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New localized excitations and cross-like fractal structures to the (2+1)-dimensional Broer–Kaup system

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Abstract. A broad general variable separation solution with two arbitrary lower-dimensional functions of the (2+1)-dimensional Broer–Kaup (BK) equations was derived by means of a projective equation method and a variable separation hypothesis. Based on the derived variable separation excitation, some new special types of localized solutions such as oscillating solitons, instanton-like and cross-like fractal structures are revealed by selecting appropriate functions of the general variable separation solution.

Keywords. Broer–Kaup equations; improved mapping method; variable separation approach; fractal structures.

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1. Introduction

Searching for explicit solutions of nonlinear evolution equations by using different methods is useful and meaningful in physical science and nonlinear science. Many powerful methods have been presented, such as inverse scattering transform [1], Hirota's bilinear form method [2], two-soliton method [3], homoclinic test technique [4–6], Bäcklund transformation method [7], three-wave type of ansätz approach [8,9], projective equation method [10], multilinear variable separation method [11] and so on.

In line with the development of symbolic computation, much work has been focussed on the various extensions and application of the known algebraic methods to construct solutions of nonlinear evolution equations [12–14].

In this paper, we shall apply a projective equation method [15] with a variable separation hypothesis to look for new families of variable separation solutions to the (2+1)-dimensional Broer–Kaup (BK) equations [16]:

$$u_{yt} + 2v_{xx} + 2(uu_x)_y - u_{xxy} = 0, \tag{1}$$

$$v_t + 2(uv)_x + v_{xx} = 0, \tag{2}$$

which were obtained from a Kadomtsev–Petviashvili equation by the symmetry constraints [17], as a concrete example to study possible oscillating soliton structures in higher-dimensional physical models. The (2+1)-dimensional BK equations have been extensively studied in several papers [18–22]. Abundant solutions, such as soliton-like solutions, triangular-like solutions, single and combined non-degenerate Jacobi elliptic wave function-like solutions, Weierstrass elliptic doubly periodic-like solutions and multisolitons have been revealed. Pang [23] gave some oscillating solitons, such as dromion, solitoff, ring, multilump and so on, by applying the variable separation approaches based on the extended homogeneous balance method.

In our work, we apply a projective equation method and a variable separation hypothesis to the BK equations (1)–(2) and obtain its exact excitations. We give some selected oscillating solitons, instanton-like and cross-like fractal structures by selecting appropriate functions in the general variable separation solution of the BK equations to demonstrate some interesting outcomes, most of which are new when compared with the solutions of the references.

2. Variable separation solution for (2+1)-dimensional BK

By letting $v = u_y$, eqs (1) and (2) can be converted into a new equation

$$u_{yt} + u_{xxy} + 2(uu_x)_y = 0. \tag{3}$$

Integrating eq. (3) with respect to y once, we have

$$u_t + u_{xx} + 2uu_x = C(x, t), \tag{4}$$

where $C(x, t)$ is an integral function about $x,\ t$.

Suppose that the solution of eq. (4) can be expressed as follows:

$$u = u_0(X) + \sum_{i=1}^{n} u_i(X) F(\phi(X)), \qquad u_n(X) \neq 0, \tag{5}$$

where $u_0(X),\ u_i(X)\ (i = 1, 2, \ldots, n)$, $\phi(X)$ are all functions of X to be determined, and $F(\phi(X))$ satisfies the Riccati equation

$$F'(\phi(X)) = \delta + F^2(\phi(X)), \tag{6}$$

with δ an arbitrary constant. Notice that, the Riccati equation (6) possesses the following solutions:

(i) When $\delta = 0$, $F(\phi(X)) = -1/\phi(X)$;
(ii) When $\delta < 0$, $F(\phi(X)) = -\sqrt{-\delta}\ \tanh(\sqrt{-\delta}\phi(X))$;
(iii) When $\delta > 0$, $F(\phi(X)) = \sqrt{\delta}\ \tan(\sqrt{\delta}\phi(X))$.

