

On solving the Schrödinger equation for a complex deictic potential in one dimension

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Abstract. Making use of an ansatz for the eigenfunction, we investigate closed-form solutions of the Schrödinger equation for an even power complex deictic potential and its variant in one dimension. For this purpose, extended complex phase-space approach is utilized and nature of the eigenvalue and the corresponding eigenfunction is determined by the analyticity property of the eigenfunction. The imaginary part of the energy eigenvalue exists only if the potential parameters are complex, whereas it reduces to zero for real coupling parameters and the result coincides with those derived from the invariance of Hamiltonian under \mathcal{PT} operations. Thus, a non-Hermitian Hamiltonian possesses real eigenvalue, if it is \mathcal{PT} -symmetric.

Keywords. Non-Hermitian Hamiltonian; \mathcal{PT} -symmetry; closed-form solutions.

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1. Introduction

The complex Hamiltonian systems have generated a lot of interest [1–7] for the better theoretical understanding of several newly discovered phenomena in different branches of physics and chemistry (such as phenomena pertaining to resonance scattering in atomic, molecular and nuclear physics and to some chemical reactions [8–10]) but such studies have not been reached up to the desired level. However, besides some general remarks on complex Hamiltonian in nonlinear domain [11,12], efforts have been made to address both classical and quantum aspects [2,3,13] of a dynamical system. In classical context, $H(x, p)$ becomes the function of two complex variables and analyticity property of the Hamiltonian leads to a class of integrable systems [1,3]. On the other hand, in quantum context, $p = -i\hbar(\partial/\partial x)$, which implies

$$p_1 = \frac{-1}{2} \frac{\partial}{\partial p_2}, \quad x_2 = \frac{-1}{2} \frac{\partial}{\partial x_1},$$

then analyticity of $H(x, p)$ is transformed into the complex potential $V(x)$ [14]. It is well known, that a complex Hamiltonian is no longer Hermitian and ordinarily does not guarantee for real eigenvalues. However, in its \mathcal{PT} -symmetric version [4,5,15–17], the system is found to exhibit real eigenvalues. The reality of the spectrum is due to the combined action of parity and time-reversal invariance of the Hamiltonian [13]. The parity operator \hat{P} and the time-reversal operator \hat{T} [12,14] are given by

$$\hat{P}(x, p, i) \rightarrow (-x, -p, i) \quad \text{and} \quad \hat{T}(x, p, i) \rightarrow (x, -p, -i),$$

whereas, the combined action of parity and time-reversal operators, i.e., $\hat{P}\hat{T}$ operator is given by

$$\hat{P}\hat{T}(x, p, i) \rightarrow (-x, p, -i). \quad (1)$$

Now, consider the general form of a \mathcal{PT} -symmetric Hamiltonian, i.e., $\hat{\mathcal{H}} = \hat{p}^2 + x^2(ix)^\epsilon$, where ϵ is a real parameter. For $\epsilon \geq 0$, all eigenvalues of such a Hamiltonian are real and positive and the corresponding parametric region has unbroken \mathcal{PT} -symmetry, while for $\epsilon < 0$, the eigenvalues are complex and the parametric region has broken \mathcal{PT} -symmetry [14]. Thus in general, we can say that a \mathcal{PT} -symmetric Hamiltonian has real eigenvalues but this conclusion is false for $\epsilon < 0$, because some of the eigenvalues have disappeared as they are complex. On the other hand, for the restricted region $\epsilon \geq 0$, this conclusion is correct and all of the eigenvalues are indeed real. The concept of broken and unbroken \mathcal{PT} -symmetries is also expressed as: if every eigenfunction of a \mathcal{PT} -symmetric Hamiltonian is also an eigenfunction of the \mathcal{PT} operator, then \mathcal{PT} -symmetry of the Hamiltonian is unbroken. Conversely, if some of the eigenfunctions of a \mathcal{PT} -symmetric Hamiltonian are not simultaneously eigenfunctions of the \mathcal{PT} operator, then \mathcal{PT} -symmetry of the Hamiltonian is broken [4].

As far as the viability of the complex Hamiltonian is concerned, it has been investigated in the context of quantum mechanics and semiclassical field theories. However, there are some other situations in the literature, where complex Hamiltonian plays an important role. For example, the energy of a complex Toda lattice is found real [7,14], Hamiltonian with an imaginary external field is used to study the population growth in biology [14] and delocalization transition in type-II superconductors [5,6,18,19]. There are various ways of complexifying a given Hamiltonian [1,3], but here we adopt the scheme given by Xavier and de Aguir [20]. In this approach, the transformation of positions and momenta variables is defined as

$$x = x_1 + ip_2, \quad p = p_1 + ix_2, \quad (2)$$

where $x_1 \equiv \text{Re } x$, $p_2 \equiv \text{Im } x$, $p_1 \equiv \text{Re } p$, and $x_2 \equiv \text{Im } p$ and the variables (x_1, p_1, x_2, p_2) in the above transformation are regarded as some sort of coordinate–momentum interaction of a dynamical system [1,7]. In this complexifying scheme, the real phase-space plane (x, p) is now replaced by (x_1, p_1, x_2, p_2) with two additional degrees of

freedom, namely, p_2 and x_2 . Such transformations (2) were used in the study of non-linear evolution equations in the context of amplitude-modulated nonlinear Langmuir waves in plasma also [12]. There are two different approaches which deal with a non-Hermitian Hamiltonian. Firstly, the non-hermiticity arising in the approach of Bender *et al* is mainly due to complexity of the coupling parameters in the potential. Secondly, the non-hermiticity arising in the approach of Kaushal *et al* is due to complexity of both coupling parameters and underlying phase-space. So, the second approach is more generalized in nature and under certain limits (for real x and p) it reduces to the conventional \mathcal{PT} -symmetry. Using the second approach, Kaushal *et al* have made a number of attempts to solve the Schrödinger equation (SE) for low-order polynomial potentials [7, 10], but no effort has been made for higher-order polynomial potentials. With the same spirit and to expand the domain of applications, we obtain ground state as well as excited state solutions of the SE for a coupled complex even power deictic potential and its variant using the extended complex phase-space (ECPS) approach.

