PRAMANA — journal of physics

Vol. 83, No. 3
September 2014
pp. 331–340

The bifurcation and peakons for the special $C(3, 2, 2)$ equation

LINA ZHANG
Department of Mathematics, Huzhou University, Huzhou, Zhejiang 313000, People's Republic of China
E-mail: zsdzln@126.com

MS received 19 November 2013; revised 25 January 2014; accepted 28 January 2014
DOI: 10.1007/s12043-014-0778-6; ***e*Publication:** 29 June 2014

Abstract. In this paper, we investigate a special $C(3, 2, 2)$ equation

$$u_t + ku_x - u_{xxt} + 3(u^3)_x = u_x(u^2)_{xx} + u(u^2)_{xxx}.$$

The bifurcation and some new exact representations of peakons, bell-shaped solitary wave solutions and periodic cusp wave solutions for the equation are obtained using the qualitative theory of dynamical systems. It is shown that the peakons are actually the limit of bell-shaped solitary waves and periodic cusp waves. Moreover, a new characteristic of non-smooth solutions, two peakons coexisting for the same wave speed, is found. Some previous results are extended.

Keywords. $C(3, 2, 2)$ equation; peakons; bell-shaped solitary waves; periodic cusp waves.

PACS Nos 05.45.Yv; 02.30.Jr; 02.30.Oz

1. Introduction

Solitons and integrable systems play important roles in nonlinear waves, dynamical systems and analytical mechanics. During the past decades, increasing attention has been paid to certain nonlinear evolution equations that support non-smooth solitons such as peaked soliton solutions. A typical representative of such equations is the well-known Camassa–Holm (CH) equation:

$$u_t + ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{1}$$

which was derived by Camassa and Holm [1] as a shallow water wave model. The aspect that has driven considerable excitement is the equation's extensive integrable properties. For instance, it admits Lax representations, bi-Hamiltonian structures, multisoliton solutions and algebraic-geometric solutions [2–7]. The most remarkable feature of the CH equation (1) is that it can admit peakon solutions. A peakon solution is a weak solution that has a discontinuous first derivative at the wave peak and thus called peakon. For $k = 0$, Camassa and Holm [1] showed that eq. (1) has peakons of the form $u(x,t) = c\mathrm{e}^{-|x-ct|}$, where c is the wave speed. For $k \neq 0$, Liu *et al* [8] showed that eq. (1) has peakons of the form

$$u(x,t) = (c+k)\mathrm{e}^{-|x-ct|} - k. \tag{2}$$

To study the bifurcations of peakons, Liu and Qian [9] suggested a generalized CH equation:

$$u_t + ku_x - u_{xxt} + au^m u_x = 2u_x u_{xx} + uu_{xxx}. \tag{3}$$

Tian and Song [10] obtained some explicit expressions of peakons of eq. (3) for $m = 1$, 2, 3. Recently, Liu and Ouyang [11] applied bifurcation method of the phase portraits to study the following mCH equation with quadratic and cubic nonlinearities:

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx}, \tag{4}$$

where the coexistence of bell-shaped solitary wave and peakon for the same wave speed was reported. In 2004, Tian and Yin [12] introduced the following fully nonlinear generalized Camassa–Holm equation $C(m,n,p)$:

$$u_t + ku_x + \beta_1 u_{xxt} + \beta_2 (u^m)_x + \beta_3 u_x (u^n)_{xx} + \beta_4 u (u^p)_{xxx} = 0, \tag{5}$$

where $k, \beta_1, \beta_2, \beta_3, \beta_4$ are arbitrary real constants and m, n, p are positive integers. By using four direct ansatzs, the authors in ref. [12] obtained kink compacton solutions, non-symmetry compacton solutions and solitary wave solutions for $C(2,1,1)$ and $C(3,2,2)$ equations.

