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Multiscale expansions in discrete world

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Abstract. In this paper, we show the attainability of KdV equation from some types of nonlinear Schrödinger equation by using multiscale expansions discretely. The power of this manageable method is confirmed by applying it to two selected nonlinear Schrödinger evolution equations. This approach can also be applied to other nonlinear discrete evolution equations. All the computations have been made with Maple computer packet program.

Keywords. Multiscale expansion; discrete evolution equation; modified nonlinear Schrödinger equation; third-order nonlinear Schrödinger equation; KdV equation.

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1. Introduction

In recent years there has been growing interest in the study of discrete nonlinear evolution equations and differential-difference equations. Also, the integrability of discrete nonlinear evolution equations is studied. The study of nonlinear differential-difference equations (NDDEs) has attracted wide interest since the first work of Fermi *et al* [1]. NDDEs, treated as models of some physical phenomena, have become the focus of common interest and play an important role in various branches of applied sciences such as biophysics, atomic and molecular physics and mechanical engineering [2–4]. In 1986, Zakharov and Kuznetsov showed how to apply multiscale expansions method to partial differential equations [5]. After that, multiscale expansions have been studied in [6,7].

Leon and Manna [8] proposed a set of tools for performing multiscale expansions on a discrete nonlinear evolution equation. These tools rely on the definition of a large grid scale by comparing the magnitude of the related difference operators, and on the expansion of the wavenumber in powers of frequency variations due to nonlinearity. Levi and Heredero also studied the discrete multiscale expansions [9]. The paper by Agrotis *et al* [10] gave us the motivation to do the present work.

In what follows we shall show the derivation of KdV equation from some types of nonlinear Schrödinger equation by using multiscale expansions. In §2, we shall give a general form of DNLS–DKdV directed multiscale expansions. Starting from modifed and third-order nonlinear Schrödinger equations, KdV equation is derived in §3 and 4, respectively. Finally, some conclusions are provided.

2. Multiscale expansions from NLS-type to KdV-type equations

The pioneering work of Agrotis *et al* [10] introduced multiscale expansions from NLS-type to KdV-type equations in discrete world. For a given discrete nonlinear evolution equation

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = K(\ldots u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}, \ldots), \tag{2.1}$$

slow variables are

$$\xi_i = \xi_i(t, \epsilon), \tag{2.2}$$

$$\tau_i = \tau_i(t, \epsilon), \tag{2.3}$$

where ϵ is a scaling parameter. Then we look for a solution in the form

$$u_n = \sum_{i=1}^{\infty} \epsilon^i u_{i,n}(\xi_0, \xi_1, \ldots, \xi_n, \tau_0, \tau_1, \ldots, \tau_n). \tag{2.4}$$

The expansion (2.4) with (2.2) and (2.3) is substituted into the discrete nonlinear evolution eq. (2.1) and then collecting the expression wih respect to the powers of the variable ϵ, we get some equations. Solving these equations simultaneously, we obtain the solution of eq. (2.1) and KdV-type equation for $u_{i,n}$. Therefore, the relationships between continuum and discrete forms of NLS and KdV equations are established.

3. From modified nonlinear Schrödinger equation to KdV equation

Let us first consider the modified Schrödinger equation [11], which is written as

$$iu_t = -u_{xx} - i(|u|^2 u)_x - 2\beta |u|^2 u \tag{3.1}$$

or

$$iu_t = -u_{xx} - iu_x |u|^2 - 2iu |u| (|u|)_x - 2\beta |u|^2 u. \tag{3.2}$$

The propagation of a temporal optical soliton in the presence of the self-steepening term can be described by the modified nonlinear Schrödinger (MNLS) equation. Note that where $u(x, t)$ represents a normalized complex amplitude of the pulse envelope, x is a normalized distance along the fibre and t is the normalized time within the frame of the reference moving along the fibre at the group velocity. Taking

$$u_x = \frac{u_{n+1} - u_{n-1}}{2h} \tag{3.3}$$

$$u_{xx} = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} \tag{3.4}$$

as the process of finite difference method, eq. (3.2) can be written in the form

$$i\frac{\mathrm{d}u_n}{\mathrm{d}t} = \frac{2u_n - u_{n+1} - u_{n-1}}{h^2} - i\frac{u_{n+1} - u_{n-1}}{2h}|u_n|^2 - 2iu_n|u_n|\frac{|u_{n+1}| - |u_{n-1}|}{2h} - 2\beta u_n|u_n|^2. \tag{3.5}$$

