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Solitary wave solutions to nonlinear evolution equations in mathematical physics

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Abstract. This paper obtains solitons as well as other solutions to a few nonlinear evolution equations that appear in various areas of mathematical physics. The two analytical integrators that are applied to extract solutions are tan–cot method and functional variable approaches. The soliton solutions can be used in the further study of shallow water waves in (1+1) as well as (2+1) dimensions.

Keywords. Evolution equations; solitons; integrability.

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1. Introduction

Nonlinear evolution equations (NLEEs) form the most fundamental fabric in mathematical physics. These equations govern various physical phenomena in industry and nature [1–25]. The dynamics of shallow water waves, propagation of pulses through optical fibres, solitons in plasmas, nuclear physics and Davydov solitons in α-helix proteins are all dictated by NLEEs. Therefore, it is mandated to take a deeper look at these NLEEs from a different perspective.

Exact solutions to NLEEs will always add elite material in this literature. While numerical simulations always give a pictorial view to these equations, it is always an analytical or exact solution that adds extra flavour to this area of research. Therefore, it is imperative to investigate various tools of integration that extract various forms of analytical solutions to these equations. This paper is devoted to the search of analytical solutions to several

NLEEs that appear in various spheres of life. Two integration tools are applied to obtain these solutions. They are tan–cot functional method and functional variable method. The overwhelming results will be widely applicable in mathematical physics wherever these equations are studied.

There are several equations that will be studied in this paper. They are: Jaulent–Miodek hierarchy [1], the Kadomtsov–Petviashvili Benjamin–Bona–Mahony (KP-BBM) equation [2], the nonlinear Zakharov–Kuznetsov Benjamin–Bona–Mahony (ZK-BBM) equation [3], the Calogero–Degasperis (CD) [4] and potential Kadomstev–Petviashvili (pKP) equations [5,6].

These equations are studied frequently in mathematical physics and they originate from various physical phenomena in daily life. Jaulent–Miodek equation is yet another NLEE that models the dynamics of shallow water waves [7]. KP-BBM equation is used to study bidirectional waves in an offshore structure, where fluid flow is relatively unbounded [8]. ZK-BBM equation is studied in the context of plasma physics. It is well known that ZK equation models are weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasmas. ZK-BBM equation is the conjunction of ZK equation and BBM equation that models shallow water waves, for order to modify the dispersion term for ZK equation [9]. CD equation describes (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis [10]. Finally, the pKP equation is another model for studying the shallow water waves in (2+1) dimensions. This model serves as a two-dimensional extension of the potential KdV equation that studies shallow water waves in (1+1) dimensions [10–12].

1.1 *Governing equations*

The governing equations for the model are, respectively, given by

$$w_t + \frac{1}{4}(w_{xx} - 2w^3)_x + \frac{3}{4}\left(\frac{1}{4}\partial_x^{-1} w_{yy} + w_x \partial_x^{-1} w_y\right) = 0, \tag{1}$$

$$(u_t + u_x - a(u^n)_x - b(u^n)_{xxt})_x + k u_{yy} = 0, \tag{2}$$

$$u_t + u_x + a(u^n)_x + b(u_{xt} + u_{yy})_x = 0, \tag{3}$$

$$u_t - 4u_x u_{xx} - 2u_y u_{xx} + u_{xxxy} = 0 \tag{4}$$

and

$$u_{xt} + \frac{3}{2} u_x u_{xx} + \frac{1}{4} u_{xxxx} + \frac{3}{4} u_{yy} = 0. \tag{5}$$

The two subsequent sections will devote to integrate these equations sequentially. The integration algorithm will be first described in both of these sections and will then be applied to extract exact solutions to these equations. The solutions are of various types ranging from singular periodic solutions to solitary wave solutions.