This paper is organized as follows. In §2, we are concerned with the mathematical formalism used for computing the eigenvalues and eigenfunctions of one-dimensional complex systems. The examples dealing with ground state and excited state solutions of the SE for a deictic potential and its variant are addressed in §3. Finally, concluding remarks are discussed in §4.

2. The formalism

For a given one-dimensional system $H(x, p)$, the SE (for $\hbar = m = 1$) is written as

$$\hat{H}(x, p)\psi(x) = E\psi(x), \quad (3)$$

where

$$\hat{H}(x, p) = -\frac{1}{2} \frac{d^2}{dx^2} + V(x). \quad (4)$$

From the transformation (1), one obtains

$$\frac{d}{dx} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial p_2} \right), \quad \frac{d}{dp} = \frac{1}{2} \left(\frac{\partial}{\partial p_1} - i \frac{\partial}{\partial x_2} \right). \quad (5)$$

The complex forms of $V(x)$, $\psi(x)$ and E are given by

$$V(x) = V_r(x_1, p_2) + i V_i(x_1, p_2), \quad (6a)$$

$$\psi(x) = \psi_r(x_1, p_2) + i \psi_i(x_1, p_2), \quad (6b)$$

$$E = E_r + i E_i, \quad (6c)$$

where subscripts r and i denote the real and imaginary parts of the corresponding quantities and other subscripts to these quantities separated by comma denote the partial

derivatives of the quantity concerned. On substituting eqs (4)–(6c) in eq. (3) and then separating the real and imaginary parts of the final expression, one gets

$$-\frac{1}{8}(\psi_{r,x_1x_1} - \psi_{r,p_2p_2} + 2\psi_{i,x_1p_2}) + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i, \quad (7a)$$

$$-\frac{1}{8}(\psi_{i,x_1x_1} - \psi_{i,p_2p_2} - 2\psi_{r,x_1p_2}) + V_r\psi_i + V_i\psi_r = E_r\psi_i + E_i\psi_r. \quad (7b)$$

Under Cauchy–Riemann conditions, the analyticity property of the wavefunction $\psi(x)$ implies

$$\psi_{r,x_1} = \psi_{i,p_2}, \quad \psi_{r,p_2} = -\psi_{i,x_1}. \quad (8)$$

After imposing the analyticity condition (8) on eqs (7a) and (7b), we have

$$-\frac{1}{2}\psi_{r,x_1x_1} + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i, \quad (9a)$$

$$-\frac{1}{2}\psi_{i,x_1x_1} + V_r\psi_i + V_i\psi_r = E_r\psi_i + E_i\psi_r. \quad (9b)$$

The ansatz for the wavefunction $\psi(x)$ is chosen as [19]

$$\psi(x) = \phi(x) \exp[g(x)], \quad (10)$$

where $\phi(x)$ and $g(x)$ are complex functions and can be expressed as

$$\phi(x_1, p_2) = \phi_r(x_1, p_2) + i\phi_i(x_1, p_2), \quad (11a)$$

$$g(x_1, p_2) = g_r(x_1, p_2) + ig_i(x_1, p_2). \quad (11b)$$

On employing eqs (6b), (11a) and (11b) in eq. (10), the real and imaginary parts of the wavefunction becomes

$$\psi_r(x_1, p_2) = e^{g_r}(\phi_r \cos g_i - \phi_i \sin g_i), \quad (12a)$$

$$\psi_i(x_1, p_2) = e^{g_r}(\phi_i \cos g_i + \phi_r \sin g_i). \quad (12b)$$

Again, Cauchy–Riemann conditions for the ansatz functions g_r and g_i become

$$g_{r,x_1} = g_{i,p_2}; \quad g_{r,p_2} = -g_{i,x_1}. \quad (13)$$

By employing eqs (12a) and (12b) in eqs (9a) and (9b), we have

$$g_{r,x_1x_1} + (g_{r,x_1})^2 - (g_{i,x_1})^2 + \frac{1}{(\phi_r^2 + \phi_i^2)}[\phi_r(\phi_{r,x_1x_1} + 2\phi_{r,x_1}g_{r,x_1} - 2\phi_{i,x_1}g_{i,x_1}) + \phi_i(\phi_{i,x_1x_1} + 2\phi_{r,x_1}g_{i,x_1} + 2\phi_{i,x_1}g_{r,x_1})] + 2(E_r - V_r) = 0, \quad (14a)$$

$$g_{i,x_1x_1} + 2g_{r,x_1}g_{i,x_1} + \frac{1}{(\phi_r^2 + \phi_i^2)}[\phi_r(\phi_{i,x_1x_1} + 2\phi_{r,x_1}g_{i,x_1} + 2\phi_{i,x_1}g_{r,x_1}) + \phi_i(-\phi_{r,x_1x_1} + 2\phi_{i,x_1}g_{i,x_1} - 2\phi_{r,x_1}g_{r,x_1})] + 2(E_i - V_i) = 0. \quad (14b)$$

For the ground state solutions, $\phi(x)$ is chosen as constant, i.e., $\phi(x) = 1$, then eqs (14a) and (14b) reduce to [5, 18]

$$g_{r,x_1x_1} + (g_{r,x_1})^2 - (g_{i,x_1})^2 + 2(E_r - V_r) = 0, \quad (15a)$$

$$g_{i,x_1x_1} + 2g_{r,x_1}g_{i,x_1} + 2(E_i - V_i) = 0. \quad (15b)$$

Thus, for a given ansatz, one can proceed with eqs (15a) and (15b) for the ground state solutions and the full forms of eqs (14a) and (14b) need to be rationalized for the excited state solutions.