From now on, we shall rescale the lattice spacing h to 1. As the MNLS equation is a type of complex function, we use the identity

$$\mathrm{e}^{i\phi_{n\pm1}} \approx \mathrm{e}^{i\phi_n}(1 + i(\phi_{n\pm1} - \phi_n)) \tag{3.6}$$

and take

$$u_n = \rho_n^{1/2}\mathrm{e}^{i\phi_n}, \tag{3.7}$$

$$u_{n-1} = \rho_{n-1}^{1/2}\mathrm{e}^{i\phi_{n-1}}, \tag{3.8}$$

$$u_{n+1} = \rho_{n+1}^{1/2}\mathrm{e}^{i\phi_{n+1}}, \tag{3.9}$$

then eq. (3.5) is transformed to

$$\begin{aligned} & i\frac{\phi_{n+1}(\rho_{n+1})^{1/2}}{h^2} - \frac{\rho_n(\rho_{n-1})^{1/2}\phi_n}{2h} - i\frac{\phi_n(\rho_{n+1})^{1/2}}{h^2} + \frac{(\rho_{n-1})^{1/2}}{h^2} \\ & - i\frac{3\rho_n(\rho_{n-1})^{1/2}}{2h} + \frac{\rho_n(\rho_{n-1})^{1/2}\phi_{n-1}}{2h} + i\frac{\phi_{n-1}(\rho_{n-1})^{1/2}}{h^2} - \frac{2(\rho_n)^{1/2}}{h^2} \\ & - \dot{\phi}_n(\rho_n)^{1/2} + \frac{i}{2}\dot{\rho}_n(\rho_n)^{-1/2} + \frac{(\rho_{n+1})^{1/2}}{h^2} - \frac{\rho_n(\rho_{n+1})^{1/2}\phi_{n+1}}{2h} \\ & + \frac{\rho_n(\rho_{n+1})^{1/2}\phi_n}{2h} - i\frac{\phi_n(\rho_{n-1})^{1/2}}{h^2} + 2\beta(\rho_n)^{3/2} + i\frac{3\rho_n(\rho_{n+1})^{1/2}}{2h} = 0. \end{aligned} \tag{3.10}$$

This last equation is actually the combination of

$$\begin{aligned} & \frac{\phi_{n+1}(\rho_{n+1})^{1/2}}{h^2} - \frac{\phi_n(\rho_{n+1})^{1/2}}{h^2} - \frac{3\rho_n(\rho_{n-1})^{1/2}}{2h} + \frac{\phi_{n-1}(\rho_{n-1})^{1/2}}{h^2} \\ & + \frac{1}{2}\dot{\rho}_n(\rho_n)^{-1/2} - \frac{\phi_n(\rho_{n-1})^{1/2}}{h^2} + \frac{3\rho_n(\rho_{n+1})^{1/2}}{2h} = 0 \end{aligned} \tag{3.11}$$

and

$$\frac{\rho_n(\rho_{n-1})^{1/2}\phi_{n-1}}{2h}-\frac{\rho_n(\rho_{n-1})^{1/2}\phi_n}{2h}+\frac{(\rho_{n-1})^{1/2}}{h^2}-\frac{2(\rho_n)^{1/2}}{h^2}-\dot{\phi}_n(\rho_n)^{1/2}$$
$$+\frac{(\rho_{n+1})^{1/2}}{h^2}-\frac{\rho_n(\rho_{n+1})^{1/2}\phi_{n+1}}{2h}+\frac{\rho_n(\rho_{n+1})^{1/2}\phi_n}{2h}+2\beta(\rho_n)^{3/2}=0. \quad (3.12)$$

