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Anisotropic spheres with Van der Waals-type equation of state

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Abstract. We study static spherically symmetric space-time to describe relativistic compact objects with anisotropic matter distribution and derive two classes of exact models to the Einstein–Maxwell system with a modified Van der Waals equation of state. We motivate a Van der Waals-type equation of state to physically signify a high-density domain of quark matter, and the generated exact solutions are shown to contain several classes of exact models reported previously that correspond to various physical scenarios. Geometrical analysis shows that the physical quantities are well behaved so that these models may be used to describe anisotropic charged compact spheres.

Keywords. Einstein–Maxwell system; anisotropic matter; equation of state; relativistic star.

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1. Introduction

The modelling of densely charged gravitating objects in strong gravitational fields has generated much interest in recent times because of its relevance in describing relativistic astrophysical objects such as neutron stars, quark stars, hybrid protoneutron stars, bare quark stars, etc. During the gravitational collapse or during an accretion process onto a compact object, the matter is likely to acquire large amounts of electric charge [1,2]. The presence of charge produces values for the redshift, luminosity and maximum mass, which are different from neutral matter [3,4].

There are several investigations of the Einstein–Maxwell system of equations for static spherically symmetric gravitational fields to include the effects of the electromagnetic field, with the interior space-time to match at the boundary, the Reissner–Nordstrom exterior model. In recent years, many exact solutions to the field equations have been generated by different approaches. Exact models have been found with generalized forms for one of the gravitational potentials that does not satisfy a particular barotropic equation of state (EoS) relating the pressure to the energy density [4–9]. On the other hand, there are solutions found for the Einstein–Maxwell system, which specify an EoS (e.g., linear, quadratic, polytropic, etc.). Many of the strange star studies have been performed within the framework of the Bag model [10–13], which is linear. Moreover, recently, there have been a few solutions to the Einstein–Maxwell system with anisotropic matter distribution with quadratic [14,15], polytropic [16] and non-linear [17] equations of state. In this paper, we have introduced a modified Van der Waals EoS that would be suitable to model a charged anisotropic compact sphere.

Van der Waals EoS is of particular significance at a regime where strange matter is viewed as a relativistic Fermi gas of quark at the extremes of densities necessary for colour deconfinement to occur. At this high-density, colour-singlet domain, the average interquark separation is expected to be smaller than the typical confining scale at which the asymptotic freedom demands that the interaction between all quarks be weak. In this regime, the only important correlation among quarks will be induced by the Pauli exclusion principle and the system should evolve into a Fermi gas of quarks where the effective nuclear forces mediated by multigluon exchange lead to a short-range force, which is considered to be in analogy with the effective Van der Waals force due to multiphoton exchange interaction between molecules (e.g., H_2O and CO_2). That is, the short-distanced colour force (the colour 'Van der Waals force') could crystallize the quark clusters at low temperatures as in the experimental analogy with H_2O: water boils at 100°C when the density is 1 g cm^{-3}, but freezes even at 300°C when the density is about 1.5 g cm^{-3}. This analogy in fact visualizes the two quark matter regimes that are of research significance: the temperature-dominated phase in the early Universe or in relativistic heavy-ion collider and the density-dominated phase in relativistic astrophysical objects. The latter domain being the focus of this work, we motivate a Van der Waals-type EoS in this theoretical study that would physically signify a model of solid quark stars. It is noted that on similar grounds, Xu [18] used a Van der Waals-type potential to model solid quark stars.

Early work of Ruderman showed that nuclear matter may become anisotropic in such high-density region when treated relativistically [19]. Though we lack a complete understanding of the microscopic origin of the pressure anisotropy, the role of pressure anisotropy in the modelling of compact stars is a field of active research. Since the pioneering work of Bowers and Liang [20], extensive literature has been devoted to the study of anisotropic, spherically symmetric, static matter distributions. The pressure difference between the radial and tangential components of such anisotropic matter becomes crucial in the calculations of surface tension of a compact star.