3. Examples

In this section, we find ground state as well as excited state solutions of the SE for a complex deictic potential and its variant using the mathematical formalism as described in §2.

3.1 Generalized deictic potential

Consider a generalized even power deictic potential of the form

$$V(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10}, \quad (16)$$

where the coupling parameters a_{2i} , $0 \leq i \leq 5$ are complex constants.

Under the transformation (2), the real and imaginary parts of the potential (16) are written as

$$\begin{aligned} V_r = & a_{0r} + a_{2r}(x_1^2 - p_2^2) - 2a_{2i}x_1p_2 \\ & + a_{4r}(x_1^4 + p_2^4 - 6x_1^2p_2^2) + 4a_{4i}(x_1p_2^3 - x_1^3p_2) \\ & + a_{6r}(x_1^6 - p_2^6 + 15x_1^2p_2^4 - 15x_1^4p_2^2) \\ & + a_{6i}(-6x_1p_2^5 - 6x_1^5p_2 + 20x_1^3p_2^3) \\ & + a_{8r}(x_1^8 - p_2^8 - 28x_1^2p_2^6 - 28x_1^6p_2^2) \\ & + a_{8i}(8x_1p_2^7 - 8x_1^7p_2 + 56x_1^5p_2^3 - 56x_1^3p_2^5) \\ & + a_{10r}(x_1^{10} - p_2^{10} + 45x_1^2p_2^8 - 45x_1^8p_2^2 \\ & + 210x_1^6p_2^4 - 210x_1^4p_2^6) \\ & + a_{10i}(-10x_1p_2^9 - 10x_1^9p_2 + 120x_1^3p_2^7 + 120x_1^7p_2^3 - 252x_1^5p_2^5), \quad (17a) \end{aligned}$$

$$\begin{aligned} V_i = & a_{0i} + a_{2i}(x_1^2 - p_2^2) + 2a_{2r}x_1p_2 \\ & + a_{4i}(x_1^4 + p_2^4 - 6x_1^2p_2^2) + 4a_{4r}(x_1p_2^3 - x_1^3p_2) \\ & + a_{6i}(x_1^6 - p_2^6 + 15x_1^2p_2^4 - 15x_1^4p_2^2) \\ & - a_{6r}(-6x_1p_2^5 - 6x_1^5p_2 + 20x_1^3p_2^3) \\ & + a_{8i}(x_1^8 - p_2^8 - 28x_1^2p_2^6 - 28x_1^6p_2^2) \\ & - a_{8r}(8x_1p_2^7 - 8x_1^7p_2 + 56x_1^5p_2^3 - 56x_1^3p_2^5) \\ & + a_{10i}(x_1^{10} - p_2^{10} + 45x_1^2p_2^8 - 45x_1^8p_2^2 + 210x_1^6p_2^4 - 210x_1^4p_2^6) \\ & - a_{10r}(-10x_1p_2^9 - 10x_1^9p_2 + 120x_1^3p_2^7 + 120x_1^7p_2^3 - 252x_1^5p_2^5), \quad (17b) \end{aligned}$$

The functional forms of $g_r(x_1, p_2)$ and $g_i(x_1, p_2)$ for the potential (16), confirming the condition (13) are given by

$$g_r = \frac{1}{2}\alpha_2(x_1^2 - p_2^2) + \beta_2 x_1 p_2 + \frac{1}{4}\alpha_4(x_1^4 + p_2^4 - 6x_1^2 p_2^2) - \beta_4(x_1 p_2^3 - x_1^3 p_2) + \frac{1}{6}\alpha_6(x_1^6 - p_2^6 - 15x_1^4 p_2^2 - 15x_1^2 p_2^4) - \frac{1}{6}\beta_6(-6x_1^5 p_2 - 6x_1 p_2^5 - p_2^5 + 20x_1^3 p_2^3), \quad (18a)$$

$$g_i = \alpha_2 x_1 p_2 - \frac{1}{2}\beta_2(x_1^2 - p_2^2) + \alpha_4(x_1 p_2^3 - x_1^3 p_2) - \frac{1}{4}\beta_4(x_1^4 + p_2^4 - 6x_1^2 p_2^2) + \frac{1}{6}\alpha_6(-6x_1^5 p_2 - 6x_1 p_2^5 - p_2^5 + 20x_1^3 p_2^3) + \frac{1}{6}\beta_6(x_1^6 - p_2^6 - 15x_1^4 p_2^2 - 15x_1^2 p_2^4), \quad (18b)$$

where α_i and β_i are real. Now, we find the ground state as well as excited state solutions of the SE for the potential (16) as follows:

Ground state solutions: For obtaining ground state solutions, substitute the above functional forms of $g_r(x_1, p_2)$ and $g_i(x_1, p_2)$ in eqs (15a) and (15b), then rationalization of the resultant expressions yield the following set of 12 non-repeating equations:

$$E_r = a_{0r} - \frac{1}{2}\alpha_2, \quad (19a)$$

$$E_i = a_{0i} + \frac{1}{2}\beta_2, \quad (19b)$$

$$\alpha_2^2 - \beta_2^2 + 3\alpha_4 = 2a_{2r}, \quad (19c)$$

$$2\alpha_2\beta_2 + 3\beta_4 = -2a_{2i}, \quad (19d)$$

$$2\alpha_2\alpha_4 - 2\beta_2\beta_4 + 5\alpha_6 = 2a_{4r}, \quad (19e)$$

$$-2\alpha_4\beta_2 - 2\alpha_2\beta_4 + 5\beta_6 = 2a_{4i}, \quad (19f)$$

$$\alpha_4^2 - \beta_4^2 + 2\alpha_2\alpha_6 + 2\beta_2\beta_6 = 2a_{6r}, \quad (19g)$$