By balancing the linear term of the highest-order with the nonlinear term in eq. (4), we get $n = 1$. Then according to ansätz (5), the solution of eq. (4) reads as

$$u(x, y, t) = u_0(x, y, t) + u_1(x, y, t) F(\phi(x, y, t)), \tag{7}$$

where $u_0(x, y, t)$, $u_1(x, y, t)$ and $\phi(x, y, t)$ are arbitrary functions of (x, y, t) to be determined, and F satisfies eq. (6). Then, on substituting (7) with (6) into (4), equating all the coefficients of F to zero yields a set of partial differential equations for $u_0(x, y, t)$, $u_1(x, y, t)$, $\phi(x, y, t)$ and $C(x, t)$. Solving the set of differential equations simultaneously, we have

$$u_0 = -\frac{\phi_{xx} + \phi_t}{2\phi_x}, \quad u_1 = -\phi_x, \tag{8}$$

$$C(x,t) = \frac{(2\phi_{xx} + \phi_t)[2\phi_x(\phi_{xxx} + \phi_{xt}) - \phi_{xx}(\phi_{xx} + \phi_t)]}{2\phi_x^3} - \frac{\phi_{xxxx} + 2\phi_{xxt} + \phi_{tt}}{\phi_x} - 2\delta a_1^2 \phi_x \phi_{xx}. \tag{9}$$

Then, by selecting the variable separation ansätz

$$\phi(x, y, t) = a_0 + a_1 p(x, t) + a_2 q(y) + a_3 p(x, t) q(y), \tag{10}$$

and substituting it into (8) and (9), we obtain the following results.

Case 1.

$$a_0 = a_0, \quad a_1 = a_1, \quad a_2 = a_2, \quad a_3 = a_3, \quad \delta = 0,$$
$$u_0 = -\frac{p_{xx} + p_t}{2p_x}, \quad u_1 = -(a_1 + a_3 q) p_x,$$
$$C(x,t) = \frac{(2p_{xx} + p_t)[2p_x(p_{xxx} + p_{xt}) - p_{xx}(p_{xx} + p_t)]}{2p_x^3} - \frac{p_{xxxx} + 2p_{xxt} + p_{tt}}{p_x}.$$

Therefore, we obtain the following variable separation solution:

$$u_1(x, y, t) = -\frac{p_{xx} + p_t}{2p_x} + \frac{(a_1 + a_3 q) p_x}{a_0 + a_1 p(x,t) + a_2 q(y) + a_3 p(x,t) q(y)}, \tag{11}$$

$$v_1(x, y, t) = \frac{(a_3 a_0 - a_1 a_2) p_x q_y}{[a_0 + a_1 p(x,t) + a_2 q(y) + a_3 p(x,t) q(y)]^2}, \tag{12}$$

where a_i $(i = 0, 1, 2, 3)$ are arbitrary constants; $p(x, t)$ and $q(y)$ are arbitrary functions of $\{x, t\}$, $\{y\}$, respectively.

Case 2.

$$a_0 = a_0, \quad a_1 = a_1, \quad a_2 = a_2, \quad a_3 = 0, \quad \delta = \delta,$$
$$u_0 = -\frac{p_{xx} + p_t}{2p_x}, \quad u_1 = -a_1 p_x,$$
$$C(x,t) = \frac{(2p_{xx} + p_t)[2p_x(p_{xxx} + p_{xt}) - p_{xx}(p_{xx} + p_t)]}{2p_x^3} - \frac{p_{xxxx} + 2p_{xxt} + p_{tt}}{p_x} - 2\delta a_1^2 p_x p_{xx}.$$

Therefore, we obtain exact solutions for eqs (1)–(2) as follows:

(I) When $\delta < 0$, we can derive the following solitary wave solutions:

$$u_2(x, y, t) = -\frac{p_{xx} + p_t}{2p_x} + \sqrt{-\delta} a_1 p_x \tanh[\sqrt{-\delta}(a_0 + a_1 p(x,t) + a_2 q(y))], \quad (13)$$

$$v_2(x, y, t) = -\delta a_1 a_2 p_x q_y \operatorname{sech}^2[\sqrt{-\delta}((a_0 + a_1 p(x,t) + a_2 q(y))], \quad (14)$$

where $a_1,\ a_2$ are arbitrary constants and $p(x,t),\ q(y)$ are arbitrary functions.