2. Tan–cot functional method

Consider the nonlinear partial differential equation in the form

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0, \tag{6}$$

where $u(x, y, t)$ is a travelling wave solution of nonlinear partial differential equation. We use the transformations $u(x, y, t) = f(\xi)$, where

$$\xi = x + y + \lambda t. \tag{7}$$

This enables us to use the following changes:

$$\begin{aligned} &\frac{\partial}{\partial t}(\cdot) = \lambda \frac{\mathrm{d}}{\mathrm{d}\xi}(\cdot), \quad \frac{\partial}{\partial x}(\cdot) = \frac{\mathrm{d}}{\mathrm{d}\xi}(\cdot), \\ &\frac{\partial}{\partial y}(\cdot) = \frac{\mathrm{d}}{\mathrm{d}\xi}(\cdot), \quad \frac{\partial^2}{\partial t^2}(\cdot) = \lambda^2 \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}(\cdot). \end{aligned} \tag{8}$$

Use eq. (7) to transfer the nonlinear partial differential equation, eq. (6), to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', ...) = 0. \tag{9}$$

The ordinary differential equation (9) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the forms [13]

$$f(\xi) = \alpha \tan^\beta(\mu\xi), \qquad |\xi| \le \frac{\pi}{2\mu} \tag{10}$$

or

$$f(\xi) = \alpha \cot^\beta(\mu\xi), \qquad |\xi| \le \frac{\pi}{2\mu}, \tag{11}$$

with the derivatives of eq. (10):

$$\begin{aligned} f'(\xi) &= \alpha\beta\mu \left\{\tan^{\beta-1}(\mu\xi) + \tan^{\beta+1}(\mu\xi)\right\}, \\ f''(\xi) &= \alpha\beta\mu^2\left\{(\beta-1)\tan^{\beta-2}(\mu\xi) + 2\beta\tan^\beta(\mu\xi) + (\beta+1)\tan^{\beta+2}(\mu\xi)\right\}, \\ f'''(\xi) &= \beta\mu^3\alpha\{(\beta-1)(\beta-2)\tan^{\beta-3}(\mu\xi) + (3\beta^2-3\beta+2)\tan^{\beta-1}(\mu\xi) \\ &\quad +(\beta+1)(\beta+2)\tan^\beta(\mu\xi) + 2\beta^2\tan^{\beta+1}(\mu\xi) \\ &\quad +(\beta+1)(\beta+2)\tan^{\beta+2}(\mu\xi)\}, \end{aligned}$$

where α, μ and β are parameters to be determined, μ and λ are the wavenumber and the wave speed, respectively. We substitute eq. (10) into the reduced equation, eq. (9), to balance the terms of the tan functions and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\tan^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns α, μ and β, and solve the subsequent system.

2.1 *Applications*

2.1.1 *Jaulent–Miodek hierarchy.* The equation generates by the Jaulent–Miodek hierarchy is in the form [1]:

$$w_t + \frac{1}{4}(w_{xx} - 2w^3)_x + \frac{3}{4}\left(\frac{1}{4}\partial_x^{-1} w_{yy} + w_x \partial_x^{-1} w_y\right) = 0. \tag{12}$$

Wafaa *et al* [1] proposed the (G'/G)-expansion method for constructing more general exact solutions of the nonlinear $(2+1)$-dimensional equation generated by the Jaulent–Miodek hierarchy. To remove the integral term, assume

$$w(x, y, t) = u_x(x, y, t). \tag{13}$$

This recasts eq. (12) to

$$u_{xt} + \frac{1}{4}u_{xxxx} - \frac{3}{2}u_x^2 u_{xx} + \frac{3}{16}u_{yy} + \frac{3}{4}u_{xx}u_y = 0. \tag{14}$$

Let $u(x, y, t) = u(\xi)$, where ξ is defined in eq. (7) and eq. (14) becomes

$$\lambda u'' + \frac{1}{4}u'''' - \frac{3}{2}(u')^2 u'' + \frac{3}{16}u'' + \frac{3}{4}u''u' = 0. \tag{15}$$

Equation (15) can be written as

$$\lambda u'' + \frac{1}{4}u'''' - \frac{1}{2}((u')^3)' + \frac{3}{16}u'' + \frac{3}{8}[(u')^2]' = 0. \tag{16}$$

Integrating eq. (16) with respect to ξ once with zero constants yields

$$\lambda u' + \frac{1}{4}u''' - \frac{1}{2}(u')^3 + \frac{3}{16}u' + \frac{3}{8}(u')^2 = 0. \tag{17}$$

Assume

$$u' = v. \tag{18}$$

Substituting eq. (18) into eq. (17), we obtain an ordinary differential equation as follows:

$$\left(\lambda + \frac{3}{16}\right)v + \frac{1}{4}v'' - \frac{1}{2}v^3 + \frac{3}{8}v^2 = 0. \tag{19}$$