In this paper we generated solutions to the Einstein–Maxwell system in the presence of an electric field with a modified Van der Waals EoS for spherically symmetric, relativistic static matter with anisotropic distribution. The motive of this work is to introduce a more general Van der Waals-type EoS that would physically signify a high-density domain of quark matter, and also to show that the generated exact solutions contain several classes of exact models reported previously that correspond to various physical scenarios.

2. Anisotropic space-time

The interior of a spherically symmetric static star can be described by the line element

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu(r)}\mathrm{d}t^2 + \mathrm{e}^{2\lambda(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{1}$$

in Schwarzschild coordinates $(x^a) = (t, r, \theta, \phi)$. We assume the energy–momentum tensor for charged anisotropic imperfect fluid sphere to be of the form

$$T_{ij} = \mathrm{diag}\left(-\rho - \frac{1}{2}E^2,\, p_{\mathrm{r}} - \frac{1}{2}E^2,\, p_{\mathrm{t}} + \frac{1}{2}E^2,\, p_{\mathrm{t}} + \frac{1}{2}E^2\right), \tag{2}$$

where ρ is the energy density, p_{r} is the radial pressure, p_{t} is the tangential pressure and E is the electric field intensity. These quantities are measured relative to the co-moving fluid velocity $u^i = \mathrm{e}^{-\nu}\delta^i_0$. For the line element (1) and matter distribution (2), the Einstein–Maxwell field equations can be expressed as

$$\frac{1}{r^2}\left[r(1-\mathrm{e}^{-2\lambda})\right]' = \rho + \frac{1}{2}E^2, \tag{3}$$

$$-\frac{1}{r^2}\left(1-\mathrm{e}^{-2\lambda}\right) + \frac{2\nu'}{r}\mathrm{e}^{-2\lambda} = p_{\mathrm{r}} - \frac{1}{2}E^2, \tag{4}$$

$$\mathrm{e}^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p_{\mathrm{t}} + \frac{1}{2}E^2, \tag{5}$$

$$\sigma = \frac{1}{r^2}\mathrm{e}^{-\lambda}(r^2E)', \tag{6}$$

where primes denote differentiation with respect to r and σ is the proper charge density. In the system of field equations (3)–(6), we are using units where the coupling constant $(8\pi G/c^4) = 1$ and the speed of light $c = 1$. The system of equations (3)–(6) governs the behaviour of an anisotropic charged imperfect fluid sphere.

Using the transformation

$$x = cr^2, \qquad Z(x) = \mathrm{e}^{-2\lambda(r)}, \qquad A^2y^2(x) = \mathrm{e}^{2\nu(r)}, \tag{7}$$

where A and c are arbitrary constants, the system (3)–(6) can be written as

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{c} + \frac{E^2}{2c}, \tag{8}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p_{\mathrm{r}}}{c} - \frac{E^2}{2c}, \tag{9}$$

$$4xZ\frac{\ddot{y}}{y} + (4Z + 2x\dot{Z})\frac{\dot{y}}{y} + \dot{Z} = \frac{p_{\mathrm{t}}}{c} + \frac{E^2}{2c}, \tag{10}$$

$$\frac{\sigma^2}{c} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2, \tag{11}$$

where dots denote differentiation with respect to the variable x.

For a physically realistic relativistic star we expect that the matter distribution should satisfy a barotropic equation of state $p_{\mathrm{r}} = p_{\mathrm{r}}(\rho)$: as explained in §1 in this paper we assume a modified Van der Waals equation of state

$$p_{\mathrm{r}} = \alpha\rho^2 + \frac{\beta\rho}{1+\gamma\rho} - B, \tag{12}$$

where α, β, γ and B are real constants. If $\gamma = 0$, (12) reduces to a quadratic EoS and if $\alpha = \gamma = 0$ this reduces to a linear EoS.