(II) When $\delta > 0$, we can derive the following periodic wave solutions:

$$u_3(x, y, t) = -\frac{p_{xx} + p_t}{2p_x} - \sqrt{\delta} a_1 p_x \tan[\sqrt{\delta}(a_0 + a_1 p(x,t) + a_2 q(y))], \quad (15)$$

$$v_3(x, y, t) = -\delta a_1 a_2 p_x q_y \sec^2[\sqrt{\delta}((a_0 + a_1 p(x,t) + a_2 q(y))], \quad (16)$$

with $a_1,\ a_2$ arbitrary constants and $p(x,t),\ q(y)$ arbitrary functions.

3. New localized excitations

Owing to the arbitrary functions, $p(x,t)$ and $q(y)$, involved in the solutions (11)–(16), it is convenient to excite soliton structure. After some calculations, we construct a new class of structures, such as the instanton-like and the cross-like fractal solitons. Here, we take the solution (12) of the BK equations (1) and (2) as an example to study the soliton structure excitation.

Case I. When $p(x,t)$ and $q(y)$ possess the following forms

$$p(x,t) = \exp(-x^2 + t^2), \quad q(y) = \operatorname{sech}(y - 0.5);$$

$$p(x,t) = 1 + 0.5 \operatorname{sech}(0.5x + t - 4) + 1.5 \operatorname{sech}(0.45x + 2t),$$

$$q(y) = 1 + \operatorname{sech}(4 + y) + \operatorname{sech}(4 - y),$$

respectively, we can obtain the multidromion soliton structures of the (2+1)-dimensional BK equation. Figures 1a and 1b clearly indicate that the multidromion moves backwards and forwards over the same path in $x-y$ plane, oscillating in x-direction.

Case II. If $p(x,t)$ and $q(y)$ are taken as

$$p(x,t) = \exp[(x - 0.5t)\cos(x + t)], \quad q(y) = \exp(-y + 0.1),$$

respectively. Then, we obtain a dromion of periodic oscillation structure which is shown in figure 2a.

Case III. When $p(x,t)$ and $q(y)$ are choosen as

$$p(x,t) = \exp(x + t) + 0.5\exp(-x + t), \quad q(y) = \exp(y) + \exp(-y),$$

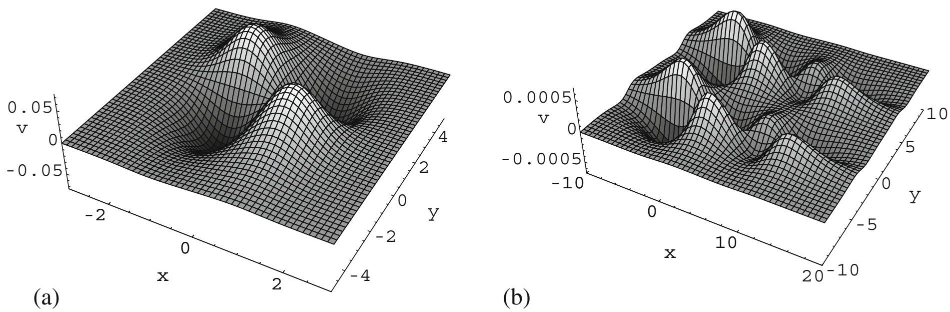

Figure 1. Multidromion structures of v_1. (**a**) $a_0 = 4,\ a_1 = 3,\ a_2 = 5,\ a_3 = 0.5,\ t = 0$ and (**b**) $a_0 = 1,\ a_1 = 3,\ a_2 = 2,\ a_3 = 4,\ t = 0$.

respectively, we can obtain solitoff solution. Figure 2b shows the structure of a 4-oscillating solitoff solution for the physical quantity v_1.