The main goal of this work is to find whether the fully nonlinear generalized Camassa–Holm equation $C(m,n,p)$ has non-smooth solutions such as peakon solutions or cuspon solutions. Generally, it is not an easy task to obtain a uniform analytic first integral of the corresponding travelling wave system of (5). Here, we consider the special case $\beta_1 = -\beta_2/3 = \beta_3 = \beta_4 = -1$, $m = 3$, $n = 2$ and $p = 2$. Then eq. (5) reduces to a special $C(3,2,2)$ equation

$$u_t + ku_x - u_{xxt} + 3(u^3)_x = u_x (u^2)_{xx} + u(u^2)_{xxx}. \tag{6}$$

In this paper, we employ bifurcation method of planar systems to investigate the dynamics of travelling wave solutions determined by the travelling wave system of (6) in their parameter space. Our method can determine in which cases of the parameter space one can obtain exact parametric representations of the travelling wave solutions of (6) and which orbits of the travelling wave system correspond to these parametric representations. We find that, in comparison with the CH equation and the mCH equation, the $C(3,2,2)$ equation has double peakons for suitable parameters.

The paper is organized as follows: In §2, we state our main results and make the numerical simulations of some travelling wave solutions. In §3, we discuss the bifurcation of phase portraits. In §4 and 5, we give theoretical derivations for the main results. A short conclusion is given in §6.

2. Main results

We state our main result as follows:

Theorem 1. *Let*

$$a = \sqrt{\frac{c}{2}}, \tag{7}$$

$$b = \sqrt{\frac{2(c-k)}{3}}, \tag{8}$$

$$A = \frac{\sqrt{3c} + \sqrt{2c+4k}}{2\sqrt{6}}, \tag{9}$$

$$B = \frac{\sqrt{3c} - \sqrt{2c+4k}}{2\sqrt{6}}, \tag{10}$$

$$T = \frac{\sqrt{3}}{2} \ln\left(\frac{\sqrt{3c} + \sqrt{2c+4k}}{\sqrt{c-4k}}\right). \tag{11}$$

Then we have the following results:

(1) *If $(c/4) < k < c$, eq. (6) has two bell-shaped solitary wave solutions of the form*:

$$\left(\frac{\sqrt{a^2-\phi^2}+\sqrt{b^2-\phi^2}}{\sqrt{a^2-b^2}}\right)\left(\frac{\sqrt{(a^2-b^2)\phi^2}}{a\sqrt{b^2-\phi^2}+b\sqrt{a^2-\phi^2}}\right)^a = \mathrm{e}^{-(\sqrt{3}/2)b|x-ct|} \tag{12}$$

and

$$\left(\frac{\sqrt{a^2-b^2}}{\sqrt{a^2-\phi^2}+\sqrt{b^2-\phi^2}}\right)\left(\frac{a\sqrt{b^2-\phi^2}+b\sqrt{a^2-\phi^2}}{\sqrt{(a^2-b^2)\phi^2}}\right)^a = \mathrm{e}^{(\sqrt{3}/2)b|x-ct|}. \tag{13}$$

(2) *If $k = c/4$, two peakon solutions coexist in eq. (6), which are expressed by*

$$\phi(x-ct) = \sqrt{\frac{c}{2}}\mathrm{e}^{-(\sqrt{3}/2)|x-ct|} \tag{14}$$

and

$$\phi(x-ct) = -\sqrt{\frac{c}{2}}\mathrm{e}^{-(\sqrt{3}/2)|x-ct|}. \tag{15}$$

(3) *If $-(c/2) < k < (c/4)$, then eq. (6) has the following exact representations of periodic cusp wave solutions*:

$$u(x,t) = A\mathrm{e}^{-(\sqrt{3}/2)|x-ct-2nT|} + B\mathrm{e}^{(\sqrt{3}/2)|x-ct-2nT|} \tag{16}$$

and

$$u(x,t) = -A\mathrm{e}^{-(\sqrt{3}/2)|x-ct-2nT|} - B\mathrm{e}^{(\sqrt{3}/2)|x-ct-2nT|}, \tag{17}$$

where $n = 0, \pm1, \pm2, \ldots,\ x-ct \in ((2n-1)T, (2n+1)T)$.

