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Approximate eigensolutions of Dirac equation for the superposition Hellmann potential under spin and pseudospin symmetries

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Abstract. The Hellmann potential is simply a superposition of an attractive Coulomb potential $-a/r$ plus a Yukawa potential $be^{-\delta r}/r$. The generalized parametric Nikiforov–Uvarov (NU) method is used to examine the approximate analytical energy eigenvalues and two-component wave function of the Dirac equation with the Hellmann potential for arbitrary spin-orbit quantum number κ in the presence of exact spin and pseudospin (p-spin) symmetries. As a particular case, we obtain the energy eigenvalues of the pure Coulomb potential in the non-relativistic limit.

Keywords. Dirac equation; Hellmann potential; spin and p-spin symmetry; Nikiforov–Uvarov method.

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1. Introduction

The Dirac equation is usually used for the description of spin-1/2 particle dynamics in relativistic quantum mechanics. It is known that the spin and pseudospin symmetries play critical roles in the shell structure and its evolution. The introduction of the spin-orbit potential to the single-particle shell model can well explain the experimentally observed existence of magic numbers for nuclei close to the valley of β stability [1,2]. To understand the near degeneracy observed in heavy nuclei between two single-particle states $(n-1, l+2, j=l+3/2)$ and $(n, l, j=l+1/2)$, the pseudospin symmetry (PSS) was introduced by defining the pseudospin doublets $(\tilde{n}=n-1, \tilde{l}=l+1, j=\tilde{l}\pm 1/2)$ [3,4], which has explained various phenomena in nuclear structure including deformation [5], superdeformation [6], identical bands [7] and magnetic moment [8]. As a result of these successes, there have been much efforts to understand their origins as well as the breaking mechanisms. For the spin symmetry (SS), the spin-orbit potential can be obtained naturally from the solution of the Dirac equation. Thus, the SS can be considered as a relativistic symmetry. On the other hand, the origin of the PSS has not been fully clarified until now. Some of the major progresses in understanding the underlying mechanism of PSS is worth reviewing. A substantial progress was accomplished by Ginocchio [9] where the relativistic feature of PSS was recognized. The pseudo-orbital angular momentum $\tilde{l}$ is nothing but the orbital angular momentum of the lower component of the Dirac spinor and the equality in magnitude, but the difference in sign of the scalar potential S and vector potential V was suggested as the exact PSS limit. Meng *et al* showed that the exact PSS occurs in the Dirac equation when the sum of the scalar S and vector V potentials is equal to a constant [10]. Unfortunately, the exact PSS cannot be met in real nuclei, and much effort has been devoted to the cause of splitting. In refs [11–13], it was argued that the observed pseudospin splitting arises from a cancellation of several energy components and the PSS in nuclei has a dynamical character. A similar conclusion was reached in refs [14,15]. In addition, it was noted that, unlike the SS, pseudospin-breaking cannot be treated as a perturbation of the PSS Hamiltonian [16]. The nonperturbation nature of the PSS has also been demonstrated in ref. [17]. Despite all these pioneering studies, the origins of SS and PSS have not been fully understood in the relativistic framework. Recently, Guo [18] has checked the PSS by using the similarity renormalization group and has shown explicitly the relativistic origin of this symmetry. However, the dependence of the quality of PSS on the relativistic effect has not been checked until now. Further, Chen and Guo [19] studied the evolution of the SS and PSS from the relativistic to the nonrelativistic, to explore the relativistic relevance of these symmetries.