Seeking the solution of the form (10), we get

$$\begin{aligned}&\left(\lambda + \frac{3}{16} + \frac{1}{2}\beta^2\mu^2\right)\tan^{\beta}(\mu\xi)\\ &+\frac{1}{4}\beta\mu^2\left[(\beta-1)\tan^{\beta-2}(\mu\xi) + (\beta+1)\tan^{\beta+2}(\mu\xi)\right]\\ &-\frac{1}{2}\alpha^2\tan^{3\beta}(\mu\xi) + \frac{3}{8}\alpha\tan^{2\beta}(\mu\xi) = 0.\end{aligned} \tag{20}$$

Equating the exponents of some terms in eq. (20),

$$3\beta = \beta + 2 \Leftrightarrow \beta = 1. \tag{21}$$

From the following system of equations

$$\begin{aligned}&\mu^2 - \alpha^2 = 0,\\ &\lambda + \frac{3}{16} + \frac{1}{2}\mu^2 = 0,\end{aligned} \tag{22}$$

one obtains

$$\alpha = \mp\mu, \quad \lambda = -\frac{1}{2}\left(\frac{3}{8} + \mu^2\right). \tag{23}$$

Then the solution of eq. (19) is

$$v(\xi) = \mp\mu\tan(\mu\xi)\,, \qquad |\xi| \le \frac{\pi}{2\mu}. \tag{24}$$

Integrating eq. (24) with respect to ξ, gives

$$u(\xi) = \mp\ln[\cos(\mu\xi)] + c \tag{25}$$

or

$$u(\xi) = \mp\ln\left[\cos\left(\mu\left(x+y-\frac{1}{2}\left(\frac{3}{8}+\mu^2\right)t\right)\right)\right] + c. \tag{26}$$

Then the solution of eq. (12) is

$$w(x,y,t) = \mp\mu\tan\left[\mu\left(x+y-\frac{1}{2}\left(\frac{3}{8}+\mu^2\right)t\right)\right],$$

$$\left(x+y-\frac{1}{2}\left(\frac{3}{8}+\mu^2\right)t\right) \le \frac{\pi}{2\mu}. \tag{27}$$

For $\mu = 1,\ t = 0.1$

$$w(x,y,0.1) = \mp\tan\left(x+y-\frac{3}{160}\right), \qquad x+y \le 1.589546. \tag{28}$$

2.1.2 *Kadomtsev–Petviashvili Benjamin–Bona–Mahony equation.* The Kadomtsev–Petviashvili Benjamin–Bona–Mahony (KP-BBM) equation is of the form in eq. (2). Wazwaz [2] used the extended tanh method to solve it. The tan–cot method will be applied now to solve the KP-BBM equation.

Let $u(x,y,t) = u(\xi)$, where ξ is defined in eq. (7), then eq. (2) becomes

$$(\lambda u' + u' - a(u^n)' - \lambda b(u^n)''')' + ku'' = 0. \tag{29}$$

Integrating eq. (29) twice with zero constant:

$$\lambda u + u - au^n - \lambda b(u^n)'' + ku = 0. \tag{30}$$

Equation (30) can be written as

$$(\lambda + k + 1)\,u - au^n - \lambda bn(n-1)u^{n-2}(u')^2 - \lambda bnu^{n-1}u'' = 0. \tag{31}$$

Seeking the solution in (10), eq. (31) becomes

$$\begin{aligned}
&(\lambda+k+1)\alpha\tan^{\beta}(\mu\xi) - a\alpha^n\tan^{n\beta}(\mu\xi) - \lambda bn(n-1)\alpha^n\beta^2\mu^2\\
&\times[\tan^{(n\beta-2)}(\mu\xi) + 2\tan^{n\beta}(\mu\xi) + \tan^{(n\beta+2)}(\mu\xi)]\\
&-\lambda bn\alpha^n\beta\mu^2[(\beta-1)\tan^{(n\beta-2)}(\mu\xi) + 2\beta\tan^{n\beta}(\mu\xi)\\
&+(\beta+1)\tan^{(n\beta+2)}(\mu\xi)] = 0.
\end{aligned} \tag{32}$$

Equating the exponents of some terms in eq. (32),

$$n\beta - 2 = \beta \Leftrightarrow \beta = \frac{2}{n-1}, \qquad n \neq 1. \tag{33}$$