3. Exact models

We have to make physically reasonable choices for any two of the independent variables and then solve the system (8)–(12) to generate exact models as there are two excess variables compared to the number of equations. Here, we choose the following forms for the gravitational potential Z and electric field intensity E as motivated by Thirukkanesh and Maharaj [12], Feroze and Siddiqui [14] and Sharma and Maharaj [21]:

$$Z = \frac{1+ax}{1+bx}, \tag{13}$$

$$\frac{E^2}{c} = \frac{k(3+bx)}{(1+bx)^2}, \tag{14}$$

where a, b and k are real constants. To integrate the system (8)–(12) with the choice in (13) and (14), we consider the following two cases

3.1 *Case* I: $a = 0$

In this case, on integration of the system we obtained the gravitational potential

$$y = D\exp[F(x)](1+bx)^{(1/16b)[\alpha c(k-2b)^2-4k]} \times \left[2(1+bx)^2 - \gamma c(k-2b)(3+bx)\right]^s [G(x)]^t, \tag{15}$$

where

$$F(x) = \frac{x}{8c}\left\{-kc(1+\beta) - 2B + b[2c(\beta+1) - Bx] - \frac{\alpha c(k-2b)^2(3+2bx)}{8b(1+bx)^2}\right\}, \tag{16}$$

$$G(x) = \frac{4(1+bx) + \gamma c(2b-k) + \sqrt{\gamma c(2b-k)[\gamma c(2b-k)-16]}}{-4(1+bx) - \gamma c(2b-k) + \sqrt{\gamma c(2b-k)[\gamma c(2b-k)-16]}}, \tag{17}$$

$$s = \frac{\beta}{32b}(2b-k)[4-\gamma c(2b-k)], \tag{18}$$

$$t = \frac{\beta\sqrt{\gamma c}(2b-k)^{3/2}[12-\gamma c(2b-k)]}{16b\sqrt{\gamma c(2b-k)-16}}, \tag{19}$$

D is a constant of integration; and the matter variables are

$$\rho = \frac{c(2b-k)(3+bx)}{2(1+bx)^2} \tag{20}$$

$$p_{\rm r} = \alpha\left[\frac{c(2b-k)(3+bx)}{2(1+bx)^2}\right]^2 + \frac{\beta c(2b-k)(3+bx)}{2(1+bx)^2 + \gamma c(2b-k)(3+bx)} - B \tag{21}$$

$$p_{\rm t} = \frac{4cx}{(1+bx)}\frac{\ddot{y}}{y} + \frac{2(2+bx)}{(1+bx)^2}\frac{\dot{y}}{y} - \frac{b}{(1+bx)^2} + \frac{k(3+bx)}{2(1+bx)^2} \tag{22}$$

$$\sigma^2 = \frac{ck(1+ac)[6+3bx+b^2x^2]}{x(3+bx)(1+bx)^5}. \tag{23}$$

It is noticed that when $\gamma = 0$ the above model will be reduced to the first category of Feroze and Siddiqui's [14] charged anisotropic matter with quadratic EoS; and when $\alpha = \gamma = 0$ we can regain the solutions of the second category for anisotropic charged anisotropic matter with linear EoS by Thirukkanesh and Maharaj [12].

3.2 *Case* II: $a \neq b \neq 0$

In this case, on integrating the system, we obtain the gravitational potential

$$y = d\exp[f(x)](1+ax)^l(1+bx)^m \times \left[2(1+bx)^2 - \gamma c[2(a-b)+k](3+bx)\right]^n [g(x)]^h, \tag{24}$$

where

$$f(x) = \frac{\alpha c[2(a-b)+k]^2}{8(a-b)^2}\frac{[a(5+4bx)-b(3+2bx)]}{(1+bx)^2} - \frac{bBx}{4ac}, \tag{25}$$

