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Compacton-like solutions for modified KdV and nonlinear Schrödinger equation with external sources

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Abstract. We present new types of compacton-like solutions for modified KdV and nonlinear Schrödinger equation with external sources, using a recently developed fractional transformation. In particular, we explicate these novel compactons for the trigonometric case, and compare their properties with those of the compactons and solitons in the case of modified KdV equation. Keeping in mind the significance of nonlinear Schrödinger equation with external source, for pulse propagation through asymmetric twin-core fibres, we hope that the newly found compacton may be launched in a long-haul telecommunication network utilizing asymmetric twin-core fibres.

Keywords. Compacton-like solutions; modified KdV equation; nonlinear Schrödinger equation; fractional transform.

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1. Introduction

Compactons are a new class of localized solutions for families of fully nonlinear, dispersive, partial differential equations. Unlike the solitons, which, although highly localized, still have infinite span, these solutions have compact support; they vanish identically outside a finite region. Hence, these solitary waves have been christened as compactons [1]. Remarkably, there is strong numerical evidence that, the collision of two compactons is elastic, a feature characterizing the solitons. These equations, arising in the context of pattern formation in nonlinear media, seem to have only a finite number of local conservation laws; yet, the behaviour of the solutions closely mimic those of the solitons of the integrable models.

The first two-parameter family of fully nonlinear, dispersive equations admitting compacton solutions, are of the form,

$$u_t + (u^m)_x + (u^n)_{3x} = 0, \quad m > 0, \quad 1 < n < 3, \tag{1}$$

with $u_t \equiv \partial u/\partial t$ and $u_x \equiv \partial u/\partial x$. These equations are formed in the process of understanding the role of nonlinear dispersion, in the formation of structures like liquid drops [2]. The compacton solution of $K(2, 2)$ reads as

$$u_{\rm c} = \frac{4\lambda}{3}\cos^2\left(\frac{x-\lambda t}{4}\right), \tag{2}$$

when $|x - \lambda t| \leq 2\pi$ and $u_{\rm c} = 0$, otherwise.

Unlike the solitons, the width here is independent of velocity; however, the amplitude depends on it. It has been shown that, $K(2, 2)$ admits only four local conservation laws. Some of the other representative compactons are

$$u_{\rm c} = [37.5\lambda - (x - \lambda t)^2]/30 \tag{3}$$

and

$$u_{\rm c} = \pm\sqrt{[3\lambda/2]}\cos((x - \lambda t)/3). \tag{4}$$

These are the solutions of $K(3, 2)$ and $K(3, 3)$ equations, respectively.

The $K(m, n)$ family cannot be derived from a first-order Lagrangian, except for $n = 1$ [3]. A generalized sequence of KdV-like equations, which could be given a Lagrangian formulation, have also been shown to admit compacton solutions. These equations

$$u_t + u_x u^{l-2} + \alpha[2u_{3x}u^p + 4pu^{p-1}u_x u_{2x} + p(p-1)u^{p-2}(u_x)^3] = 0, \tag{5}$$

have the same terms as in eq. (1); the relative weights of the terms are different. Further generalizations to one-parameter generalized KdV equation [4] and two-parameter odd order KdV equations [5] enlarged the class of evolution equations, which admitted solutions with compact support. These types of solutions have also appeared in the context of baby Skyrmions [6].

The stability of the compacton solutions was considered in [7]; it was shown by linear stability analysis as well as by Lyapunov stability criterion that, these solutions are stable for arbitrary values of nonlinear parameters. Recently, in [8], envelope compacton and solitary pattern solutions of a generalized NLSE were described. Also, envelope compactons and solitary patterns were studied [9]. More recently, new exact special solutions including compactons in a generalized NLSE model were studied [10].

None of the evolution equations possessing compacton solutions have been shown to be integrable; in fact, some are non-integrable and possess only a finite number of conserved quantities. Hence, it is of great interest to search for compacton-like solutions of integrable nonlinear equations. In particular, one would like to compare their properties with the solitons on one hand, and the compactons on the other. Furthermore, the possibility of these compact solutions arising from the nonlinear equations, relevant for physical problems, will make them amenable for experimental detection.

2. Compacton-like solution for MKdV equation

Here, we first show the existence of fractional-transform solutions to modified KdV (MKdV) equation [11],

$$u_t + u^2 u_x + u_{3x} = 0. \tag{6}$$

We start with a fractional transfrom

$$u(\xi) = \frac{A + Bf^2}{1 + Df^2}, \tag{7}$$

where the travelling wave variable $\xi = x - 4k^2 t$ and the determinant of the fractional transform should not be zero. Then f can be chosen as any of the twelve Jacobian elliptic functions and exact solutions of the MKdV equation can be found. For demonstrating the existence of compacton-like solution for this equation, we choose $f = \mathrm{cn}(\xi, m)$ and obtain the following trigonometric solution:

$$u_{\mathrm{p}}(x,t) = \frac{\sqrt{32}}{3} k \frac{\cos^2 k\xi}{(1 - \frac{2}{3}\cos^2 k\xi)}. \tag{8}$$

One immediately notices that, the mKdV equation supports a compacton solution,

$$u_{\mathrm{c}}(x,t) = \frac{\sqrt{32}}{3} k \frac{\cos^2 k\xi}{(1 - \frac{2}{3}\cos^2 k\xi)} \Theta, \tag{9}$$

where

$$\Theta = \left[\theta\left(\xi + \frac{\pi}{2k}\right) - \theta\left(\xi - \frac{\pi}{2k}\right)\right],$$

if a source term of the following type

$$\Pi = 2\frac{\sqrt{32}}{3} kk^3 \left[\delta\left(\xi + \frac{\pi}{2k}\right) - \delta\left(\xi - \frac{\pi}{2k}\right)\right] \tag{10}$$

is included in the MKdV equation.