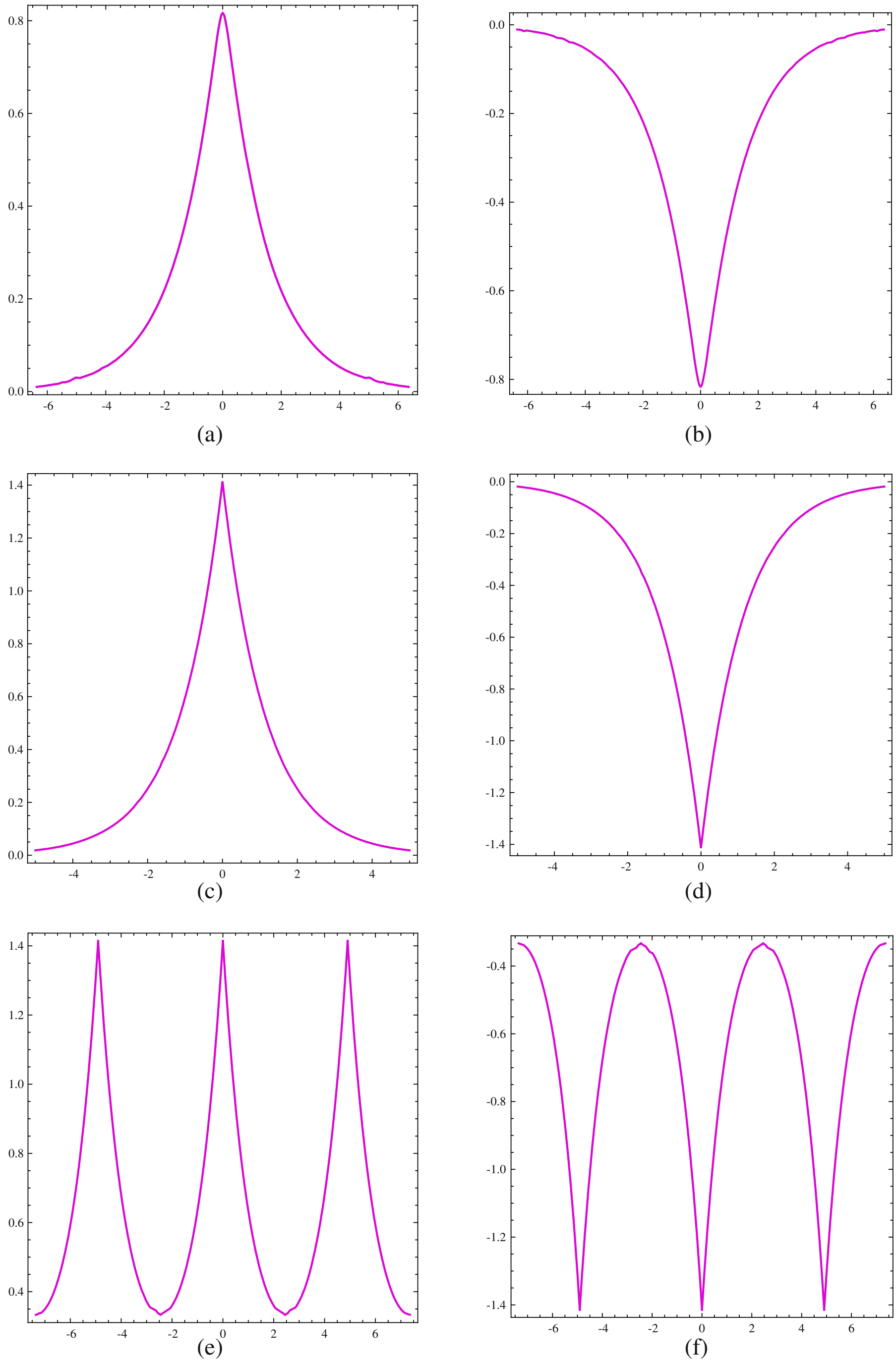

Figure 1. (**a**) Bell-shaped solitary wave of peak form, (**b**) bell-shaped solitary wave of valley form, (**c**) peakon, (**d**) valleyon, (**e**) periodic peak cusp wave and (**f**) periodic valley cusp wave.

By using the contour plot or plot of Wolfram Mathematica 8, planar graphs of the bell-shaped solitary wave solutions, the peakon solutions and the periodic cusp wave solutions can be displayed.

Example 1.