In the present work, we shall explore the SS and PSS in Hellmann potential [20,21], which is a superposition of the Coulomb plus Yukawa potential suggested by Hellmann as

$$V(r) = -\frac{a}{r} + b\frac{\mathrm{e}^{-\delta r}}{r}, \tag{1}$$

where a and b are the strengths of the Coulomb and the Yukawa potentials, respectively, and δ is the screening parameter [20,21]. The Hellmann potential has been used by various authors to represent the electron–core [22,23] or the electron–ion [24,25] interaction. Varshni and Shukla [26] have used this model potential for alkali hydride molecules. Das and Chakravarty [27] have proposed that such a potential is suitable for the study of inner-shell ionization problems. Adamowski [28] studied the bound-state energies of the Hellmann potential for various sets of values of b and δ in a variational framework using ten variational parameters. Dutt *et al* [29] have also investigated the bound-state energies as well as the wave functions of this potential using the large-N expansion technique. Hall and Katatbeh have used potential envelopes method to analyse the bound-state spectrum of the Schrödinger Hamiltonian with a Hellmann potential [30]. Ikhdair and Sever have investigated the energy levels of neutral atoms by applying an alternative perturbative scheme in solving the Schrödinger equation for the Yukawa potential model with a modified screening parameter [31]. Ikhdair and Sever have also studied the bound states of the Hellmann potential with arbitrary strength b and screening parameter δ by using a perturbative approach [32]. Roy *et al* studied the Hellman problem using a generalized pseudospectral method [33]. Nasser and Abdelmonem, using the J-matrix approach, studied the trajectories of the poles of the S-matrix for a Hellmann potential in the complex energy plane near the critical screening parameter [34].

The objective of this work is to obtain approximate analytical solution of the Dirac particle in the scalar and vector Hellmann potentials in the presence of spin and p-spin symmetries in the framework of the NU method.

The structure of the paper is as follows. In §2, the parametric generalization of the NU method is displayed. In §3, in the context of spin and p-spin symmetries, we briefly introduce the Dirac equation with scalar and vector Hellmann potentials for arbitrary spin-orbit quantum number κ. In the presence of the spin and p-spin symmetries, the approximate energy eigenvalue equations and the corresponding two-component wave functions of the Dirac–Hellmann problem are obtained. The nonrelativistic limit of the problem is discussed in this section, too. Finally, our concluding remarks are given in §4.

2. Parametric NU method

Let us consider the following differential equation [35–37]:

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \tag{2}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without the need to check the validity of the solution, we present a shortcut for the method. At first, we write the general form of the Schrödinger-like equation (2) in a more general form as

$$\psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(1-c_3 s)}\right)\psi_n'(s) + \left(\frac{-p_2 s^2 + p_1 s - p_0}{s^2(1-c_3 s)^2}\right)\psi_n(s) = 0, \tag{3}$$

satisfying the wave functions

$$\psi_n(s) = \phi(s) y_n(s). \tag{4}$$

Comparing (3) with its counterpart (2), we obtain the following identifications:

$$\tilde{\tau}(s) = c_1 - c_2 s, \quad \sigma(s) = s(1-c_3 s), \quad \tilde{\sigma}(s) = -p_2 s^2 + p_1 s - p_0. \tag{5}$$

Following the NU method [35], we obtain the bound-state energy condition [36,37]

$$\begin{aligned} c_2 n &- (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + n(n-1)c_3 \\ &+ c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0, \end{aligned} \tag{6}$$

resulting also in

$$\begin{aligned} &\rho(s) = s^{c_{10}}(1-c_3 s)^{c_{11}}, \quad \phi(s) = s^{c_{12}}(1-c_3 s)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0, \\ &y_n(s) = P_n^{(c_{10}, c_{11})}(1-2c_3 s), \quad c_{10} > -1, \quad c_{11} > -1, \end{aligned} \tag{7a}$$

so that the wave function becomes

$$\psi_{nl}(s) = N_{nl} s^{c_{12}}(1-c_3 s)^{c_{13}} P_n^{(c_{10}, c_{11})}(1-2c_3 s), \tag{7b}$$

where $P_n^{(\mu,\nu)}(x)$, $\mu > -1$, $\nu > -1$, $x \in [-1, 1]$ are Jacobi polynomials with parameters