$$\alpha_4\beta_4 + \beta_6\alpha_6 - \alpha_2\beta_6 + \alpha_6\beta_2 = -a_{6i}, \quad (19h)$$

$$\alpha_4\alpha_6 + \beta_4\beta_6 = a_{8r}, \quad (19i)$$

$$\beta_4\alpha_6 - \alpha_4\beta_6 = -a_{8i}, \quad (19j)$$

$$\alpha_6^2 - \beta_6^2 = 2a_{10r}, \quad (19k)$$

$$\alpha_6\beta_6 = a_{10i}. \quad (19l)$$

Here, eqs (19c), (19d), (19g) and (19h) provide constraining relations among the potential parameters, whereas pair of eqs [(19k), (19l)], [(19i), (19j)] and [(19e), (19f)] can be immediately solved for the six arbitrary constants arising in the ansatz (18a) and (18b). The results obtained are

$$\alpha_6 = -a_{10_+}, \quad \beta_6 = a_{10_-}, \quad (20a)$$

$$\alpha_4 = -\frac{1}{2|a_{10}|} [a_{8r}a_{10_+} + a_{8i}a_{10_-}], \quad \beta_4 = \frac{1}{2|a_{10}|} [a_{8r}a_{10_-} - a_{8i}a_{10_+}], \quad (20b)$$

$$\alpha_2 = \frac{1}{|a_8|^2} [(a_{8r}a_{4r} + a_{8i}a_{4i})a_{10_+} + (a_{8i}a_{4r} - a_{8r}a_{4i})a_{10_-} - 5(a_{8r}a_{10r} + a_{8i}a_{10i})], \quad (20c)$$

$$\beta_2 = \frac{1}{|a_8|^2} [-(a_{8r}a_{4i} + a_{8i}a_{4r})a_{10_+} + (a_{8r}a_{4r} - a_{8i}a_{4i})a_{10_-} + 5(a_{8r}a_{10i} - a_{8i}a_{10r})]. \quad (20d)$$

Employing the results of eqs (20c) and (20d) in eqs (19a) and (19b), the real and imaginary parts of the energy eigenvalues become

$$E_r^{(0)} = a_{0r} + \frac{1}{2|a_8|^2} [(a_{8r}a_{4r} + a_{8i}a_{4i})a_{10_+} + (a_{8i}a_{4r} - a_{8r}a_{4i})a_{10_-} - 5(a_{8r}a_{10r} + a_{8i}a_{10i})], \quad (21a)$$

$$E_i^{(0)} = a_{0i} + \frac{1}{2|a_8|^2} [-(a_{8r}a_{4i} + a_{8i}a_{4r})a_{10_+} + (a_{8r}a_{4r} - a_{8i}a_{4i})a_{10_-} + 5(a_{8r}a_{10i} - a_{8i}a_{10r})]. \quad (21b)$$

After using the ansatz parameters (20a)–(20d) in eqs (18a)–(18b), eq. (10) yields

$$\psi^{(0)}(x) = N \exp \left[\frac{1}{2|a_8|^2} \{ (a_{8r}a_4 + ia_{8i}a_4^*)a_{10_+} + (a_{8i}a_4 - ia_{8r}a_4^*)a_{10_-} - 5a_8^*a_{10r} \} x^2 + \frac{1}{8|a_{10}|} (a_{10_+} - ia_{10_-})a_8x^4 + \frac{1}{6} (a_{10_+} - ia_{10_-})x^6 \right], \quad (22)$$

where $a_4^* = a_{4r} - ia_{4i}$ and $a_8^* = a_{8r} - ia_{8i}$.

Special case: Under the \mathcal{PT} -symmetric condition (1), the potential (16) reduces to

$$V(x) = a_{0r} + a_{2r}x^2 + a_{4r}x^4 + a_{6r}x^6 + a_{8r}x^8 + a_{10r}x^{10}. \quad (23)$$

Also, under transformation (1), the real and imaginary parts of the potential (23) are written as

$$\begin{aligned}
 V_r = & a_{0r} + a_{2r} (x_1^2 - p_2^2) + a_{4r} (x_1^4 + p_2^4 - 6x_1^2 p_2^2) \\
 & + a_{6r} (x_1^6 - p_2^6 + 15x_1^2 p_2^4 - 15x_1^4 p_2^2) \\
 & + a_{8r} (x_1^8 - p_2^8 - 28x_1^2 p_2^6 - 28x_1^6 p_2^2) \\
 & + a_{10r} (x_1^{10} - p_2^{10} + 45x_1^2 p_2^8 - 45x_1^8 p_2^2 \\
 & + 210x_1^6 p_2^4 - 210x_1^4 p_2^6), \quad (24a)
 \end{aligned}$$

$$\begin{aligned}
 V_i = & 2a_{2r} x_1 p_2 + 4a_{4r} (x_1 p_2^3 - x_1^3 p_2) \\
 & - a_{6r} (-6x_1 p_2^5 - 6x_1^5 p_2 + 20x_1^3 p_2^3) \\
 & - a_{8r} (8x_1 p_2^7 - 8x_1^7 p_2 + 56x_1^5 p_2^3 - 56x_1^3 p_2^5) \\
 & - a_{10r} (-10x_1 p_2^9 - 10x_1^9 p_2 + 120x_1^3 p_2^7 + 120x_1^7 p_2^3 - 252x_1^5 p_2^5), \quad (24b)
 \end{aligned}$$