Case IV. When $p(x, t)$ and $q(y)$ possess the following forms:

$$p(x, t) = 1 + 0.5 \sec(x + t) + \operatorname{sech}(x - t), \qquad q(y) = 1 + \operatorname{sech}(2y),$$

respectively, we have breather solutions. Their plots are presented in figures 3a and 3b. Figure 3 shows periodic breather-type of two soliton structures with amplitudes oscillated as time changed.

Case V. If we select $p(x, t)$ and $q(y)$ which are exponentially decreasing with t and y, such as

$$p(x, t) = \exp(0.3x^3 + t^2 - 10), \qquad q(y) = \exp(-0.05y^4 - 10),$$

respectively, we obtain the instanton-like structures. In figure 4, as $|t|$ goes from 0 to 10, the amplitude decreases from $|1 \times 10^{-37}|$ to $|4 \times 10^{-56}|$.

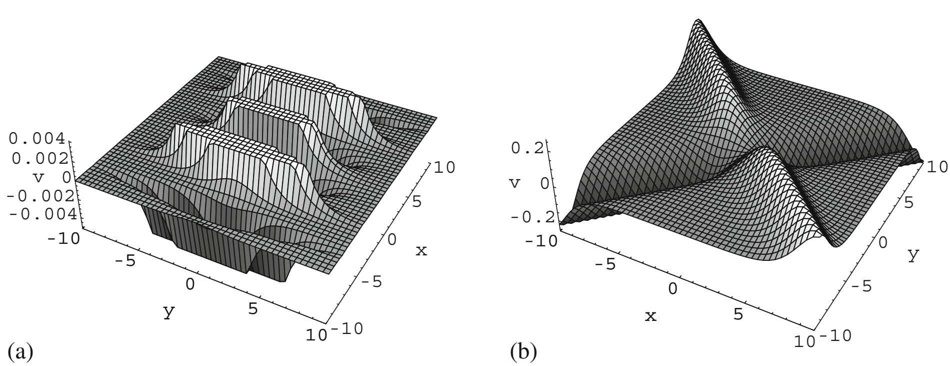

Figure 2. (**a**) Dromion of periodic oscillation structure ($a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 1,\ t = 0$) and (**b**) solitoff solution structure ($a_0 = 2,\ a_1 = 3,\ a_2 = 2,\ a_3 = 0;\ t = 0$).

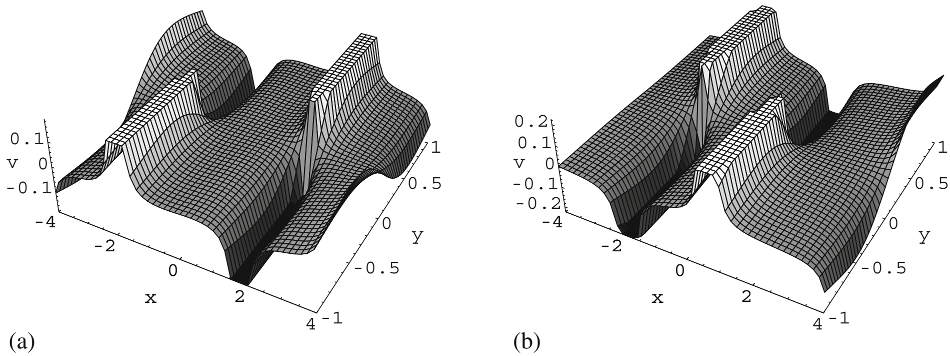

Figure 3. Periodic breather-type of two soliton structures of v_1: $a_0 = 1$, $a_1 = 3$, $a_2 = 2$, $a_3 = 1$. (**a**) $t = 0$ and (**b**) $t = 10$.