$$g(x) = \frac{-4(1+bx)+\gamma c[2(a-b)+k]+\sqrt{\gamma c[2(a-b)+k][16+\gamma c[2(a-b)+k]]}}{4(1+bx)-\gamma c[2(a-b)+k]+\sqrt{\gamma c[2(a-b)+k][16+\gamma c[2(a-b)+k]]}}, \tag{26}$$

$$l = \frac{1}{16}\left[\frac{\alpha c(b-3a)^2[2(a-b)+k]^2}{(a-b)^3} - \frac{4(a-b)(B+ac)}{a^2c} + \frac{2k(b-3a)}{a(a-b)} + \frac{4\beta[2(a-b)+k](3a^2-4ab+b^2)}{a\left[\gamma ac(3a-b)[2(a-b)+k]-2(a-b)^2\right]}\right] \tag{27}$$

$$m = \frac{4k(a-b)^2 - \alpha c(b-3a)^2[2(a-b)+k]^2}{16(a-b)^3}, \tag{28}$$

$$n = \frac{-\beta[2(a-b)+k]\left[4(a-b)+\gamma c(3a-b)[2(a-b)+k]\right]}{16\left[\gamma ac(3a-b)[2(a-b)+k]-2(a-b)^2\right]}, \tag{29}$$

$$h = \frac{-\beta\sqrt{\gamma c}[2(a-b)+k]^{3/2}\left[4(7a-3b)+\gamma c(3a-b)[2(a-b)+k]\right]}{16\sqrt{16+\gamma c[2(a-b)+k]}\left[\gamma ac(3a-b)[2(a-b)+k]-2(a-b)^2\right]}, \tag{30}$$

d is a constant of integration; and the matter variables are

$$\rho = \frac{c[2(b-a)-k](3+bx)}{2(1+bx)^2} \tag{31}$$

$$p_{\rm r} = \alpha\left[\frac{c[2(b-a)-k](3+bx)}{2(1+bx)^2}\right]^2 + \frac{\beta c[2(b-a)-k](3+bx)}{2(1+bx)^2+\gamma c[2(b-a)-k](3+bx)} - B \tag{32}$$

$$p_t = \frac{4cx(1+ax)}{(1+bx)}\frac{\ddot{y}}{y} + \left[\frac{4(1+ax)}{(1+bx)} - \frac{2(b-a)x}{(1+bx)^2}\right]\frac{\dot{y}}{y} - \frac{2(b-a)+k(3+bx)}{2(1+bx)^2} \tag{33}$$

$$\sigma^2 = \frac{ck(1+ac)[6+3bx+b^2x^2]}{x(3+bx)(1+bx)^5}. \tag{34}$$

We see that the above model is reduced to the second category of Feroze and Siddiqui's [14] charged anisotropic matter with quadratic EoS when $\gamma = 0$; and for $\alpha = \gamma = 0$ it contains the third category of Thirukkanesh and Maharaj [12] anisotropic charged anisotropic matter with linear EoS. Thirukkanesh and Maharaj [12] have shown that the uncharged anisotropic model of Sharma and Maharaj [21], dark energy stars of Lobo [22], de Sitter isotropic model and the familiar Einstein model can be regained from their model.

4. Discussion

We have derived two classes of exact models to describe charged anisotropic relativistic compact objects with a generalized EoS, which contain models reported previously. It is observed that the models generated with modified Van der Waals EoS reduce to the solutions by Feroze and Siddiqui [14] for charged anisotropic matter with quadratic EoS when $\gamma = 0$; and further reduced to solutions by Thirukkanesh and Maharaj [12] for charged anisotropic matter with linear EoS when $\alpha = \gamma = 0$. Moreover, it is shown [12] that the uncharged anisotropic model of Sharma and Maharaj [21], the dark energy stars of Lobo [22], the de Sitter isotropic model and Einstein's model can be regained from their model with linear EoS. Observe that our approach has combined both the charged and uncharged cases for a relativistic star: when $k = 0$ we directly obtain new classes of solutions for the uncharged matter distribution. The radial dependence of the physical