$$\begin{aligned}
&c_4 = \frac{1}{2}(1-c_1), \qquad c_5 = \frac{1}{2}(c_2 - 2c_3), \\
&c_6 = c_5^2 + p_2; \qquad c_7 = 2c_4c_5 - p_1, \\
&c_8 = c_4^2 + p_0, \qquad c_9 = c_3(c_7 + c_3c_8) + c_6, \\
&c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1, \\
&c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1, \quad c_3 \neq 0, \\
&c_{12} = c_4 + \sqrt{c_8} > 0, \\
&c_{13} = -c_4 + \frac{1}{c_3}\left(\sqrt{c_9} - c_5\right) > 0, \qquad c_3 \neq 0,
\end{aligned} \tag{8}$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

In the rather more special case of $c_3 = 0$, the wave function (24) becomes

$$\lim_{c_3\to 0} P_n^{(c_{10},c_{11})}(1-2c_3s) = L_n^{c_{10}}(c_{11}s), \quad \lim_{c_3\to 0}(1-c_3s)^{c_{13}} = \mathrm{e}^{c_{13}s},$$

$$\psi(s) = Ns^{c_{12}}\mathrm{e}^{c_{13}s}L_n^{c_{10}}(c_{11}s), \tag{9}$$

where $L_n(x)$ is a Laguerre polynomial.

3. Dirac equation

The Dirac equation for fermionic massive spin-1/2 particles moving in an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ is (in units of $\hbar = c = 1$)

$$\left[\vec{\alpha}\cdot\vec{p} + \beta(M + S(r))\right]\psi(\vec{r}) = [E - V(r)]\psi(\vec{r}), \tag{10}$$

where E is the relativistic energy of the system, $\vec{p} = -i\vec{\nabla}$ is the three-dimensional momentum operator and M is the mass of the fermionic particle [38]. $\vec{\alpha}$ and β are the 4×4 usual Dirac matrices given as

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{11}$$

where I is 2×2 unitary matrix and $\vec{\sigma}$ are three-vector spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

The total angular momentum operator $\vec{J}$ and spin-orbit $K = (\vec{\sigma}\cdot\vec{L} + 1)$, where $\vec{L}$ is the orbital angular momentum operator of the spherical nucleons which commute with the Dirac Hamiltonian. The eigenvalues of the spin-orbit coupling operator are

$$\kappa = \left(j + \frac{1}{2}\right) > 0 \quad \text{and} \quad \kappa = -\left(j + \frac{1}{2}\right) < 0$$

for unaligned spin $j = l - (1/2)$ and the aligned spin $j = l + (1/2)$, respectively. (H, K, J^2, J_z) can be taken as the complete set of the conservative quantities. Thus, the spinor wave functions can be classified according to their angular momentum j, spin-orbit quantum number κ and the radial quantum number n, and can be written as follows:

$$\psi_{n\kappa}(\vec{r}) = \begin{pmatrix} f_{n\kappa}(\vec{r}) \\ g_{n\kappa}(\vec{r}) \end{pmatrix} = \begin{pmatrix} \dfrac{F_{n\kappa}(r)}{r} Y^l_{jm}(\theta, \varphi) \\ i\dfrac{G_{n\kappa}(r)}{r} Y^{\tilde{l}}_{jm}(\theta, \varphi) \end{pmatrix}, \tag{13}$$

where $f_{n\kappa}(\vec{r})$ is the upper (large) component and $g_{n\kappa}(\vec{r})$ is the lower (small) component of the Dirac spinors. $Y^l_{jm}(\theta, \varphi)$ and $Y^{\tilde{l}}_{jm}(\theta, \varphi)$ are spin and pseudospin spherical harmonics, respectively, and m is the projection of the angular momentum on the z-axis. Substituting eq. (13) into eq. (10), one obtains two coupled differential equations for upper and lower radial wave functions $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ as