Case VI. When $p(x, t)$ and $q(y)$ possess the following forms:

$$p(x, t) = (2x + t) \ln(x + 2t)^2, \qquad q(y) = -2y \ln y^2,$$

respectively, we obtain the cross-like fractal structures. Figures 5a and 5b give the figures of the solution (12) with the settings below, but x, y in, respectively, $[-5\times 10^{-6}, 5\times 10^{-6}]$, $[-5 \times 10^{-12}, 5 \times 10^{-12}]$. The essential property of the fractal structures is the similarity of the figures in different axis scales. Figure 5 demonstrates that the cross-like fractal soliton holds its similarity in different ranges of x, y.

Case VII. If $p(x, t)$ and $q(y)$ are selected to be the solution of the following chemical dynamic chaotic system:

$$\begin{aligned} m(j) &= m(A_1 - k_1 - n - l) + k_2 n^2 + A_3, \\ n(j) &= n(m - k_2 n - A_5) + A_2, \\ l(j) &= l(A_4 - m - k_3 l) + A_3, \end{aligned}$$

where m, n and l are functions of j ($j = x$ or $j = x - ct$), we can also have the chaotic behaviours for the BK system.

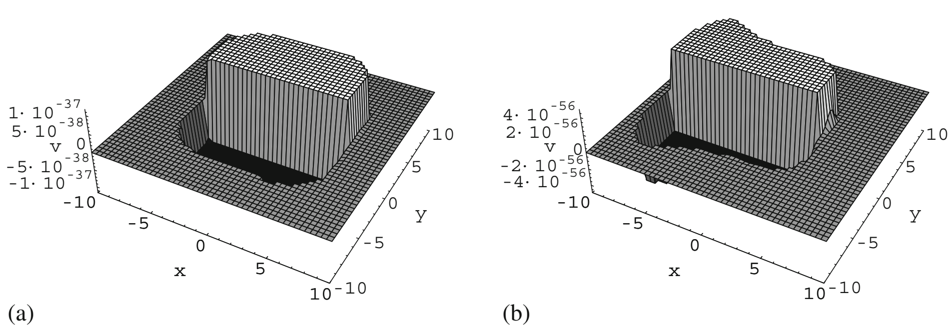

Figure 4. Instanton structures of v_1: $a_0 = 5$, $a_1 = 5$, $a_2 = 8$, $a_3 = 3$. (**a**) $t = 0$ and (**b**) $t = 10$.

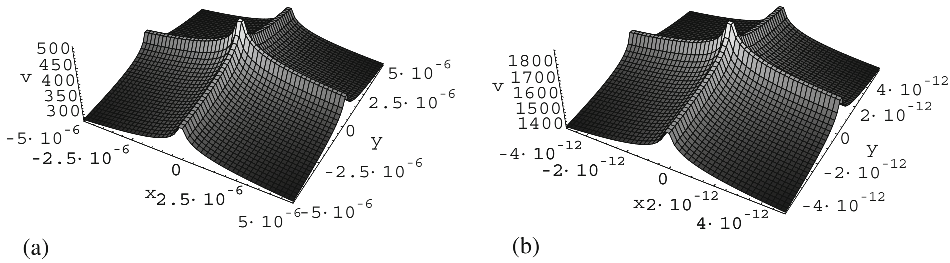

Figure 5. Cross-like fractal structures of v_1: $a_0 = 5,\ a_1 = 4,\ a_2 = 8,\ a_3 = 5,\ t = 0$. (**a**) $x, y \in [-5 \times 10^{-6}, 5 \times 10^{-6}]$ and (**b**) $x, y \in [-5 \times 10^{-12}, 5 \times 10^{-12}]$.

4. Conclusion

In this paper, we applied an improved mapping method and a variable separation hypothesis to the (2+1)-dimensional BK equations and obtained a general variable separation with two arbitrary functions. Based on the general variable separation solution, abundant novel localized excitations, such as oscillating soliton, instanton-like and cross-like fractal structures have been constructed. The arbitrary functions in the obtained solutions imply that these solutions have rich spatial structures which may be helpful in future studies of intricate nature. This method can also be extended to other higher-dimensional nonlinear equations.

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