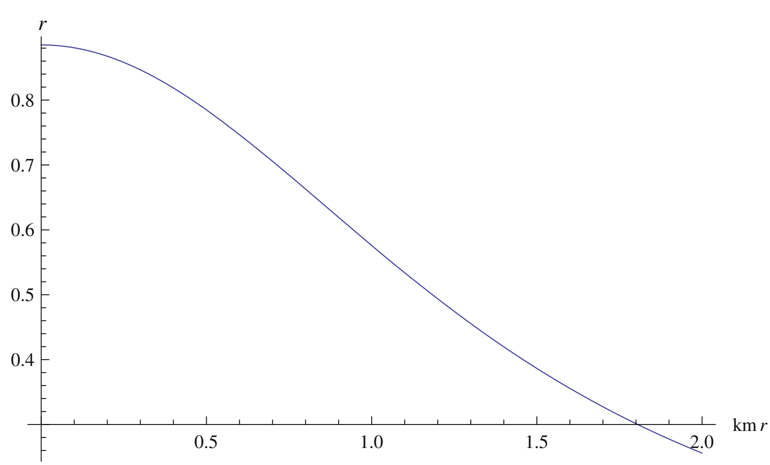

Figure 1. Radial dependence of matter density for Case I.

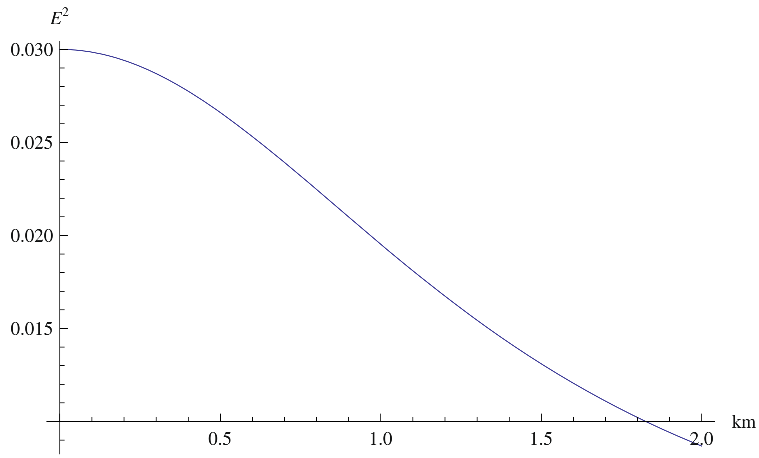

Figure 2. Diminishing electric field for Case I.

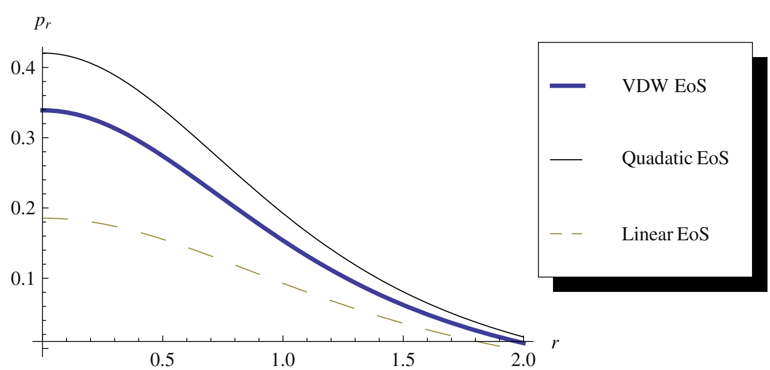

Figure 3. Radial pressure for Case I.