We next collect all terms with the same power and set their coefficients to zero to get a system of algebraic equations:

$$(\lambda + k + 1)(n-1) - 2\lambda bn\left[2\mu^2 + \mu^2\left(\frac{3-n}{n-1}\right)\right]\alpha^{n-1} = 0,$$

$$a\alpha^n + 2\lambda bn\alpha^n \frac{4}{(n-1)}\mu^2 + 2\lambda bn\alpha^n \frac{4}{(n-1)^2}\mu^2 = 0,$$

$$4\lambda bn\alpha^n\mu^2 + 2\lambda bn\alpha^n\mu^2\left(\frac{n+1}{n-1}\right) = 0. \tag{34}$$

Solving system (34), then:

$$\alpha = \left[\frac{4n(\lambda+k+1)}{a(n+2)}\right]^{1/(n-1)}, \qquad \mu = \mp i\frac{n-1}{2n}\sqrt{\frac{a}{2\lambda b}},$$

$$u(x, y, t) = \left[\frac{4n(\lambda+k+1)}{a(n+2)}\right]^{1/(n-1)} \times \tan^{2/(n-1)}\left(\mp i\frac{n-1}{2n}\sqrt{\frac{a}{2\lambda b}}\,(x+y+\lambda t)\right) \tag{35}$$

or

$$u(x, y, t) = \mp i\left[\frac{4n(\lambda+k+1)}{a(n+2)}\right]^{1/(n-1)} \times \tanh^{2/(n-1)}\left(\frac{n-1}{2n}\sqrt{\frac{a}{2\lambda b}}\,(x+y+\lambda t)\right). \tag{36}$$

For $n = \frac{1}{3},\ a = 2,\ k = b = \lambda = 1,\ t = 0.5$

$$u(x, y, t) = \mp i\left[\frac{7}{6}\right]^{3/2}\coth^3(x + y + 0.5)\,. \tag{37}$$

2.1.3 *Zakharov–Kuznetsov Benjamin–Bona–Mahony equation.* The Zakharov–Kuznetsov Benjamin–Bona–Mahony (ZK-BBM) equation is of the form in eq. (3) [2]. Mahmoudi *et al* [3] studied and solved this equation by using the exp-function method, while Wazwaz [2] used the extended tanh method to solve it. The tan–cot method will be applied now to solve the ZK-BBM equation.

Let $u(x, y, t) = u(\xi)$, where ξ is defined in eq. (7), and eq. (3) becomes

$$\lambda u' + u' + a(u^n)' + b(\lambda u'' + u'')' = 0. \tag{38}$$

Integrating eq. (38) once with zero constant:

$$(\lambda + 1)u + au^n + b(\lambda + 1)u'' = 0. \tag{39}$$

Seeking the solution as the form given by (10), eq. (39) becomes

$$(\lambda+1)\alpha\tan^{\beta}(\mu\xi)+a\alpha^{n}\tan^{n\beta}(\mu\xi)$$
$$+b(\lambda+1)\alpha\beta\mu^{2}\left[(\beta-1)\tan^{\beta-2}(\mu\xi)+2\beta\tan^{\beta}(\mu\xi)\right.$$
$$\left.+(\beta+1)\tan^{\beta+2}(\mu\xi)\right]=0. \tag{40}$$

Equating the exponents of some terms in eq. (40),

$$\beta+2=n\beta\Leftrightarrow\beta=\frac{2}{n-1},\qquad n\neq1. \tag{41}$$

We next collect all terms with the same power and set their coefficients to zero to get a system of algebraic equations:

$$a\alpha^{n}+2b(\lambda+1)\alpha\mu^{2}\frac{(n+1)}{(n-1)^{2}}=0,$$
$$(\lambda+1)+2b(\lambda+1)\beta^{2}\mu^{2}=0. \tag{42}$$

Solving system (42), gives

$$\alpha=\left[(\lambda+1)\frac{(n+1)}{4a}\right]^{1/(n-1)},\qquad\mu=\mp i\sqrt{\frac{1}{2b}}\frac{(n-1)}{2}, \tag{43}$$

$$u(x,y,t)=\left[(\lambda+1)\frac{(n+1)}{4a}\right]^{1/(n-1)}$$
$$\times\tan^{2/(n-1)}\left(\mp i\sqrt{\frac{1}{2b}}\frac{(n-1)}{2}(x+y+\lambda t)\right) \tag{44}$$

or

$$u(x,y,t)=\mp i\left[(\lambda+1)\frac{(n+1)}{4a}\right]^{1/(n-1)}$$
$$\times\tanh^{2/(n-1)}\left(\mp i\sqrt{\frac{1}{2b}}\frac{(n-1)}{2}(x+y+\lambda t)\right). \tag{45}$$