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right) F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r)) G_{n\kappa}(r), \tag{14a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right) G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r)) F_{n\kappa}(r), \tag{14b}$$

where

$$\Delta(r) = V(r) - S(r), \tag{15a}$$

$$\Sigma(r) = V(r) + S(r). \tag{15b}$$

Eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ from eqs (14a) and (14b), we obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components, respectively:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2}\right] F_{n\kappa}(r) + \frac{\mathrm{d}\Delta(r)/\mathrm{d}r}{M + E_{n\kappa} - \Delta(r)} \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right) F_{n\kappa}(r)$$
$$= [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))] F_{n\kappa}(r), \tag{16}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2}\right] G_{n\kappa}(r) + \frac{\mathrm{d}\Sigma(r)/\mathrm{d}r}{M - E_{n\kappa} + \Sigma(r)} \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right) G_{n\kappa}(r)$$
$$= [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))] G_{n\kappa}(r), \tag{17}$$

where

$$\kappa(\kappa - 1) = \tilde{l}(\tilde{l} + 1) \quad \text{and} \quad \kappa(\kappa + 1) = l(l + 1).$$

The quantum number κ is related to the quantum numbers for spin symmetry l and pseudospin symmetry $\tilde{l}$ as

$$\kappa = \begin{cases} -(l+1) = -\left(j + \frac{1}{2}\right) \left(s_{1/2}, p_{3/2}, \text{etc.}\right), \ j = l + \frac{1}{2}, \text{aligned spin } (\kappa < 0) \\ +l = +\left(j + \frac{1}{2}\right) \left(p_{1/2}, d_{3/2}, \text{etc.}\right), \ j = l - \frac{1}{2}, \text{unaligned spin } (\kappa > 0), \end{cases}$$

and the quasidegenerate doublet structure can be expressed in terms of a pseudospin angular momentum $\tilde{s} = \frac{1}{2}$ and pseudo-orbital angular momentum $\tilde{l}$, which is defined as

$$\kappa = \begin{cases} -\tilde{l} = -\left(j + \frac{1}{2}\right) \left(s_{1/2}, p_{3/2}, \text{etc.}\right), \ j = \tilde{l} - \frac{1}{2}, \text{aligned p-spin } (\kappa < 0) \\ +(\tilde{l}+1) = +\left(j + \frac{1}{2}\right) \left(d_{3/2}, f_{5/2}, \text{etc.}\right), \ j = \tilde{l} + \frac{1}{2}, \text{unaligned p-spin } (\kappa > 0), \end{cases}$$

where $\kappa = \pm 1, \pm 2, \ldots$. For example, $\left(1s_{1/2}, 0d_{3/2}\right)$ and $\left(1p_{3/2}, 0f_{5/2}\right)$ can be considered as pseudospin doublets.

3.1 *Bound states in SS limit*

In the spin symmetric limitation, $(\mathrm{d}\Delta(r)/\mathrm{d}r = 0)$ or $\Delta(r) = C_s =$ constant [39–42], then eq. (16) with $\Sigma(r)$ as Hellmann potential becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} - \gamma\left(-\frac{a}{r} + b\frac{\mathrm{e}^{-\delta r}}{r}\right) - \beta^2\right]F_{n\kappa}(r) = 0, \tag{18a}$$

$$\gamma = M + E_{n\kappa} - C_s \quad \text{and} \quad \beta^2 = (M - E_{n\kappa})(M + E_{n\kappa} - C_s), \tag{18b}$$

where $\kappa = l$ and $\kappa = -l - 1$ for $\kappa < 0$ and $\kappa > 0$, respectively. The Schrödinger-like eq. (18a) that results from the Dirac equation is a second-order differential equation that needs to be treated very carefully while performing the approximation. Therefore, we resort to use an appropriate approximation scheme to deal with the centrifugal potential term as

$$\frac{1}{r^2} \approx \frac{\delta^2}{(1 - \mathrm{e}^{-\delta r})^2}, \tag{19}$$