(1) Given $c = 2$ and $k = 1$, it follows that $a = 1,\ b = \sqrt{\frac{3}{2}} \doteq 0.816497$. From (12) and (13), we obtain the profiles of bell-shaped solitary wave solutions as shown in figures 1a and 1b, respectively.
(2) Taking $c = 4$ and $k = 1$, corresponding to (14) and (15), we obtain the profiles of peakon solutions as shown in figures 1c and 1d, respectively.
(3) Taking $c = 2$ and $k = \frac{1}{3}$, we get the the approximations of $A,\ B$ and T in eqs (16) and (17), where $A \doteq 0.971405,\ B \doteq 0.0285955$ and $T \doteq 2.03545$. Further, we obtain the profiles of periodic cusp wave solutions, which are shown in figures 1e and 1f, respectively.

3. Bifurcation conditions and possible phase portraits

Substituting $u(x,t) = \phi(\xi)$ with $\xi = x - ct$ into (6) yields

$$(k-c)\phi' + c\phi''' + 3(\phi^3)' = \phi'(\phi^2)'' + \phi(\phi^2)'''. \tag{18}$$

Integrating (18) once and setting the integration constant as zero, we have

$$(k-c)\phi + c\phi'' + 3\phi^3 = 2\phi(\phi')^2 + 2\phi^2\phi''. \tag{19}$$

Let $y = \phi'$, we get a planar integrable system

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{3\phi^3 - (c-k)\phi - 2\phi y^2}{2\phi^2 - c} \tag{20}$$

with the first integral

$$H(\phi, y) = (2(c-k)\phi^2 - 3\phi^4 - 2cy^2 + 4\phi^2 y^2)/8 = h. \tag{21}$$

We notice that the right-hand side of the second equation in eq. (20) is discontinuous when $\phi = \pm\sqrt{c/2}\ \ (c > 0)$. In other words, the function ϕ'' is not well defined on such straight lines of the phase plane (ϕ, y). It implies that a smooth travelling wave equation may have non-smooth travelling wave solutions. During the past decades, the majority of literature has been focussed on discussing the reasons causing the appearance of non-smooth travelling wave solutions under the condition that there exists only one singular straight line in the corresponding phase plane (see [8,9,11,13,14] and references therein). Here, we are interested in a study of the occurence of non-smooth travelling wave solutions when there exist two singular straight lines in the ordinary differential equation (20). Hence, we assume $c > 0$ in the rest of this paper.

Let $\mathrm{d}\xi = (2\phi^2 - c)\mathrm{d}\tau$, then (20) is changed to a regular system

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = (2\phi^2 - c)y, \qquad \frac{\mathrm{d}y}{\mathrm{d}\tau} = 3\phi^3 - (c-k)\phi - 2\phi y^2. \tag{22}$$