Subsequently, we use multiscale expansions with the series given below:

$$\rho_n = 1+\epsilon\rho_{1,n}+\epsilon^2\rho_{2,n}+\epsilon^3\rho_{3,n}+\cdots, \quad (3.13)$$

$$\phi_n = -t+\epsilon^{1/2}\phi_{1,n}+\epsilon^{3/2}\phi_{2,n}+\epsilon^{5/2}\phi_{3,n}+\cdots, \quad (3.14)$$

$$t'=\epsilon^{3/2}t, \qquad x'=-c\epsilon^{1/2}t, \qquad h'=\epsilon^{1/2}h, \quad (3.15)$$

in eqs (3.11) and (3.12). It is obvious that eqs (3.11) and (3.12) give

$$\cdots+\Big[\phi_{2,n+1}-2\phi_{2,n}+\frac{1}{2}\rho_{1,n+1}\left(\phi_{1,n+1}-\phi_{1,n}\right)+\frac{1}{2}\left(\rho_{1,n}\right)_{t'}$$
$$-\frac{c}{4}\left(\rho_{2,n+1}-\rho_{2,n-1}\right)+\frac{1}{8}c\rho_{1,n}\left(\rho_{1,n+1}-\rho_{1,n-1}\right)+\frac{3}{4}\rho_{2,n+1}$$
$$+\frac{1}{2}\rho_{1,n-1}\left(\phi_{1,n-1}-\phi_{1,n}\right)-\frac{3}{16}\left(\rho_{1,n+1}\right)^2+\frac{3}{16}\left(\rho_{1,n-1}\right)^2$$
$$-\frac{3}{4}\rho_{2,n-1}+\frac{3}{4}\rho_{1,n}\left(\rho_{1,n+1}-\rho_{1,n-1}\right)+\phi_{2,n-1}\Big]\epsilon^{5/2}+[\phi_{1,n+1}$$
$$+\phi_{1,n-1}+\frac{3}{4}\rho_{1,n+1}-\frac{3}{4}\rho_{1,n-1}-\frac{1}{4}c\left(\rho_{1,n+1}-\rho_{1,n-1}\right)$$
$$-2\phi_{1,n}]\epsilon^{3/2}+\cdots=0 \quad (3.16)$$

and

$$\cdots+\Big[-\left(\phi_{1,n}\right)_{t'}+\frac{c}{2}\left(\phi_{2,n+1}-\phi_{2,n-1}\right)+\frac{c}{4}\rho_{1,n}\left(\phi_{1,n+1}-\phi_{1,n-1}\right)$$
$$+\frac{1}{2}\rho_{2,n}-\frac{1}{8}\left(\rho_{1,n}\right)^2-\frac{1}{2}\left(\frac{1}{2}\rho_{1,n-1}+\rho_{1,n}\right)\left(\phi_{1,n}-\phi_{1,n-1}\right)$$
$$-\frac{1}{2}\left(\frac{1}{2}\rho_{1,n+1}+\rho_{1,n}\right)\left(\phi_{1,n+1}-\phi_{1,n}\right)+2\beta\left(\frac{3}{2}\rho_{2,n}+\frac{3}{8}\left(\rho_{1,n}\right)^2\right)$$
$$-\rho_{1,n}-\frac{1}{2}\phi_{2,n+1}+\frac{1}{2}\rho_{1,n-1}+\frac{1}{2}\phi_{2,n-1}+\frac{1}{2}\rho_{1,n+1}\Big]\epsilon^2+\Big[\frac{1}{2}\rho_{1,n}$$
$$+\frac{c}{2}\left(\phi_{1,n+1}-\phi_{1,n-1}\right)+\frac{1}{2}\phi_{1,n-1}-\frac{1}{2}\phi_{1,n+1}+3\beta\rho_{1,n}\Big]\epsilon=0, \quad (3.17)$$