or equivalently

$$\frac{1}{r} \approx \frac{\delta}{1 - \mathrm{e}^{-\delta r}}, \tag{20}$$

which is valid for $\delta r \ll 1$ [43,44]. In [44], authors have used approximations given by eqs (19), (20) and showed that approximated results have good agreement with numerical calculations for lowest states [50]. Furthermore, very similar approximations have been used before in [45–47] providing a very accurate energy spectrum for the lowest states if compared with the numerical ones calculated from the numerical solution of the Schrödinger equation (see table 1 of [45]). Therefore, the Hellmann potential in (1) reduces to [44–49]

$$V(r) \approx -\frac{\delta a}{1 - \mathrm{e}^{-\delta r}} + \frac{\delta b \mathrm{e}^{-\delta r}}{1 - \mathrm{e}^{-\delta r}}. \tag{21}$$

To see the accuracy of our approximation, we plotted the Hellmann potential (1) and its approximation (21) with parameters $a = 2$, $b = -4$ and $\delta = 0.01$ [32], in figure 1. Thus, employing such an approximation scheme, we can then write eq. (18a) as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \kappa(\kappa+1)\frac{\delta^2}{(1 - \mathrm{e}^{-\delta r})^2} - \gamma\left(-\frac{\delta a}{1 - \mathrm{e}^{-\delta r}} + \frac{\delta b \mathrm{e}^{-\delta r}}{1 - \mathrm{e}^{-\delta r}}\right) - \beta^2\right]F_{n\kappa}(r) = 0. \tag{22}$$

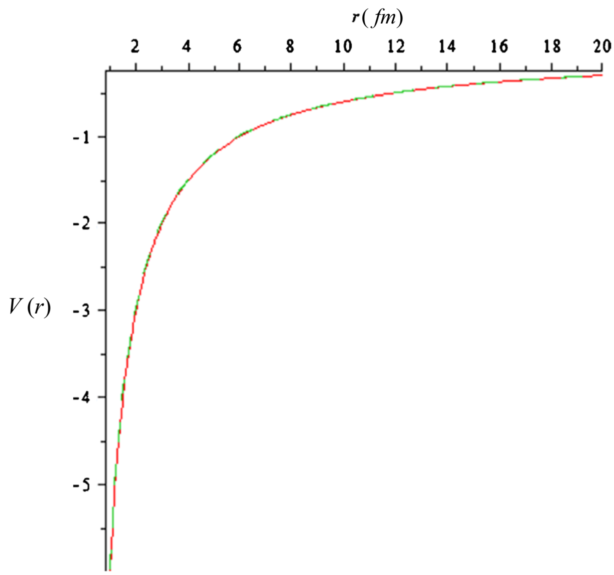

Figure 1. The Hellmann potential (red line) and its approximation in eq. (21) (green line).

Followed by making a new change of variables $s = \mathrm{e}^{-\delta r}$, this allows us to decompose the spin-symmetric Dirac equation (16) into the Schrödinger-type equation satisfying the upper-spinor component $F_{n,\kappa}(s)$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{1-s}{s(1-s)}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{1}{s^2(1-s)^2}\left[-As^2 + Bs - C\right]\right\} F_{n,\kappa}(s) = 0,$$

$$A = -\frac{\gamma b}{\delta} + \frac{\beta^2}{\delta^2},$$

$$B = -\frac{\gamma a}{\delta} - \frac{\gamma b}{\delta} + \frac{2\beta^2}{\delta^2},$$

$$C = \kappa(\kappa+1) - \frac{\gamma a}{\delta} + \frac{\beta^2}{\delta^2}, \tag{23}$$

where $F_{n\kappa}(r) \equiv F_{n,\kappa}(s)$ is used. If the above equation is compared with eq. (3), we can obtain the specific values for constants $c_i (i = 1, 2, 3)$ as

$$c_1 = 1, \quad c_2 = 1 \quad \text{and} \quad c_3 = 1. \tag{24}$$

In order to obtain the bound-state solutions of eq. (23), it is necessary to calculate the remaining parametric constants, that is, $c_i (i = 4, 5, \ldots, 13)$ by means of the relation (8). Their specific values are displayed in table 1 for the relativistic Hellmann potential model.