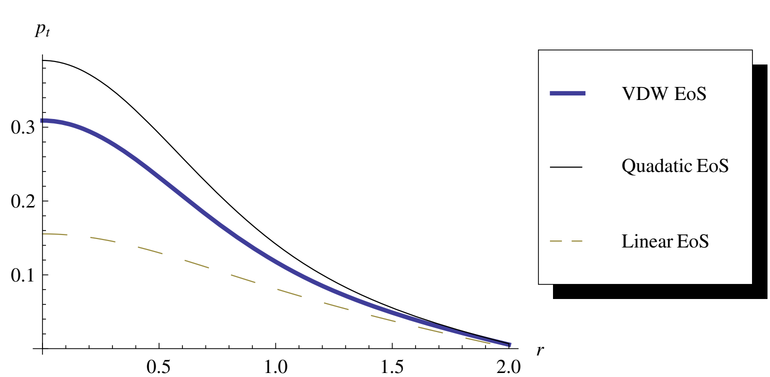

Figure 4. Tangential pressure for Case I.

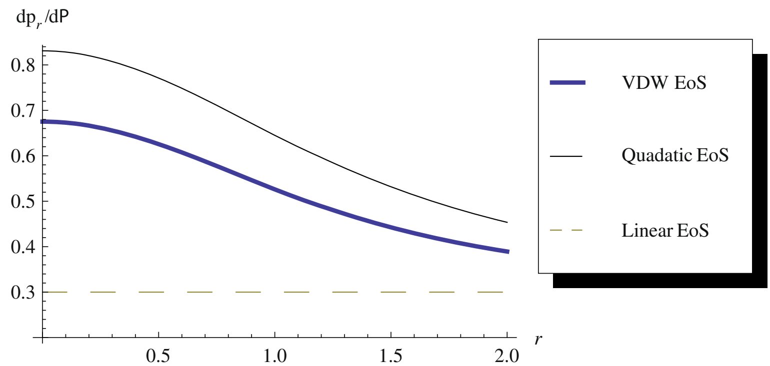

Figure 5. Derivative of radial pressure with respect to energy density for Case I.

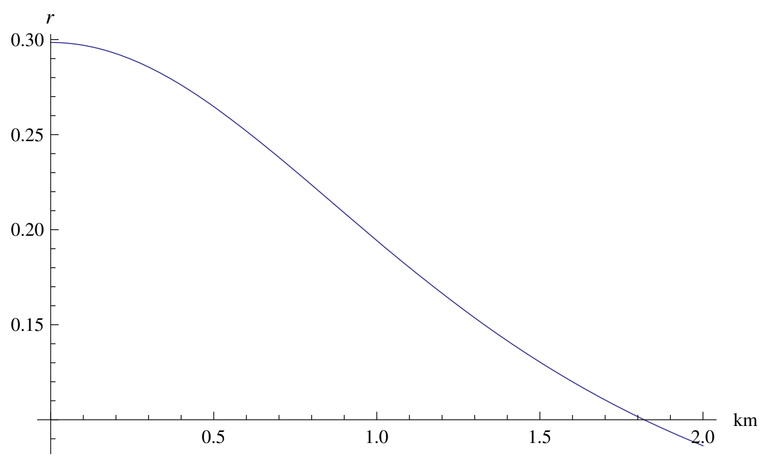

Figure 6. Radial dependence of matter density for Case II.

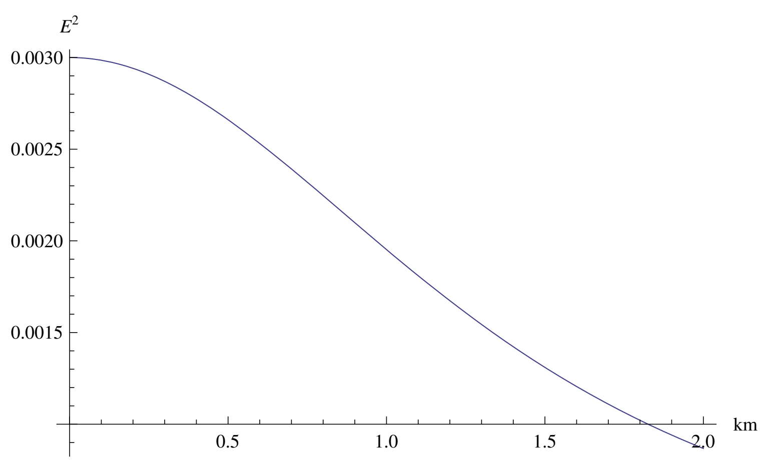

Figure 7. Diminishing electric field for Case II.