Now, the functional forms of g_r and g_i in consonance with condition (13) are written as

$$\begin{aligned}
 g_r = & \frac{1}{2}\alpha_2 (x_1^2 - p_2^2) + \frac{1}{4}\alpha_4 (x_1^4 + p_2^4 - 6x_1^2 p_2^2) \\
 & + \frac{1}{6}\alpha_6 (x_1^6 - p_2^6 - 15x_1^2 p_2^4 - 15x_1^4 p_2^2), \quad (25a)
 \end{aligned}$$

$$\begin{aligned}
 g_i = & \alpha_2 x_1 p_2 + \alpha_4 (x_1 p_2^3 - x_1^3 p_2) \\
 & + \frac{1}{6}\alpha_6 (-6x_1^5 p_2 - 6x_1 p_2^5 - p_2^5 + 20x_1^3 p_2^3), \quad (25b)
 \end{aligned}$$

As before, after substituting eqs (25a) and (25b) in eqs (15a) and (15b), the rationalization of the resultant expressions yields

$$E_r = a_{0r} - \frac{1}{2}\alpha_2, \quad (26a)$$

$$E_i = 0, \quad (26b)$$

$$\alpha_2^2 + 3\alpha_4 = 2a_{2r}, \quad (26c)$$

$$2\alpha_2\alpha_4 + 5\alpha_6 = 2a_{4r}, \quad (26d)$$

$$\alpha_4^2 + 2\alpha_2\alpha_6 = 2a_{6r}, \quad (26e)$$

$$\alpha_4\alpha_6 = a_{8r}, \quad (26f)$$

$$\alpha_6^2 = 2a_{10r}. \quad (26g)$$

After solving eqs (26c)–(26g), ansatz parameters are written as

$$\alpha_6 = -\sqrt{2a_{10r}}, \quad \alpha_4 = -\frac{a_{8r}}{4\sqrt{2a_{10r}}}, \quad (27a)$$

$$\alpha_2 = \frac{a_{4r}\sqrt{2a_{10r}} - 5a_{10r}}{2a_{8r}}. \quad (27b)$$

On using the ansatz parameters in eqs (26a), (26b) and (10), the eigenspectra become

$$E_{r(\mathcal{PT})}^{(0)} = a_{0r} - \frac{a_{4r}\sqrt{2a_{10r}} - 5a_{10r}}{2a_{8r}}, \quad E_{i(\mathcal{PT})}^{(0)} = 0, \quad (28a)$$

$$\psi^{(0)}(x)_{(\mathcal{PT})} = \exp\left[\frac{a_{4r}\sqrt{2a_{10r}} - 5a_{10r}}{2a_{8r}} x^2 - \frac{a_{8r}}{4\sqrt{2a_{10r}}} x^4 + \sqrt{2a_{10r}} x^6\right]. \quad (28b)$$

From eq. (28a), we see that imaginary part of the energy eigenvalue vanishes, thus a non-Hermitian Hamiltonian possesses real eigenvalue if it is \mathcal{PT} -symmetric.

Excited state solutions: For computing the first excited state solutions of the potential (16), the functional form of $\phi(x)$ is taken as

$$\phi(x) = \alpha x + \eta. \quad (29)$$

Under transformation (2), eq. (29) yields

$$\phi_r(x_1, p_2) = \alpha x_1 + \eta, \quad \phi_i(x_1, p_2) = \alpha p_2, \quad (30)$$

where α and η are considered as real constants. Then inserting eqs (18a), (18b) and (30) in eqs (14a) and (14b), the rationalization of the resultant expressions yields the following non-repeating equations in addition to eqs (19g)–(19l):

$$E_r^{(1)} = a_{0r} - \frac{3}{2}\alpha_2, \quad (31a)$$

$$E_i^{(1)} = a_{0i} + \frac{3}{2}\beta_2, \quad (31b)$$

$$5\alpha_4 + \alpha_2^2 - \beta_2^2 = 2a_{2r}, \quad (31c)$$

$$5\beta_4 + 2\alpha_2\beta_2 = -2a_{2i}, \quad (31d)$$

$$2\alpha_2\alpha_4 - 2\beta_2\beta_4 + 7\alpha_6 = 2a_{4r}, \quad (31e)$$

$$2\beta_2\alpha_4 + 2\alpha_2\beta_4 + 7\beta_6 = -2a_{4i}. \quad (31f)$$

Here, the ansatz parameters α_4 , β_4 , α_6 and β_6 are the same as obtained in ground state solutions. However, α_2 and β_2 are obtained from eqs (31e)–(31f) as

$$\alpha_2 = \frac{1}{|a_8|^2} [(a_{8r}a_{4r} + a_{8i}a_{4i})a_{10+} + (a_{8i}a_{4r} - a_{8r}a_{4i})a_{10-} - 7(a_{8r}a_{10r} + a_{8i}a_{10i})], \quad (32a)$$

$$\beta_2 = \frac{1}{|a_8|^2} [-(a_{8r}a_{4i} + a_{8i}a_{4r})a_{10+} + (a_{8r}a_{4r} - a_{8i}a_{4i})a_{10-} + 7(a_{8r}a_{10i} - a_{8i}a_{10r})]. \quad (32b)$$

Substituting the values of α_2 and β_2 in eqs (31a) and (31b), the real and imaginary parts of the eigenvalue become

$$E_r^{(1)} = a_{0r} - \frac{3}{2|a_8|^2} [(a_{8r}a_{4r} + a_{8i}a_{4i})a_{10+} + (a_{8i}a_{4r} - a_{8r}a_{4i})a_{10-} - 5(a_{8r}a_{10r} + a_{8i}a_{10i})], \quad (33a)$$

$$E_i^{(1)} = a_{0i} + \frac{3}{2|a_8|^2} [-(a_{8r}a_{4i} + a_{8i}a_{4r})a_{10+} + (a_{8r}a_{4r} - a_{8i}a_{4i})a_{10-} + 5(a_{8r}a_{10i} - a_{8i}a_{10r})], \quad (33b)$$

and eigenfunction for the same is obtained as

$$\psi^{(1)}(x) = N \exp \left[\frac{1}{2|a_8|^2} \{ (a_{8r}a_4 + ia_{8i}a_4^*)a_{10+} + (a_{8i}a_4 - ia_{8r}a_4^*)a_{10-} - 7a_{8i}^*a_{10r} \} x^2 + \frac{1}{8|a_{10}|} (a_{10+} - ia_{10-})a_8x^4 + \frac{1}{6} (a_{10+} - ia_{10-})x^6 \right]. \quad (34)$$