Table 1. Specific values for the parametric constants used to calculate the energy eigenvalues and wave function.

Constant	Analytic value
c_4	0
c_5	$-\frac{1}{2}$
c_6	$\frac{1}{4}+A$
c_7	$-B$
c_8	C
c_9	$\left(\kappa+\frac{1}{2}\right)^2$
c_{10}	$2\sqrt{C}$
c_{11}	$2\kappa+1$
c_{12}	$\sqrt{C}$
c_{13}	$\kappa+1$

Further, using these constants along with (6), we can readily obtain the energy eigenvalue equation for the Dirac–Hellmann problem as

$$\kappa(\kappa+1)+(n+\kappa+1)^2+\left(2n+\kappa+\frac{3}{2}\right)$$
$$\times\sqrt{\kappa(\kappa+1)-\frac{\gamma a}{\delta}+\frac{\beta^2}{\delta^2}}-\frac{\gamma}{\delta}(b-a)=0. \tag{25}$$

To show the procedure for determining the energy eigenvalues from eq. (25), we take a set of physical parameter values, $M=5\ \text{fm}^{-1}$, $a=1$, $b=-4$, $C_s=5.5\ \text{fm}^{-1}$ and $\delta=0.01$ [32].

In table 2, we present the energy spectrum for the spin symmetric case. Obviously, the pairs $(np_{1/2},\ np_{3/2})$, $(nd_{3/2}, nd_{5/2})$, $(nf_{5/2}, nf_{7/2})$, $(ng_{7/2}, ng_{9/2})$, and so on, are degenerate states.

Table 2. Bound-state energy eigenvalues in unit of fm^{-1} of the spin-symmetry Hellmann potential for several values of n and κ.

l	$n,\kappa<0$	$(l, j=l+1/2)$	$E_{n,\kappa<0}$	$n,\kappa>0$	$(l, j=l-1/2)$	$E_{n,\kappa>0}$
1	0, −2	$0p_{3/2}$	2.266823746	0, 1	$0p_{1/2}$	2.266823746
2	0, −3	$0d_{5/2}$	3.174420713	0, 2	$0d_{3/2}$	3.174420713
3	0, −4	$0f_{7/2}$	3.760219205	0, 3	$0f_{5/2}$	3.760219205
4	0, −5	$0g_{9/2}$	4.127994487	0, 4	$0g_{7/2}$	4.127994487
1	1, −2	$1p_{3/2}$	3.167838743	1, 1	$1p_{1/2}$	3.167838743
2	1, −3	$1d_{5/2}$	3.753448611	1, 2	$1d_{3/2}$	3.753448611
3	1, −4	$1f_{7/2}$	4.121562668	1, 3	$1f_{5/2}$	4.121562668
4	1, −5	$1g_{9/2}$	4.358895657	1, 4	$1g_{7/2}$	4.358895657

On the other hand, in order to establish the upper-spinor component of the wave functions $F_{n,\kappa}(r)$, namely eq. (18a), the relations (7a) and (7b) are used. First, we find the first part of the wave function as

$$\rho(s) = s^{2\sqrt{C}}(1-s)^{2\kappa+1}. \tag{26}$$

Secondly, we calculate the weight function as

$$\phi(s) = s^{\sqrt{C}}(1-s)^{\kappa+1} \tag{27}$$

which gives the second part of the wave function as

$$y_n(s) = P_n^{(2\sqrt{C},2\kappa+1)}(1-2s), \tag{28}$$

where $P_n^{(a,b)}(y)$ are the orthogonal Jacobi polynomials. Finally, the upper-spinor component for arbitrary κ can be found through the relation (7b)