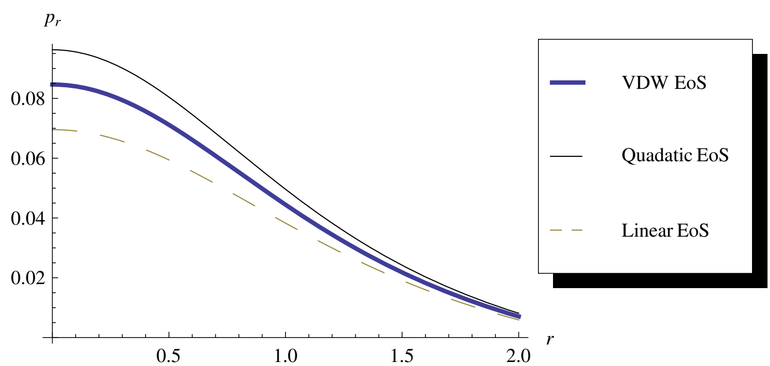

Figure 8. Radial pressure for Case II.

quantities of the two classes of models ($a = 0$ and $a \neq b \neq 0$) generated for the Einstein–Maxwell system in spherically symmetric static space-time with the modified Van der Waals EoS have been analysed in §3.1 and 3.2, respectively, and plotted in figures 1–10. The parameters were chosen to be $b = \alpha = \beta = 0.3$, $\gamma = 0.5$, $k = 0.01$, $c = 1$ and $B = 0.08$ for plots in figures 1–5 and $a = 0.2$, $b = \alpha = \beta = 0.3$, $\gamma = 0.5$, $k = 0.001$, $c = 1$ and $B = 0.02$ for plots in figures 6–10 for Van der Waals EoS, and we have set $\gamma = 0$ for quadratic EoS and $\alpha = \gamma = 0$ for linear EoS. A pleasing feature of our plots is that we can conveniently distinguish the role of equations of state on the physical quantities.

Figures 1 and 6 illustrate the radial dependence of matter density $\rho(r)$ for the two cases, respectively, which decreases monotonically towards the surface of the sphere and the central density $\rho(0) > 0$. Moreover, figures 2 and 7 illustrate the diminishing electric field towards the surface. Setting parameter $\gamma = 0$ for the first case with $a = 0$, the model is reduced to the first category by Feroze and Siddiqui's [14] the charged anisotropic matter with quadratic equation of state, for which the radial dependence of $p_{\mathrm{r}}(r)$ and $p_{\mathrm{t}}(r)$ are shown by plots of thin curves in figures 3 and 4, respectively. Moreover, the case

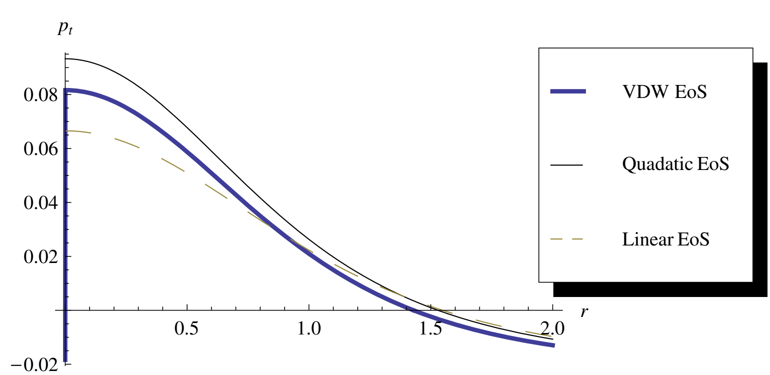

Figure 9. Tangential pressure for Case II.