Special case: Under the \mathcal{PT} -symmetric condition (1), the first excited state eigenspectra for potential (16) turn out to be

$$E_{r(\mathcal{PT})}^{(1)} = a_{0r} - \frac{3a_{4r}\sqrt{2a_{10r}} - 7a_{10r}}{2a_{8r}}, \quad E_{i(\mathcal{PT})}^{(1)} = 0, \quad (35)$$

$$\psi^{(1)}(x)_{(\mathcal{PT})} = N \exp \left[\frac{a_{4r}\sqrt{2a_{10r}} - 7a_{10r}}{2a_{8r}} x^2 + \frac{a_{8r}}{4\sqrt{2a_{10r}}} x^4 + \sqrt{2a_{10r}} x^6 \right]. \quad (36)$$

3.2 Variant of deictic potential

Here, we consider a deictic potential with inverse harmonic term as [5]

$$V(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10} + \frac{A}{x^2}, \quad (37)$$

where parameters a_i and A are complex constants.

Again, under the transformation (2), the real and imaginary parts of the potential (37) becomes

$$V_r = V_{r1} + A_r \frac{(x_1^2 - p_2^2)}{(x_1^2 + p_2^2)^2} + A_i \frac{(2x_1 p_2)}{(x_1^2 + p_2^2)^2}, \quad (38a)$$

$$V_i = V_{i1} + A_i \frac{(x_1^2 - p_2^2)}{(x_1^2 + p_2^2)^2} - A_r \frac{(2x_1 p_2)}{(x_1^2 + p_2^2)^2}, \quad (38b)$$

where V_{r1} and V_{i1} are the same as given by eqs (17a) and (17b). The functional forms of $g_r(x_1, p_2)$ and $g_i(x_1, p_2)$ complying with analyticity condition (13) turn out to be

$$g_r = g_{r1} + \gamma \tan^{-1}(x_1/p_2) - \frac{\delta}{2} \ln(x_1^2 + p_2^2), \quad (39a)$$

$$g_i = g_{i1} + \delta_1 \tan^{-1}(x_1/p_2) + \frac{\gamma}{2} \ln(x_1^2 + p_2^2), \quad (39b)$$

where g_{r1} and g_{i1} are the same as given by eqs (18a) and (18b). Now, the ground state and excited state solutions of the SE for the potential (37) are computed as follows:

Ground state solutions: In order to calculate the ground state solutions for the potential (37), substitute eqs (38a)–(39b) in eqs (15a) and (15b), then equating the coefficients of various terms to zero, one obtains the following set of non-repeating equations in addition to eqs (19g)–(19l) as before, namely,

$$E_r = a_{0r} - \frac{1}{2}\alpha_2 + \alpha_2\delta - \beta_2\gamma, \quad (40a)$$

$$E_i = a_{0i} + \frac{1}{2}\beta_2 - \alpha_2\gamma - \beta_2\delta, \quad (40b)$$

$$\alpha_2^2 - \beta_2^2 + 3\alpha_4 - 2\alpha_4\delta + 2\beta_4\gamma = 2a_{2r}, \quad (40c)$$

$$2\alpha_2\beta_2 + 3\beta_4 - 2\alpha_4\gamma - 2\beta_4\delta = -2a_{2i}, \quad (40d)$$

$$2\alpha_2\alpha_4 - 2\beta_2\beta_4 + 5\alpha_6 - 2\alpha_6\delta - 2\beta_6\gamma = 2a_{4r}, \quad (40e)$$

$$2\beta_2\alpha_4 + 2\alpha_2\beta_4 - 5\beta_6 - 2\alpha_6\gamma + 2\beta_6\delta = -2a_{4i}, \quad (40f)$$

$$\gamma + 2\delta\gamma = -A_i, \quad (40g)$$

$$\delta - \gamma^2 + \delta^2 = 2A_r. \quad (40h)$$

The value of ansatz parameters $\alpha_4, \beta_4, \alpha_6$ and β_6 are the same as obtained in §3.1. However, the rest of the parameters are obtained from eqs (40e)–(40h) as

$$\begin{aligned} \alpha_2 = & \frac{1}{|a_8|^2} [(a_{8r}a_{4r} + a_{8i}a_{4i})a_{10+} \\ & + (a_{8i}a_{4r} - a_{8r}a_{4i})a_{10-} - 5(a_{8r}a_{10r} + a_{8i}a_{10i}) \\ & + 4A_r\{a_{10r}(a_{8r} - a_{8i}) + a_{10i}(a_{8r} + a_{8i})\}], \end{aligned} \quad (41a)$$

$$\beta_2 = \frac{1}{|a_8|^2} [-(a_{8r}a_{4i} + a_{8i}a_{4r})a_{10+} + (a_{8r}a_{4r} - a_{8i}a_{4i})a_{10-} + 5(a_{8r}a_{10i} - a_{8i}a_{10r}) + 4A_r\{a_{10r}(a_{8r} + a_{8i}) - a_{10i}(a_{8r} - a_{8i})\}], \quad (41b)$$

$$\gamma = \delta = 2A_r. \quad (41c)$$

Finally, substituting the values of various ansatz parameters in eqs (40a) and (40b), the real and imaginary parts of energy eigenvalue become