Note that, for this compact solution, both the width and amplitude are proportional to the square root of the velocity. This behaviour is reminiscent of the solitons. Though the second derivative of the solution is discontinuous at the boundaries, it is a strong solution of the equation of motion, a feature similar to the compactons. We would like to point out that, $u_{\mathrm{c}}(x,t)$, when unconstrained, is a periodic solution of the MKdV equation, which to the best of our knowledge, has not appeared in earlier works. Judicious truncation of the periodic solution, by confining it to the fundamental strip, gives compact support for this solution.

The Hamiltonian from which MKdV equation can be derived by the variational principle,

$$H = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2 \mathrm{d}x - \frac{1}{12}\int_{-\infty}^{\infty} u^4 \mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} u\Pi \mathrm{d}x, \tag{11}$$

and the momentum expression given by

$$P = \frac{1}{2}\int_{-\infty}^{\infty} u^2 \mathrm{d}x, \tag{12}$$

are well defined at the edges.

Explicitly, for the above solution, the energy and momentum are given, respectively by $E_{\rm c} = -(16/3)\pi k^3 + 2.07\pi$ and $P_{\rm c} = -4\pi k$. For the soliton solution of the MKdV equation

$$u_{\rm s}(x,t) = \sqrt{6}\, k\, \mathrm{sech} k(x - k^2 t), \tag{13}$$

the corresponding quantities are $E_{\rm s} = -2k^3$ and $P_{\rm s} = 6k$. Hence, the compacton for positive values of k has lower energy compared to the soliton. The situation is opposite for negative values of k. Other conserved quantities, involving the even derivatives of the solutions, are not well defined at the boundaries. In this sense, these solutions are similar to compactons.

3. Compacton-like solution for nonlinear Schrödinger equation with an external source

Next, we show that this compacton-like solution of the MKdV equation is also a strong solution of the nonlinear Schrödinger equation (NLSE) if an appropriate source term is included.

The NLSE with an external source reads [12] as

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q - \eta = 0. \tag{14}$$

Using the following ansatz,

$$q(x,t) = \mathrm{e}^{i[\psi(\xi') - \omega t]} a(\xi), \tag{15}$$

where $\xi' = x - vt$, and choosing the source term as $\eta(\xi') = K\tilde{\Pi}\mathrm{e}^{i[\psi(\xi')-\omega t]}$, we can separate the real and the imaginary parts of the equation as

$$v\psi' a + \omega a + \frac{a''}{2} - \frac{\psi'^2 a}{2} + a^3 - K\tilde{\Pi} = 0, \tag{16}$$

$$-va' + \frac{\psi'' a}{2} + \psi' a' = 0. \tag{17}$$

Equation (17) can be straightforwardly solved to give

$$\psi' = v + \frac{P}{a^2}, \tag{18}$$

where P is the integration constant. Choosing $P = 0$, we arrive at the following solutions for the functions $\psi(\xi')$ and $a(\xi')$:

$$\psi(\xi') = v\xi' \tag{19}$$

and

$$a(\xi') = \left(\frac{16K}{27}\right)^{1/3} \frac{\cos^2[(\frac{27}{16})^{1/6} K^{1/3} \xi']}{(1 - \frac{2}{3}\cos^2[(\frac{27}{16})^{1/6} K^{1/3}\xi'])}\tilde{\Pi}, \tag{20}$$

where

$$\tilde{\Pi} = \left[\theta\left(\xi' + \frac{\pi}{2k}\right) - \theta\left(\xi' - \frac{\pi}{2k}\right)\right].$$

Here, ω is related to v by

$$2\omega + v^2 = -\frac{27}{4}(16K/27)^{2/3}. \tag{21}$$

Note that, the solution exists for values of $\omega \leq -\frac{27}{4}(16K/27)^{2/3}$. We would like to add that, we have taken the simplest solution of the equation involving the phase. In principle, one can choose $P \neq 0$, which gives rise to periodic solutions obtained numerically [13].

It is worth emphasizing that the NLSE plays a significant role in nonlinear optics [14,15]. Since q represents the electric field, the corresponding source term η can be understood as dipole sources. Hence, a fluid nonlinear medium with moving dipoles can be a plausible source of these types of solitary waves.

4. Conclusion

In conclusion, we have shown the existence of solutions with compact support for MKdV and nonlinear Schrödinger equations with appropriate sources. As the MKdV equation manifests in diverse physical phenomena, it will be exciting, if these waves can be realized in an experimental situation. Although not presented here, the stability of this compacton solution has been checked using centred finite difference method. We have found that the compacton becomes unstable as we refine the grid size. The present authors have shown the utility of the NLSE with an external source for compression of solitary wave solutions in asymmetric twin-core fibres [16–18]. We have also checked the stability of the compacton solution of NLSE with an external source using Crank–Nicolson finite difference algorithm and found that the compacton remains quite stable against finite perturbations. Finally, we hope that this compacton signal may be launched in long-haul telecommunication networks involving asymmetric twin-core fibres.

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