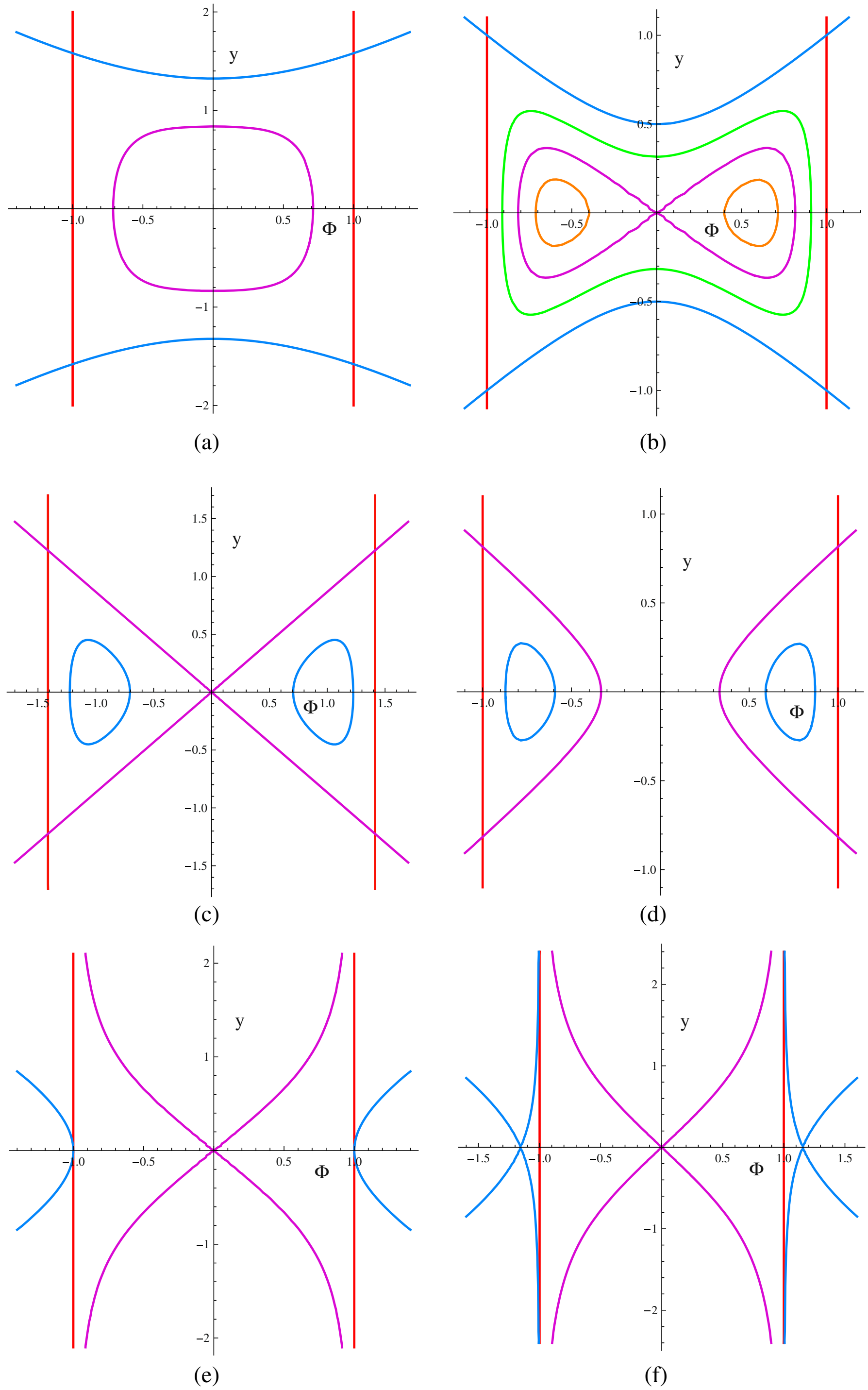

Figure 2. The phase portrait bifurcation of system (22).

System (22) has the same phase orbits as system (20) except for the straight lines $\phi = \pm\sqrt{c/2}$.

It is easy to see that system (22) has three equilibrium points $O(0,0)$ and $E_{1,2}(\pm\phi_1, 0)$ on the ϕ-axis when $k < c$, where $\phi_1 = \sqrt{(c-k)/3}$. For $k > -(c/2)$, system (22) has two equilibrium points $N_{1,2}(\sqrt{(c/2)}, \pm Y_0)$ on the straight line $\phi = \sqrt{(c/2)}$ and two equilibrium points $N_{3,4}(-\sqrt{(c/2)}, \pm Y_0)$ on the straight line $\phi = -\sqrt{(c/2)}$, where $Y_0 = \sqrt{(c+2k)}/2$.

Let $M(\phi, y)$ be the coefficient matrix of the linearized system of (22) at the equilibrium point (ϕ, y). We have

$$\det M(\pm\phi_1, 0) = 2(c-k)(c+2k)/3,$$

$$\det M(\pm\sqrt{c/2}, \pm Y_0) = -2c(c+2k).$$

Thus, when $-(c/2) < k < c$, the equilibrium points E_1 and E_2 are both centre points, and when $k > -(c/2)$, the equilibrium points N_i $(i = 1,2,3,4)$ are four saddle points.

According to the qualitative theory of dynamical systems, we draw the bifurcation of phase portraits of system (22) in figures 2a–2f.

4. The derivations on the bell-shaped solitary waves and peakons for the $C(3,2,2)$ equation

For $(c/4) < k < c$, there are two homoclinic orbits with ∞-shape determined by $H(\phi, y) = 0$ (see figure 2b). These two homoclinic orbits can be represented as

$$y = \pm\sqrt{\frac{3\phi^4 - 2(c-k)\phi^2}{2(2\phi^2 - c)}}. \tag{23}$$

Let $y = 0$, from (23), we get $\phi = \sqrt{2(c-k)/3}$, in which $\left(\sqrt{2(c-k)/3}, 0\right)$ and $\left(-\sqrt{2(c-k)/3}, 0\right)$ are the intersection points of the homoclinic orbits with the positive and negative ϕ-axes, respectively. Substituting (23) into the first equation of (20) and integrating along the homoclinic orbits, we get