$$E_r^{(0)} = a_{0r} - \frac{1}{2|a_8|^2} [(a_{8r}a_{4r} + a_{8i}a_{4i})a_{10+} + (a_{8i}a_{4r} - a_{8r}a_{4i})a_{10-} - 5(a_{8r}a_{10r} + a_{8i}a_{10i}) + 4A_r\{a_{10r}(a_{8r} - a_{8i}) + a_{10i}(a_{8r} + a_{8i})\}] + \frac{2A_r}{|a_8|^2} [(a_{4r} + a_{4i})\{(a_{8r} + a_{8i})a_{10+} - (a_{8r} - a_{8i})a_{10-}\} + 5\{-(a_{10r} + a_{10i})a_{8r} + (a_{10r} - a_{10i})a_{8i}\}] + 8A_r\{a_{10i}a_{8r} - a_{10r}a_{8i}\}, \quad (42)$$

$$E_i^{(0)} = a_{0i} + \frac{1}{2|a_8|^2} [-(a_{8r}a_{4i} + a_{8i}a_{4r})a_{10+} + (a_{8r}a_{4r} - a_{8i}a_{4i})a_{10-} + 5(a_{8r}a_{10i} - a_{8i}a_{10r}) + 4A_r\{a_{10r}(a_{8r} + a_{8i}) - a_{10i}(a_{8r} - a_{8i})\}] - \frac{2A_r}{|a_8|^2} [(a_{4r} - a_{4i})\{(a_{8r} - a_{8i})a_{10+} - (a_{8r} + a_{8i})a_{10-}\} + 5\{-(a_{10r} - a_{10i})a_{8r} - (a_{10r} + a_{10i})a_{8i}\}] + 8A_r\{a_{10r}a_{8r} + a_{10i}a_{8i}\} \quad (43)$$

and the corresponding eigenfunction turns out to be

$$\psi^{(0)}(x) = N(x_1^2 + p_2^2)^{(i-1)A_r} \times \exp \left[\frac{1}{2|a_8|^2} \{(a_{8r}a_4 + ia_{8i}a_4^*)a_{10+} + (a_{8i}a_4 - ia_{8r}a_4^*)a_{10-} - 5a_{8r}^*a_{10r} + 4A_r[a_{10r}(1-i)a_{10+}a_8^* + (1+i)a_{10-}a_8^*]\}x^2 + \frac{1}{8|a_{10}|}(a_{10+} - ia_{10-})a_8x^4 + \frac{1}{6}(a_{10+} - ia_{10-})x^6 + 2A_r(1+i)\tan^{-1}\frac{x_1}{p_2} \right]. \quad (44)$$

Under the \mathcal{PT} -symmetric condition (1), the eigenspectra becomes

$$E_{r(\mathcal{PT})}^{(0)} = a_{0r} + \frac{(-2 + \sqrt{1+8A_r})}{2} \left[\frac{a_{4r}\sqrt{2a_{10r}} - a_{10r}(6 - \sqrt{1+8A_r})}{2a_{8r}} \right], \quad (45a)$$

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$$E_{i(\mathcal{PT})}^{(0)} = 0, \quad (45b)$$

$$\psi^{(0)}(x)_{(\mathcal{PT})} = (x_1^2 + p_2^2)^{-\delta/2} \exp \left[\frac{a_{4r} \sqrt{2a_{10r}} - a_{10r} (6 - \sqrt{1 + 8A_r})}{a_{8r}} x^2 - \frac{a_{8r}}{4\sqrt{2a_{10r}}} x^4 + \sqrt{2a_{10r}} x^6 \right]. \quad (45c)$$

Excited state solutions: To compute the energy eigenvalue and associated eigenfunction for the first excited state of potential (37), imply eqs (38a)–(39b) and (30) in eqs (14a) and (14b), then rationalization of the resultant expressions yields the following equations in addition to eqs (19g)–(19l) as

$$E_r = a_{0r} - \frac{3}{2}\alpha_2 + \alpha_2\delta - \beta_2\gamma, \quad (46a)$$

$$E_i = a_{0i} + \frac{3}{2}\beta_2 - \alpha_2\gamma - \beta_2\delta, \quad (46b)$$

$$\alpha_2^2 - \beta_2^2 + 5\alpha_4 - 2\alpha_4\delta + 2\beta_4\gamma = 2a_{2r}, \quad (46c)$$

$$2\alpha_2\beta_2 + 5\beta_4 - 2\alpha_4\gamma - 2\beta_4\delta = -2a_{2i}, \quad (46d)$$

$$2\alpha_2\alpha_4 - 2\beta_2\beta_4 + 7\alpha_6 - 2\alpha_6\delta - 2\beta_6\gamma = 2a_{4r}, \quad (46e)$$

$$2\beta_2\alpha_4 + 2\alpha_2\beta_4 - 7\beta_6 - 2\alpha_6\gamma + 2\beta_6\delta = -2a_{4i}, \quad (46f)$$

$$\delta^2 - \delta - \gamma^2 = 2A_r \quad (46g)$$

$$\gamma - 2\delta\gamma = -A_i. \quad (46h)$$

The ansatz parameters, α_4 , β_4 , α_6 and β_6 are the same as obtained in the earlier section and others are obtained from eqs (46e)–(46h) as

$$\begin{aligned} \alpha_2 = \frac{1}{|a_8|^2} & [(a_{8r}a_{4r} + a_{8i}a_{4i})a_{10+} \\ & + (a_{8i}a_{4r} - a_{8r}a_{4i})a_{10-} - 7(a_{8r}a_{10r} + a_{8i}a_{10i}) \\ & + 4A_r \{a_{10r}(a_{8r} - a_{8i}) + a_{10i}(a_{8r} + a_{8i})\}], \end{aligned} \quad (47a)$$

$$\begin{aligned} \beta_2 = \frac{1}{|a_8|^2} & [-(a_{8r}a_{4i} + a_{8i}a_{4r})a_{10+} \\ & + (a_{8r}a_{4r} - a_{8i}a_{4i})a_{10-} + 7(a_{8r}a_{10i} - a_{8i}a_{10r}) \\ & + 4A_r \{a_{10r}(a_{8r} + a_{8i}) - a_{10i}(a_{8r} - a_{8i})\}], \end{aligned} \quad (47b)$$