For $n=2,\ a=2,\ b=0.5,\ \lambda=1,\ t=0.5$

$$u(x,y,t)=\mp i\frac{3}{4}\tanh^{2}\left(\frac{1}{2}(x+y+0.5)\right). \tag{46}$$

2.1.4 *Calogero–Degasperis equation.* The Calogero–Degasperis (CD) equation, also known as breaking soliton equation, is used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis [4]. Solitary waves are wave packets or pulses which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects, these waves retain a stable waveform. A soliton is a very special type of solitary wave, which also keeps its waveform after collision with other solitons. Anwar *et al* [6] solved the CD equation by using tanh–coth method.

The tan–cot method is applied in this paper to solve the CD equation as in the following: Let $u(x, y, t) = u(\xi)$, where ξ is defined in eq. (7), eq. (4) becomes

$$\lambda u' - 6u'u'' + u'''' = 0. \tag{47}$$

Equation (47) can be written as

$$\lambda u' - 3[(u')^2]' + u'''' = 0. \tag{48}$$

Integrating eq. (48) once with zero constants

$$\lambda u - 3(u')^2 + u''' = 0. \tag{49}$$

Seeking the solution in (10), eq. (49) becomes

$$\begin{aligned}\lambda \tan^{\beta}(\mu\xi)–3\alpha\beta^2\mu^2 &\left[\tan^{2\beta-2}(\mu\xi) + 2\tan^{2\beta}(\mu\xi) + \tan^{2\beta+2}(\mu\xi)\right] \\ &+ \beta\mu^3[(\beta-1)(\beta-2)\tan^{\beta-3}(\mu\xi) + \left(3\beta^2 - 3\beta + 2\right)\tan^{\beta-1}(\mu\xi) \\ &+ (\beta+1)(\beta+2)\tan^{\beta}(\mu\xi) + 2\beta^2\tan^{\beta+1}(\mu\xi) \\ &+ (\beta+1)(\beta+2)\tan^{\beta+2}(\mu\xi)] = 0.\end{aligned} \tag{50}$$

Equating the exponents of some terms in eq. (50),

$$\beta = 2\beta - 2 \Leftrightarrow \beta = 2. \tag{51}$$

Collecting all terms with the same power and set their coefficients to zero to get a system of algebraic equations:

$$\begin{aligned}&\lambda - 12\alpha\mu^2 + 24\mu^3 = 0, \\ &-\alpha + \mu = 0.\end{aligned} \tag{52}$$

Solving system (52), then gives

$$\lambda = -12\mu^3, \qquad \alpha = \mu. \tag{53}$$

$$u(x, y, t) = \mu\tan^2\left[\mu\left(x + y - 12\mu^3 t\right)\right]. \tag{54}$$

For $\mu = 1,\ t = 0.5$

$$u(x, y, t) = \tan^2(x + y - 6). \tag{55}$$

2.1.5 *Potential Kadomtsev–Petviashvili equation.* New soliton solutions are obtained for (2+1)-dimensional potential Kadomtsev–Petviashvili (PKP) equation by using the tanh–coth method by Anwar *et al* [6]. The tan–cot method is applied in this paper to solve the PKP equation as follows:

Let $u(x, y, t) = u(\xi)$, where ξ is defined in eq. (7), eq. (5) becomes

$$\lambda u'' + \frac{3}{2}u'u'' + \frac{1}{4}u'''' + \frac{3}{4}u'' = 0. \tag{56}$$

Equation (56) can be written as

$$\lambda u' + \frac{3}{4}[(u')^2]' + \frac{1}{4}u'''' + \frac{3}{4}u'' = 0. \tag{57}$$

Integrating eq. (57) once with zero constants

$$\lambda u + \frac{3}{4}(u')^2 + \frac{1}{4}u''' + \frac{3}{4}u' = 0. \tag{58}$$