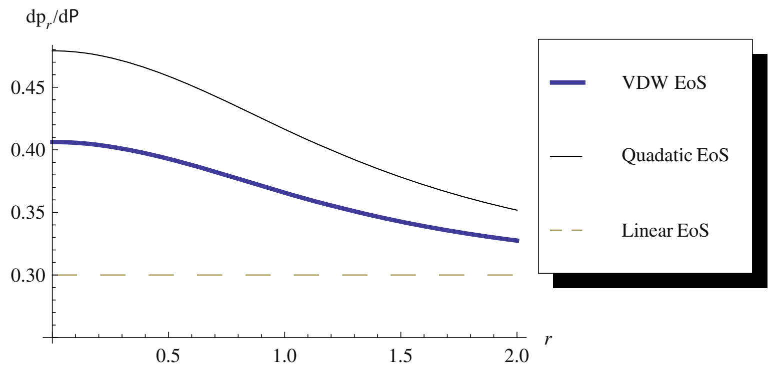

Figure 10. Derivative of radial pressure with respect to energy density for Case II.

regains the second category by Thirukkanesh and Maharaj [12]: the charged anisotropic matter with linear equation of state, for which the radial dependence of $p_{\rm r}(r)$ and $p_{\rm t}(r)$ are shown by plots of dashed curves in figures 3 and 4, respectively. These illustrate a good comparison with the $p_{\rm r}(r)$ and $p_{\rm t}(r)$ behaviour with Van der Waals EoS for the corresponding case that is represented by the thick-line curves in figures 3 and 4, which take an intermediate behaviour between quadratic and linear equations of state. It is noted that the radial pressure $p_{\rm r}$ decreases towards the surface from a positive value at the centre of the sphere for all three equations of state.

On the other hand, for Case II with $a \neq b \neq 0$, the model reduces to the second category by Feroze and Siddiqui [14] for $\gamma = 0$, and for $\alpha = \gamma = 0$ the model contains the third category by Thirukkanesh and Maharaj [12]. The thin (quadratic EoS) and the dashed (linear EoS) curves, respectively, correspond to these cases and the respective $p_{\rm r}(r)$ and $p_{\rm t}(r)$ behaviour are illustrated in figures 8 and 9. These illustrate a good comparison with Van der Waals EoS of $p_{\rm r}(r)$ and $p_{\rm t}(r)$ behaviour for the case that is plotted with thick line in figures 8 and 9, which also takes an intermediate range between quadratic and linear equations of state curves as for Case I. Moreover, as described for Case I, the radial pressure satisfies the positive definiteness condition for all types of equations of state. Figures 5 and 10 illustrate that $0 < {\rm d}p_{\rm r}/{\rm d}\rho < 1$ throughout the interior of the sphere and hence causality condition is maintained (i.e., speed of sound is less than the speed of light) within the sphere for all type of EoS for $a = 0$ and $a \neq b \neq 0$.

The derived exact solutions signify the possible representation of the Van der Waals EoS for the description of weakly interacting relativistic Fermi gas of quark. In such a scenario, the short-distanced colour Van der Waals force can crystallize the quark clusters at low temperatures and the model can physically signify a solid quark star in a high-density, colour-singlet domain. Figures 3, 4, 8 and 9 signify a high-pressure domain with Van der Waals EoS compared with linear Bag model (linear EoS by dashed curves) that conventionally represent a strange quark star [12], which is expected to be in a relatively lower density regime than the scenario considered in this work with charge confinement. Hence, the Van der Waals EoS with interior charge is proposed to be a potential model to describe a special regime of quark stars.

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