$$\gamma = \delta = 2A_r. \quad (47c)$$

Finally, substituting the values of various ansatz parameters in eqs (46a) and (46b), the real and imaginary parts of energy eigenvalue are written as

$$\begin{aligned}
 E_r^{(1)} = & a_{0r} - \frac{1}{2|a_8|^2} [(a_{8r}a_{4r} + a_{8i}a_{4i})a_{10_+} \\
 & + (a_{8i}a_{4r} - a_{8r}a_{4i})a_{10_-} - 7(a_{8r}a_{10r} + a_{8i}a_{10i}) \\
 & + 4A_r \{a_{10r}(a_{8r} - a_{8i}) + a_{10i}(a_{8r} + a_{8i})\}] \\
 & + \frac{2A_r}{|a_8|^2} [(a_{4r} + a_{4i})\{(a_{8r} + a_{8i})a_{10_+} \\
 & - (a_{8r} - a_{8i})a_{10_-}\} + 7\{-(a_{10r} + a_{10i})a_{8r} \\
 & + (a_{10r} - a_{10i})a_{8i}\} + 8A_r \{a_{10i}a_{8r} - a_{10r}a_{8i}\}], \tag{48a}
 \end{aligned}$$

$$\begin{aligned}
 E_i^{(1)} = & a_{0i} + \frac{1}{2|a_8|^2} [-(a_{8r}a_{4i} + a_{8i}a_{4r})a_{10_+} \\
 & + (a_{8r}a_{4r} - a_{8i}a_{4i})a_{10_-} + 7(a_{8r}a_{10i} - a_{8i}a_{10r}) \\
 & + 4A_r \{a_{10r}(a_{8r} + a_{8i}) - a_{10i}(a_{8r} - a_{8i})\}] \\
 & - \frac{2A_r}{|a_8|^2} [(a_{4r} - a_{4i})\{(a_{8r} - a_{8i})a_{10_+} \\
 & - (a_{8r} + a_{8i})a_{10_-}\} + 7\{-(a_{10r} - a_{10i})a_{8r} \\
 & - (a_{10r} + a_{10i})a_{8i}\} + 8A_r \{a_{10r}a_{8r} + a_{10i}a_{8i}\}] \tag{48b}
 \end{aligned}$$

and the eigenfunction is given by

$$\begin{aligned}
 \psi^{(1)}(x) = & N (x_1^2 + p_2^2)^{(1-i)A_r} x_1 \exp \left[\frac{1}{2|a_8|^2} \{ (a_{8r}a_4 + ia_{8i}a_4^*) a_{10_+} \right. \\
 & + (a_{8i}a_4 - ia_{8r}a_4^*) a_{10_-} - 7a_{8r}^* a_{10r} \\
 & \left. - 4A_r \{ a_{10r}(1-i)a_{10_+}a_8^* + (1+i)a_{10_-}a_8^* \} \right] x^2 + \frac{1}{8|a_{10}|} (a_{10_+} \\
 & - ia_{10_-}) a_8 x^4 + \frac{1}{6} (a_{10_+} - ia_{10_-}) x^6 - 2A_r (1+i) \tan^{-1} \frac{x_1}{p_2} \Big]. \tag{49}
 \end{aligned}$$

Under condition (1), the \mathcal{PT} -symmetric solutions of the potential (37) reduce to

$$E_{r(\mathcal{PT})}^{(1)} = a_{0r} + \frac{(-2 + \sqrt{1 + 8A_r})}{2} \left[\frac{a_{4r} \sqrt{2a_{10r}} - a_{10r} (6 - \sqrt{1 + 8A_r})}{2a_{8r}} \right], \tag{50}$$

$$E_{i(\mathcal{PT})}^{(1)} = 0, \tag{51}$$

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$$\psi^{(1)}(x)_{(\mathcal{PT})} = N(x_1^2 + p_2^2)^{-\delta/2} \exp \left[\frac{a_{4r} \sqrt{2a_{10r}} - a_{10r} (6 - \sqrt{1 + 8A_r})}{a_{8r}} x^2 - \frac{a_{8r}}{4\sqrt{2a_{10r}}} x^4 + \sqrt{2a_{10r}} x^6 \right]. \quad (52)$$

4. Conclusions

In this paper, we have described a general procedure to calculate the ground state, as well as excited state solutions of the SE for an even power deictic potential and its variant in one dimension. For this purpose the ECPS approach is utilized [1,18] and besides the complexity of the phase-space produced by (2), the complexity of the potential parameters is also taken into account [5,12]. The eigenspectra are real in the respective domain and imaginary part can be made to zero by imposing some restrictions on the coupling parameters in the potential. Thus, the present method suggests another degree of freedom to obtain real eigenspectra for a non-Hermitian Hamiltonian [3,4] and restrictions on the potential parameters are in addition to the constraining relations that give rise to exact solutions of SE. It is also emphasized that solutions of the SE are obtained in the presence of some constraining relations among the potential parameters, which give rise to the bound states of a system. It is found that the imaginary part of the energy eigenvalues exists for complex coupling parameters and it vanishes for real coupling parameters. However, for complex coupling parameters of a non-Hermitian Hamiltonian, the imaginary part of the energy eigenvalue can be made to zero if the Hamiltonian is \mathcal{PT} -symmetric. Thus, a non-Hermitian Hamiltonian possesses real eigenvalues if it is \mathcal{PT} -symmetric. For the first excited state solutions, it is found that the real and imaginary parts of the eigenvalues follow just the opposite ordering for discrete energy levels by retaining the conventional ordering for the magnitude of the eigenvalues.

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