$$\int_{\sqrt{\frac{2(c-k)}{3}}}^{\phi} \frac{1}{\phi}\sqrt{\frac{c-2\phi^2}{2(c-k)-3\phi^2}}\mathrm{d}\phi = -\int_0^{\xi} \frac{\sqrt{2}}{2}\mathrm{sgn}(\xi)\mathrm{d}\xi,$$
$$0 < \phi \le \sqrt{\frac{2(c-k)}{3}}, \tag{24}$$

$$\int_{-\sqrt{\frac{2(c-k)}{3}}}^{\phi} \frac{1}{\phi}\sqrt{\frac{c-2\phi^2}{2(c-k)-3\phi^2}}\mathrm{d}\phi = -\int_0^{\xi} \frac{\sqrt{2}}{2}\mathrm{sgn}(\xi)\mathrm{d}\xi,$$
$$-\sqrt{\frac{2(c-k)}{3}} \le \phi < 0. \tag{25}$$

From eqs (24) and (25), we then have the implicit representations of two bell-shaped solitary wave solutions, which are given in (12) and (13). Letting $(c/4) < k < c$ and $k \to c/4$ in eqs (24) and (25), we get

$$\phi(x-ct) \to \sqrt{\frac{c}{2}}\mathrm{e}^{-(\sqrt{3}/2)|x-ct|}, \qquad 0 < \phi \le \sqrt{\frac{2(c-k)}{3}} \tag{26}$$

and

$$\phi(x-ct) \to -\sqrt{\frac{c}{2}}\mathrm{e}^{-(\sqrt{3}/2)|x-ct|}, \qquad -\sqrt{\frac{2(c-k)}{3}} \le \phi < 0, \tag{27}$$

which implies that for $k = c/4$, eq. (6) has two peakons

$$u(x,t) = \sqrt{\frac{c}{2}}\mathrm{e}^{-(\sqrt{3}/2)|x-ct|} \tag{28}$$

and

$$u(x,t) = -\sqrt{\frac{c}{2}}\mathrm{e}^{-(\sqrt{3}/2)|x-ct|}. \tag{29}$$

Hence we obtain peakons from the limit of bell-shaped solitary wave solutions .

5. The derivations on the periodic cusp waves and peakons for the $C(3, 2, 2)$ equation

From eq. (18) and figure 2d, we know that when $-(c/2) < k < (c/4)$ and $h \in (c(c-4k)/32, (c-k)^2/24)$, system (20) has two families of uncountably infinite many periodic orbits. When $k \to c/4$, the periodic orbits lose their smoothness and evolve to periodic cusp orbits. Let $\Gamma_{1,2}^k$ be the two limiting curves of the periodic orbits as $h \to c(c-4k)/32$. By eq. (18) $\Gamma_{1,2}^k$ consists of

$$y^2 = \frac{6\phi^2 - c + 4k}{8}$$

for

$$-\sqrt{\frac{c}{2}} < \phi \le -\sqrt{\frac{c-4k}{6}} \quad \text{or} \quad \sqrt{\frac{c-4k}{6}} \le \phi < \sqrt{\frac{c}{2}} \tag{30}$$

and

$$\phi = \pm\sqrt{\frac{c}{2}} \qquad \text{for} \quad |y| \le \frac{\sqrt{c+2k}}{2}.$$

Substituting (30) into the the first equation of system (15) and integrating along Γ_1^k and Γ_2^k, respectively, we have

$$\int_{\sqrt{c/2}}^{\phi} -\sqrt{\frac{8}{6\phi^2-(c-4k)}}\mathrm{d}\phi = \int_0^{\xi} \mathrm{sgn}(\xi)\mathrm{d}\xi,$$
$$\phi \in \left[\sqrt{\frac{c}{2}}\sqrt{\frac{c-4k}{6}}\right] \tag{31}$$

and

$$\int_{-\sqrt{c/2}}^{\phi} \sqrt{\frac{8}{6\phi^2 - (c-4k)}} \mathrm{d}\phi = \int_0^{\xi} \mathrm{sgn}(\xi) \mathrm{d}\xi,$$
$$\phi \in \left[-\sqrt{\frac{c}{2}}, -\sqrt{\frac{c-4k}{6}} \right]. \tag{32}$$