Seeking the solution in (10), eq. (58) becomes

$$\begin{aligned} &4\lambda \tan^{\beta}(\mu\xi) + 3\alpha\beta^2\mu^2\left[\tan^{2\beta-2}(\mu\xi) + 2\tan^{2\beta}(\mu\xi) + \tan^{2\beta+2}(\mu\xi)\right] \\ &\quad + \beta\mu^3[(\beta-1)(\beta-2)\tan^{\beta-3}(\mu\xi) + \left(3\beta^2-3\beta+2\right)\tan^{\beta-1}(\mu\xi) \\ &\quad + (\beta+1)(\beta+2)\tan^{\beta}(\mu\xi) + 2\beta^2\tan^{\beta+1}(\mu\xi) \\ &\quad + (\beta+1)(\beta+2)\tan^{\beta+2}(\mu\xi)] + 3\beta\mu[\tan^{\beta-1}(\mu\xi) \\ &\quad + \tan^{\beta+1}(\mu\xi)] = 0. \end{aligned} \tag{59}$$

Equating the exponents of some terms in eq. (59),

$$\beta = 2\beta - 2, \qquad \beta = 2\beta + 2 \Leftrightarrow \beta = 2. \tag{60}$$

Collect all terms with the same power and set their coefficients to zero to get a system of algebraic equations:

$$\begin{aligned} &4\lambda + 12\alpha\mu^2 + 24\mu^3 = 0, \\ &24\alpha\mu^2 + 24\mu^3 = 0, \\ &16\mu^3 + 6\mu = 0. \end{aligned} \tag{61}$$

Solving system (61), then gives

$$\lambda = \mp\frac{9}{16}\sqrt{\frac{3}{2}}i, \quad \alpha = \pm\frac{i}{2}\sqrt{\frac{3}{2}}, \quad \mu = \mp\frac{i}{2}\sqrt{\frac{3}{2}}. \tag{62}$$

Then

$$u(x,y,t) = \pm\frac{i}{2}\sqrt{\frac{3}{2}}\tan^2\left(\mp\frac{i}{2}\sqrt{\frac{3}{2}}\left(x + y \mp \frac{9}{16}\sqrt{\frac{3}{2}}it\right)\right) \tag{63}$$

or

$$u(x,y,t) = \pm\sqrt{\frac{3}{8}}\tanh^2\left(\mp\sqrt{\frac{3}{8}}\left(x + y \mp \frac{9}{16}\sqrt{\frac{3}{2}}it\right)\right). \tag{64}$$

3. Functional variable method

The functional variable method, which is a direct and effective algebraic method for the computation of compactons, solitons, solitary patterns and periodic solutions, was first proposed by Zerarka *et al* [14]. This method was further developed by many authors in

[10,15–17]. We now summarize the functional variable method, established by Zerarka *et al* [14], the details of which can be found in [10,15–17] among many others.

Consider a general nonlinear PDE in the form

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t \partial x}, \ldots\right) = 0, \tag{65}$$

where P is a polynomial in u and its partial derivatives. Using a wave variable, $\xi = \alpha_0 t + \alpha_1 x + \alpha_2 y + \delta$, so that

$$u(x, y, t) = U(\xi), \tag{66}$$

eq. (65) can be converted to an ordinary differential equation (ODE) as

$$Q\left(U, U', U'', U''', \ldots\right) = 0, \tag{67}$$

where Q is a polynomial in $U = U(\xi)$ and prime denotes derivative with respect to ξ. If all terms contain derivatives, then eq. (67) is integrated, where integration constants are considered zeros.

Let us make a transformation in which the unknown function $U(\xi)$ is considered as a functional variable in the form

$$U_\xi = F(U), \tag{68}$$

and some successively derivatives of U are

$$\begin{aligned} U_{\xi\xi} &= \frac{1}{2}(F^2)', \\ U_{\xi\xi\xi} &= \frac{1}{2}(F^2)''\sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'], \end{aligned} \tag{69}$$

where $' = \mathrm{d}/\mathrm{d}U$.

The ODE (67) can be reduced in terms of U, F and their derivatives upon using the expressions of eq. (69) into eq. (67) give

$$R(U, F, F', F'', F''', \ldots) = 0. \tag{70}$$

The key idea of this particular form of eq. (70) is of special interest, because it admits analytical solutions for a large class of nonlinear wave-type equations. After integration, eq. (70) provides the expression of F, and this together with eq. (68), give the relevant solutions to the original problem.

Remark

The functional variable method definitely can be applied to nonlinear PDEs which can be converted to a second-order ordinary differential equation (ODE) through the travelling wave transformation.

3.1 *Applications*

In this section, we present two examples to illustrate the applicability of the functional variable method to establish compactons, solitons and periodic solutions of nonlinear PDEs.