$$F_{n\kappa}(s) = N_{n\kappa}s^{\sqrt{C}}(1-s)^{\kappa+1}P_n^{(2\sqrt{C},2\kappa+1)}(1-2s), \tag{29}$$

or

$$F_{n\kappa}(r) = N_{n\kappa}e^{-\delta\sqrt{C}r}\left(1-e^{-\delta r}\right)^{\kappa+1}P_n^{(2\sqrt{C},2\kappa+1)}\left(1-2e^{-\delta r}\right), \tag{30}$$

where $N_{n\kappa}$ is the normalization constant. Further, the lower-spinor component of the wave function can be calculated by using

$$G_{n\kappa}(r) = \frac{1}{M+E_{n\kappa}-C_s}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}\right)F_{n\kappa}(r), \tag{31}$$

where $E \neq -M + C_s$ and only positive energy states do exist.

3.2 *Bound states in PSS limit*

The pseudospin symmetry can be achieved when the relationship between the vector and scalar potential is given by $V(r) = -S(r)$ [10,11,50–53]. Further, Meng *et al* showed that if

$$\frac{\mathrm{d}[V(r)+S(r)]}{\mathrm{d}r} = \frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r} = 0,$$

then $\Sigma(r) = C_{ps} =$ constant, the p-spin symmetry is exact in the Dirac equation. Thus, choosing $\Delta(r)$ as Hellmann potential, eq. (17) under this symmetry becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}-\tilde{\gamma}\left(-\frac{a}{r}+b\frac{\mathrm{e}^{-\delta r}}{r}\right)-\tilde{\beta}^2\right]G_{n\kappa}(r) = 0, \tag{32a}$$

$$\tilde{\gamma} = E_{n\kappa}-M-C_{ps} \quad \text{and} \quad \tilde{\beta}^2 = (M+E_{n\kappa})\left(M-E_{n\kappa}+C_{ps}\right), \tag{32b}$$

where $\kappa = -\tilde{l}$ and $\kappa = \tilde{l}+1$ for $\kappa < 0$ and $\kappa > 0$, respectively. Employing the new approximation, the p-spin Dirac equation (27a) can be written as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\kappa(\kappa-1)\frac{\delta^2}{(1-\mathrm{e}^{-\delta r})^2}\right.$$
$$\left.-\tilde{\gamma}\left(-\frac{\delta a}{1-\mathrm{e}^{-\delta r}}+\frac{\delta b\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}}\right)-\tilde{\beta}^2\right]G_{n\kappa}(r) = 0. \tag{33}$$

To avoid repetition, the solution of eq. (28) can be easily obtained from the spin-symmetric solution by means of the following parametric mappings:

$$F_{n\kappa}(r) \leftrightarrow G_{n\kappa}(r), \quad \kappa \to \kappa - 1, \quad V(r) \to -V(r), \tag{34}$$

i.e.,

$$V_1 \to -V_1, \quad V_2 \to -V_2,$$

$$E_{n\kappa} \to -E_{n\kappa}, \quad C_s \to -C_{ps}.$$

Following the previous procedure, one can obtain the p-spin symmetric energy equation as

$$\kappa(\kappa-1) + (n+\kappa-1)^2 + \left(2n+\kappa+\frac{1}{2}\right) \times \sqrt{\kappa(\kappa-1) - \frac{\tilde{\gamma}a}{\delta} + \frac{\tilde{\beta}^2}{\delta^2}} - \frac{\tilde{\gamma}}{\delta}(b-a) = 0. \tag{35}$$

And also, by using transformation (34), the corresponding lower wave function can easily be obtained from eq. (30).