Thus we get the following exact parametric representations of travelling wave solutions:

$$u(x,t) = \pm \left(A\mathrm{e}^{-(\sqrt{3}/2)|x-ct|} + B\mathrm{e}^{(\sqrt{3}/2)|x-ct|} \right),$$
$$0 \leq |x-ct| \leq \frac{\sqrt{3}}{2} \ln\left(\frac{\sqrt{3c} + \sqrt{2c+4k}}{\sqrt{c-4k}} \right), \tag{33}$$

where

$$A = \frac{\sqrt{3c} + \sqrt{2c+4k}}{2\sqrt{6}} \quad \text{and} \quad B = \frac{\sqrt{3c} - \sqrt{2c+4k}}{2\sqrt{6}}.$$

Let

$$T = \frac{\sqrt{3}}{2} \ln\left(\frac{\sqrt{3c} + \sqrt{2c+4k}}{\sqrt{c-4k}} \right),$$

we obtain the the following periodic cusp wave solutions:

$$u(x,t) = \pm \left(A\mathrm{e}^{-(\sqrt{3}/2)|x-ct-2nT|} + B\mathrm{e}^{(\sqrt{3}/2)|x-ct-2nT|} \right),$$
$$(2n-1)T \leq x-ct \leq (2n+1)T. \tag{34}$$

When $-(c/2) < k < (c/4)$, let $k \to (c/4)$, then $T \to \infty$, $A \to \sqrt{c/2}$ and $B \to 0$, thus we have

$$\lim_{k \to c/4} u(x,t) = \pm \sqrt{\frac{c}{2}} \mathrm{e}^{-(\sqrt{3}/2)|x-ct|}. \tag{35}$$

Hence we obtain the peakons (14) and (15) again from the limit of the periodic cusp waves as $k \to c/4$.

6. Conclusion

In this paper, we have obtained the exact parametric representations of peakons, periodic cusped wave solutions and bell-shaped solitary wave solutions for the $C(3, 2, 2)$ equation (6) by using the bifurcation method and qualitative theory of dynamical systems. A new phenomenon in which these travelling wave solutions always appear in pairs, due to the existence of two singular straight lines $\phi = \sqrt{c/2}$ and $\phi = -\sqrt{c/2}$ in the corresponding phase plane, is discovered. In particular, we demonstrate that the peakons can be obtained from two different approaches as given in §4 and 5.

Acknowledgement

This work is supported by the National Natural Science Foundation of China under Grant No. 11326131 and Zhejiang Provincical Natural Science Foundation of China under Grant No. LQ14A010009.

References

[1] R Camassa and D D Holm, *Phys. Rev. Lett.* **71**, 1661 (1993)
[2] R Camassa, D D Holm and J M Hyman, *Adv. Appl. Mech.* **31**, 1 (1994)
[3] P J Olver and P Rosenau, *Phys. Rev. E* **53**, 1900 (1996)
[4] A S Fokas and Q M Liu, *Phys. Rev. Lett.* **77**, 2347 (1996)
[5] H R Dullin, G A Gottwald and D D Holm, *Phys. Rev. Lett.* **87**, 194501 (2001)
[6] Z Qiao, *Commun. Math. Phys.* **239**, 309 (2003)
[7] Y Li and J Zhang, *Proc. R. Soc. London Ser. A* **460**, 2617 (2004)
[8] Z Liu, W Wang and Z Jing, *Chaos, Solitons and Fractals* **19**, 77 (2004)
[9] Z Liu and T Qian, *Int. J. Bifurcat. Chaos* **11**, 781 (2001)
[10] L Tian and X Song, *Chaos, Solitons and Fractals* **19**, 621 (2004)
[11] Z Liu and Z Ouyang, *Phys. Lett. A* **366**, 377 (2007)
[12] L Tian and J Yin, *Chaos, Solitons and Fractals* **20**, 289 (2004)
[13] B He, *Nonlinear Anal.: Real World Appl.* **11**, 364 (2010)
[14] J Li and H H Dai, *On the study of singular nonlinear traveling wave equations: Dynamical system approach* (Science Press, Beijing, 2007) (in English)