3.1.1 *(3+1)-Dimensional generalized Kadomtsev–Petviashvili equation.* The $(3+1)$-dimensional generalized Kadomtsev–Petviashvili (gKP) equation, given by [18,19]

$$(q_t + 6q^n q_x + q_{xxx})_x + 3q_{yy} + 3q_{zz} = 0, \tag{71}$$

describes the dynamics of solitons and nonlinear waves in plasma physics and fluid dynamics [20].

Under the travelling wave transformation

$$u(x, y, t) = U(\xi), \qquad \xi = ax + by + cz - vt, \tag{72}$$

we have

$$a(-vU' + 6aU^n U' + a^3 U''')' + 3b^2 U'' + 3c^2 U'' = 0, \tag{73}$$

where $U = U(\xi)$ and prime denotes derivative with respect to ξ.

Integrating eq. (73) twice with respect to ξ and neglecting the constants of integration, yields

$$(3(b^2 + c^2) - av)U + \frac{6a^2}{n+1} U^{n+1} + a^4 U'' = 0. \tag{74}$$

Then, we use the transformation

$$U_\xi = F(U), \tag{75}$$

that will convert eq. (74) to

$$(3(b^2 + c^2) - av)U + \frac{6a^2}{n+1} U^{n+1} + \frac{a^4}{2}(F^2(U))' = 0. \tag{76}$$

Thus, we get from eq. (76), the expression of the function $F(U)$ which reads as

$$F(U) = \frac{\sqrt{av - 3(b^2 + c^2)}}{a^2}\, U \sqrt{1 - \frac{12a^2}{(av - 3(b^2 + c^2))(n+1)(n+2)} U^n}. \tag{77}$$

After making the change of variables

$$Z = \frac{12a^2}{(av - 3(b^2 + c^2))(n+1)(n+2)} U^n, \tag{78}$$

and using the relation (75), the solution of eq. (74) is in the following form:

$$U(\xi) = \left\{ \frac{(av - 3(b^2 + c^2))(n+1)(n+2)}{12a^2} \times \operatorname{sech}^2\left(\frac{n\sqrt{av - 3(b^2 + c^2)}}{2a^2} (\xi + \xi_0) \right) \right\}^{1/n}. \tag{79}$$

Using the travelling wave transformation (72), we obtain the following soliton solutions of the $(3+1)$-dimensional gKP equation:

$$q_1(x, y, z, t) = \left\{ \frac{(av - 3(b^2 + c^2))(n+1)(n+2)}{12a^2} \times \operatorname{sech}^2\left(\frac{n\sqrt{av - 3(b^2 + c^2)}}{2a^2} (ax + by + cz - vt + \xi_0) \right) \right\}^{1/n} \tag{80}$$

and

$$q_2(x,y,z,t) = \left\{\frac{(3(b^2+c^2)-av)(n+1)(n+2)}{12a^2} \times \operatorname{csch}^2\left(\frac{n\sqrt{av-3(b^2+c^2)}}{2a^2}(ax+by+cz-vt+\xi_0)\right)\right\}^{1/n}. \tag{81}$$

It is easy to see that solutions (80) and (81) can reduce to singular periodic solutions as follows:

$$q_3(x,y,z,t) = \left\{\frac{(av-3(b^2+c^2))(n+1)(n+2)}{12a^2} \times \sec^2\left(\frac{n\sqrt{3(b^2+c^2)-av}}{2a^2}(ax+by+cz-vt+\xi_0)\right)\right\}^{1/n} \tag{82}$$

and

$$q_4(x,y,z,t) = \left\{\frac{(av-3(b^2+c^2))(n+1)(n+2)}{12a^2} \times \csc^2\left(\frac{n\sqrt{3(b^2+c^2)-av}}{2a^2}(ax+by+cz-vt+\xi_0)\right)\right\}^{1/n}. \tag{83}$$

3.1.2 *Generalized Benjamin equation.* We consider nonlinear generalized Benjamin equation which is given by

$$(q^m)_{tt} + a(q^n q_x)_x + \beta(q^m)_{xxxx} = 0, \tag{84}$$

where α and β are constants.

When $m = 1$, we have the following equation [18,19]:

$$q_{tt} + a(q^n q_x)_x + \beta q_{xxxx} = 0. \tag{85}$$

This kind of equation is one of the most important NLPDEs, used in the analysis of long waves in shallow water [21].

