+\cdots+M}x\sum_{n=1}^{g-1}M^n\tilde{E}_n(x). \end{aligned} \tag{5}$$

From the description in [9], we have the following equation:

$$\tilde{E}_g(1) = \tilde{H}_g(1) = 1. \tag{6}$$

After some algebraic operations, eq. (4) can be recast as

$$\tilde{E}_g(x)\left[\frac{Ma^{g-1}+a^{g-2}+\cdots+a+1}{x}-(M-1)a^{g-1}-\sum_{n=1}^{g-1}a^{g-n}\tilde{H}_n(x)\right]=1 \tag{7}$$

and eq. (5) can be expressed as follows:

$$\tilde{H}_g(x)\left[\frac{M^g+M^{g-1}+\cdots+M}{M^g x}-\sum_{n=1}^{g-1}M^{n-g}\tilde{E}_n(x)\right]=1. \tag{8}$$

We define two intermediate relations, T_g^E and T_g^H, again to calculate the MFPT $\langle T\rangle_g$. T_g^E stands for the first-passage time for a walker starting from an arbitrary node in **E** to reach the hub for the first time, which is in fact the number of steps for the walker originating from any node in **E** to first visit the hub. T_g^H stands for the FPT needed for a particle initially located at hub to first reach any node in **E**. Then, according to the property of generating functions, the two quantities T_g^E and T_g^H can be given separately by

$$T_g^E = \frac{\mathrm{d}}{\mathrm{d}x}\tilde{E}_g(x)|_{x=1} \tag{9}$$

and

$$T_g^H = \frac{\mathrm{d}}{\mathrm{d}x}\tilde{H}_g(x)|_{x=1}. \tag{10}$$

Next, we differentiate both sides of eq. (7) with respect to x and set $x = 1$, in addition, we insert eq. (6) into eq. (7), and we obtain the following formula:

$$T_g^E = Ma^{g-1}+a^{g-2}+\cdots+a+1+\sum_{n=1}^{g-1}a^{g-n}T_n^H. \tag{11}$$

Following the same procedure for eq. (8), we have

$$T_g^H = \frac{M^g+M^{g-1}+\cdots+M}{M^g}+\sum_{n=1}^{g-1}M^{n-g}T_n^E. \tag{12}$$

From the above two equations, we get the following formulae easily:

$$T_{g+1}^E - aT_g^E = 1 + aT_g^H \tag{13}$$

and

$$MT_{g+1}^H - T_g^H = M + T_g^E. \tag{14}$$

Considering the initial conditions $T_2^E = Ma + a + 1$ and $T_2^H = (2M+1)/M$, we can solve the simultaneous equations, i.e. eqs (13) and (14). Thus, we can obtain the following generating functions:

$$T_g^E = \left(\frac{aM+1}{M}\right)^{g-2} T_2^E + \frac{aM+M-1}{M-aM-1}\left[1-\left(\frac{aM+1}{M}\right)^{g-2}\right] \tag{15}$$

and

$$T_g^H = \left(\frac{aM+1}{M}\right)^{g-2} T_2^H + \frac{M-aM+1}{M-aM-1}\left[1-\left(\frac{aM+1}{M}\right)^{g-2}\right]. \tag{16}$$

T_g^E and T_g^H, the intermediate expressions obtained, are very important. We shall use them to determine MFPT $\langle T\rangle_g$. Hence, to obtain an exact solution for $\langle T\rangle_g$, all that is left is to evaluate the sum in eq. (1), with a goal to find a recursive relation for $\langle T\rangle_g$. From the weighted network's structure and properties (see figures 1 and 2), the sum term on the right-hand side of eq. (1) can be established as follows:

$$\begin{aligned}\sum_{i\in\overline{\lambda}_g} T_i(g) &= T_g^E|\mathbf{E}| + \frac{|\mathbf{E}|}{M}(T_g^E+1)\\ &\quad + \sum_{n=1}^{g-2} M^{g-n-1}\left[(N_n-1)\langle T\rangle_n + N_n T_{n+1}^H + N_n T_g^E\right].\end{aligned} \tag{17}$$

Substituting the previously obtained equations for the expressions of related quantities in eq. (17) and combining with eq. (1), we obtain the following recurrence relation for $\langle T\rangle_g$:

$$\begin{aligned}&(N_{g+1}-1)\langle T\rangle_{g+1} - (M+1)(N_g-1)\langle T\rangle_g\\ &= \left[M(Ma+a+1)^2 + 2M + 1 - \frac{a^2M^3 - M^3 + 3M^2 + a^2M^2 + M - 1}{M-aM-1}\right]\\ &\quad \times \frac{(M+1)^{g-1}(aM+1)^{g-2}}{M^{g-2}} + \frac{2M^2}{M-aM-1}(M+1)^{g-1}.\end{aligned} \tag{18}$$

Let

$$A := \left[M(Ma+a+1)^2 + 2M + 1 - \frac{a^2M^3 - M^3 + 3M^2 + a^2M^2 + M - 1}{M-aM-1}\right],$$

$$B := \frac{2M^2}{M-aM-1}.$$

Thus, we use the initial condition,

$$\langle T\rangle_2 = \frac{M^2a + 2Ma + 2M + a + 2}{M+2},$$

eq. (18) is solved inductively to obtain the rigorous expression for the MFPT:

$$\langle T\rangle_g = \left[\frac{(N_2-1)\langle T\rangle_2+B(g-2)}{(M+1)^2}+\frac{AM}{(M-aM-1)(M+1)^2}\right] \times\frac{(M+1)^g}{(M+1)^g-1}-\frac{AM^3}{(M-aM-1)(M+1)^2(aM+1)^2} \cdot\frac{(M+1)^g(aM+1)^g}{M^g[(M+1)^g-1]}. \tag{19}$$