In table 3, we give the numerical results for the p-spin symmetric case. In this case, we take the set of parameter values, $M = 5\,\mathrm{fm}^{-1}$, $a = -1$, $b = 4$, $c_{ps} = -5.5\,\mathrm{fm}^{-1}$ and $\delta = 0.01$ [32]. We observe the degeneracy in the following doublets: $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$, and so on. Thus, each pair is considered as p-spin doublet and has negative energy. Also, in the presence of the p-spin symmetry, only negative energy states exist.

3.3 *The non-relativistic limiting case*

In this section, we study the energy eigenvalue equation (25) and upper-spinor component of the wave function (30) of the Dirac–Hellmann problem under the non-relativistic limits: $C_s = 0$, $\kappa \to l$, $E_{n\kappa} - M \simeq E_{nl}$ and $M + E_{n\kappa} \simeq 2m$. Thus, we obtain the energy equation

Table 3. Bound-state energy eigenvalues in unit of fm^{-1} of the p-spin symmetry Hellmann potential for several values of n and κ.

$\tilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}$	$n-1, \kappa > 0$	$(l+2, j+1)$	$E_{n-1,\kappa>0}$
1	1, −1	$1s_{1/2}$	−3.167838743	0, 2	$0d_{3/2}$	−3.167838743
2	1, −2	$1p_{3/2}$	−3.753448611	0, 3	$0f_{5/2}$	−3.753448611
3	1, −3	$1d_{5/2}$	−4.121562668	0, 4	$0g_{7/2}$	−4.121562668
4	1, −4	$1f_{7/2}$	−4.358895657	0, 5	$0h_{9/2}$	−4.358895657
1	2, −1	$2s_{1/2}$	−3.748920980	1, 2	$1d_{3/2}$	−3.748920980
2	2, −2	$2p_{3/2}$	−4.116720149	1, 3	$1f_{5/2}$	−4.116720149
3	2, −3	$2d_{5/2}$	−4.354112751	1, 4	$1g_{7/2}$	−4.354112751
4	2, −4	$2f_{7/2}$	−4.513208345	1, 5	$1h_{9/2}$	−4.513208345

of the Schrödinger equation with any arbitrary orbital state for the Hellmann potential as

$$E_{nl} = -\frac{\delta^2}{2m}\left[\left(\frac{\frac{2m}{\delta}(a-b)-(n+l+1)^2-l(l+1)}{2(n+l+1)}\right)^2 - l(l+1)+\frac{2ma}{\delta}\right]. \tag{36}$$

And also the radial functions can be obtained as

$$R_{nl}(r) = N_{nl}\mathrm{e}^{-\sqrt{(\varepsilon-A)+l(l+1)\delta^2}r}\left(1-\mathrm{e}^{-\delta r}\right)^{l+1} \times P_n^{\left(2\sqrt{\frac{1}{\delta^2}(\varepsilon-A)+l(l+1)},2l+1\right)}\left(1-2\mathrm{e}^{-\delta r}\right), \tag{37}$$

where N_{nl} is the normalization constant.

Finally, when the screening parameter δ approaches zero and also $b=0$, the potential (1) reduces to a Coulomb potential. Thus, in this limit, the energy eigenvalues of (36) become the energy levels of the pure Coulomb interaction, i.e.

$$E_{n,l_{\mathrm{Coulomb}}} = -\frac{1}{2}m\frac{a^2}{n'^2}, \tag{38}$$

where $n' = n+l+1$ [46].

4. Concluding remarks

To sum up, in this work, we have explored the bound-state solutions of the Dirac equation with Hellmann potential for any spin-orbit quantum number κ. By making an appropriate approximation to deal with the spin-orbit centrifugal (pseudocentrifugal) coupling term, we have obtained the approximate energy eigenvalue equation and the unnormalized two components of the radial wave function expressed in terms of the Jacobi polynomials using the NU method. We obtained the non-relativistic case of the problem and also found the energy levels of the familiar pure Coulomb potential in the low screening regime when the screening parameter δ goes to zero.

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