To look for the exact solutions of eq. (84), we make transformation

$$q(x,t) = U(\xi), \qquad \xi = \alpha_0 t + \alpha_1 x + \delta, \tag{86}$$

and generate the reduced nonlinear ODE in the form

$$\alpha_0^2 (U^m)'' + a\alpha_1^2 (U^n U')' + \beta\alpha_1^4 (U^m)'''' = 0. \tag{87}$$

Integrating (85) twice with respect to ξ and setting the constants of integration to be zero, we find

$$\alpha_0^2 U^m + \frac{a\alpha_1^2}{n+1} U^{n+1} + \beta\alpha_1^4 (U^m)'' = 0. \tag{88}$$

We use the transformation

$$U(\xi) = V^{1/m}(\xi), \tag{89}$$

that will reduce eq. (88) into the ODE

$$\alpha_0^2 V + \frac{a\alpha_1^2}{n+1} V^{(n+1)/m} + \beta\alpha_1^4 V'' = 0. \tag{90}$$

Following eq. (69), it is easy to deduce from (90) the expression of the function $F(V)$ which reads as

$$F(V) = \sqrt{-\frac{\alpha_0^2}{\beta\alpha_1^4}} V \sqrt{1 + \frac{2am\alpha_1^2}{(n+1)(n+m+1)\alpha_0^2} V^{(n+1-m)/m}}. \tag{91}$$

Using the change of variables

$$Z = -\frac{2am\alpha_1^2}{(n+1)(n+m+1)\alpha_0^2} V^{(n+1-m)/m}, \tag{92}$$

and using the relation (68), the solution of eq. (90) is in the following form:

$$V(\xi) = \left\{ -\frac{(n+1)(n+m+1)\alpha_0^2}{2am\alpha_1^2} \times \mathrm{sech}^2 \left(\frac{n+1-m}{2m} \sqrt{-\frac{\alpha_0^2}{\beta\alpha_1^4}} (\xi + \xi_0) \right) \right\}^{m/(n+1-m)}. \tag{93}$$

Using the transformation (89), we can obtain the following soliton solutions of eq. (84):

$$q_1(x,t) = \left\{ -\frac{(n+1)(n+m+1)\alpha_0^2}{2am\alpha_1^2} \times \mathrm{sech}^2 \left(\frac{n+1-m}{2m} \sqrt{-\frac{\alpha_0^2}{\beta\alpha_1^4}} (\alpha_0 t + \alpha_1 x + \delta + \xi_0) \right) \right\}^{1/(n+1-m)}, \tag{94}$$

$$q_2(x,t) = \left\{ \frac{(n+1)(n+m+1)\alpha_0^2}{2am\alpha_1^2} \times \mathrm{csch}^2 \left(\frac{n+1-m}{2m} \sqrt{-\frac{\alpha_0^2}{\beta\alpha_1^4}} (\alpha_0 t + \alpha_1 x + \delta + \xi_0) \right) \right\}^{1/(n+1-m)}. \tag{95}$$

It is easy to see that solutions (94) and (95) can reduce to singular periodic solutions as follows:

$$q_3(x,t) = \left\{ -\frac{(n+1)(n+m+1)\alpha_0^2}{2am\alpha_1^2} \times \sec^2\left(\frac{n+1-m}{2m}\sqrt{\frac{\alpha_0^2}{\beta\alpha_1^4}}(\alpha_0 t + \alpha_1 x + \delta + \xi_0)\right)\right\}^{1/(n+1-m)}, \tag{96}$$

$$q_4(x,t) = \left\{ -\frac{(n+1)(n+m+1)\alpha_0^2}{2am\alpha_1^2} \times \csc^2\left(\frac{n+1-m}{2m}\sqrt{\frac{\alpha_0^2}{\beta\alpha_1^4}}(\alpha_0 t + \alpha_1 x + \delta + \xi_0)\right)\right\}^{1/(n+1-m)}. \tag{97}$$

4. Conclusions

This paper obtained soliton, singular periodic as well as other solutions to several NLEEs that appear in mathematical physics. While several integration architectures are available, only two of them were employed in this paper. These results are going to be extremely useful in various areas of future research.

Several perturbation terms will be considered to integrate the perturbed versions of these equations. The soliton perturbation theory will be developed for these equations. In addition to deterministic perturbation terms, stochastic perturbation terms will also be taken into account. The Langevin equation will also determine the mean free soliton speed. These results will be discussed in detail in future publications and will be reported later.

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