The formulation of $\langle T\rangle_g$ has been obtained. Moreover, we continue to show how clearly MFPT exhibits dependence on network order N_g. Recalling $N_g = (M+1)^g$, we have $g = \log_{M+1} N_g$. Hence, eq. (19) can be rewritten as follows:

$$\langle T\rangle_g = \left[\frac{(N_2-1)\langle T\rangle_2+B(g-2)}{(M+1)^2}+\frac{AM}{(M-aM-1)(M+1)^2}\right]\frac{N_g}{N_g-1} -\frac{AM^3}{(M-aM-1)(M+1)^2(aM+1)^2}\cdot\frac{N_g(aM+1)^g}{(N_g-1)M^g}. \tag{20}$$

When the network system is very large, i.e. $N_g \to \infty$,

$$\langle T\rangle_g \sim (N_g)^{\theta(a)} = (N_g)^{\ln\left(\frac{aM+1}{M}\right)/\ln(M+1)}. \tag{21}$$

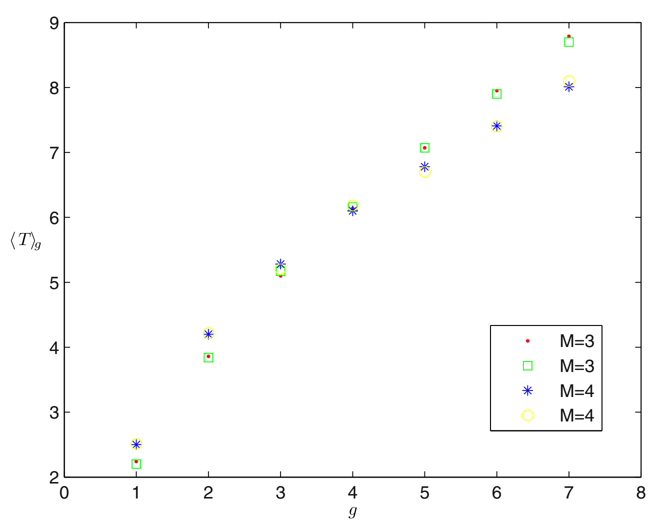

Figure 3. Mean first-passage time, $\langle T\rangle_g$, as a function of the iteration g and the edge weight $a(0 < a < 1)$ on the weighted hierarchical modular networks for $M = 3$ and $M = 4$, when $a = 0.2$. The filled symbols are the data from genuine simulations of the weighted trapping process; the empty circles and squares represent the exact values given by eq. (20).

This confirms that in the large g limit, the MFPT $\langle T\rangle_g$ increases as a power-law function of the network order with the exponent, represented by $\theta(a) = \ln(\frac{aM+1}{M})/\ln(M+1)$, being an increasing function of a. When a increases from 0 to $(M-1)/M$, the exponent increases from $-[\ln M/\ln(M+1)]$ and approaches 0, indicating that $\langle T\rangle_g$ increases as negative power-law with the network order. When a increases from $(M-1)/M$ to 1, the exponent increases from 0 and approaches $\ln\left(\frac{M+1}{M}\right)/\ln(M+1)$, indicating that $\langle T\rangle_g$ increases sublinearly with the network order. We find that the asymptotic behaviour of $\langle T\rangle_g$ coincides with $\langle T\rangle_t$, if $a = 1$ (see [9]). This also means that the efficiency of the trapping process depends on is the parameter a: the smaller the value of a, the more efficient is the trapping process.

We have made comparisons of theoretical predictions and computer simulation results of the mean first-passage time (MFPT) when $a = 0.2$ (see figure 3). It shows that mean first-passage time $\langle T\rangle_g$ is a function of the iteration g and the edge weight a $(0 < a < 1)$ on the weighted hierarchical modular networks for $M = 3$ and $M = 4$. The filled symbols are the data from genuine simulations of the weighted trapping process; the empty triangles and squares represent the exact values given by eq. (20). The empty circles and squares are the analytical expressions, while filled symbols are the corresponding data fitting of the distribution.

4. Conclusion

In the preceding text, we have investigated trapping on the weighted hierarchical networks. The weighted networks can mimic some real-world natural and social systems to some extent [20–22]. We focus on the MFPT for random walks on the weighted hierarchical networks, with the only trap located at the hub node having the largest degree. Due to the generating functions, we derived the recursive relations governing the evolution of the MFPT. By means of these recursive relations, we determined the solution for the MFPT, which shows that the MFPT $\langle T\rangle_g$ varies algebraically with the weighted network order N_g as $\langle T\rangle_g \sim (N_g)^{\theta(a)}$. When a increases from 0 to $(M-1)/M$, the exponent increases from $-[\ln M/\ln(M+1)]$ and approaches 0, indicating that $\langle T\rangle_g$ increases as negative power-law with the network order. Thus, we conclude that when a increases from $(M-1)/M$ to 1, the exponent increases from 0 and approaches $\ln\left(\frac{M+1}{M}\right)/\ln(M+1)$, indicating that $\langle T\rangle_g$ increases sublinearly with the network order. When $0 < a < 1$, the trapping process is more efficient than $a = 1$.

Acknowledgements

This research is supported by the National Science Foundation of China (11071224) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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