respectively. With the necessary computations, the equation

$$-(3+2\sqrt{3})(\phi_1)_{t'x'} + \frac{1+\sqrt{3}}{2}(\phi_1)_{x'x'x'x'} + \left(11+11\sqrt{3}\right)(\phi_1)_{x'}(\phi_1)_{x'x'} = 0 \tag{3.18}$$

which is a type of KdV equation, is obtained. Integrating eq. (3.18) with respect to x' once and setting the integration constant to zero we find the equation

$$(\phi_1)_{t'} = \frac{1+\sqrt{3}}{6+4\sqrt{3}}(\phi_1)_{x'x'x'} + \frac{11+11\sqrt{3}}{6+4\sqrt{3}}\left((\phi_1)_{x'}\right)^2. \tag{3.19}$$

4. From third-order nonlinear Schrödinger to KdV equation

As it is well known, the continuum form of the third-order nonlinear Schrödinger equation is

$$iu_t = u_{xx} - 2|u|^2 u + i\beta u_{xxx}. \tag{4.1}$$

Equation (4.1) appears in a wide variety of physical systems. For instance, propagation of picosecond optical pulses near the zero second-order dispersion point in an optical fibre is governed by this equation [12]. Femtosecond pulses in a fibre laser cavity are modelled by this equation as well [13]. In ref. [14], eq. (4.1) with term $|u|^2u$ appears in water waves near a caustic. With the aid of the identity

$$u_{xx} = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}, \tag{4.2}$$

$$u_{xxx} = \frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h^3}, \tag{4.3}$$

discretization of eq. (4.1) implies that

$$i\dot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} - 2|u_n|^2 u_n + i\beta\frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h^3}. \tag{4.4}$$

Taking into account (3.6) and

$$\begin{aligned} u_n &= \rho_n^{1/2} e^{i\phi_n}, \\ u_{n+1} &= \rho_{n+1}^{1/2} e^{i\phi_{n+1}}, \\ u_{n-1} &= \rho_{n-1}^{1/2} e^{i\phi_{n-1}}, \\ u_{n+2} &= \rho_{n+2}^{1/2} e^{i\phi_{n+2}}, \\ u_{n-2} &= \rho_{n-2}^{1/2} e^{i\phi_{n-2}}, \end{aligned} \tag{4.5}$$

eq. (4.4) can be written as

$$
\begin{aligned}
& i\,\frac{\beta\phi_n\phi_{n+1}\,(\rho_{n+2})^{1/2}}{h^3} - \frac{\beta\phi_n\,(\rho_{n+2})^{1/2}}{h^3} + i\,\frac{\beta\phi_n\phi_{n-2}\,(\rho_{n-2})^{1/2}}{h^3} \\
& - i\,\frac{\beta\phi_n\phi_{n+2}\,(\rho_{n+2})^{1/2}}{h^3} + \frac{\beta\phi_{n+2}\,(\rho_{n+2})^{1/2}}{h^3} + i\,\frac{\beta\phi_{n+2}\phi_{n+1}\,(\rho_{n+2})^{1/2}}{h^3} \\
& - \frac{2\beta\phi_{n+1}\,(\rho_{n+1})^{1/2}}{h^3} + i\,\frac{\beta\,(\rho_{n-2})^{1/2}}{h^3} - i\,\frac{2\beta\,(\rho_{n-1})^{1/2}}{h^3} - \frac{2\beta\phi_n\,(\rho_{n-1})^{1/2}}{h^3} \\
& - i\,\frac{\beta\phi_n\phi_{n-1}\,(\rho_{n-2})^{1/2}}{h^3} + \frac{\beta\phi_n\,(\rho_{n-2})^{1/2}}{h^3} - i\,\frac{\beta\phi_{n-2}\phi_{n-1}\,(\rho_{n-2})^{1/2}}{h^3} \\
& - i\,\frac{\beta\,(\rho_{n+2})^{1/2}}{h^3} - \frac{\beta\phi_{n-2}\,(\rho_{n-2})^{1/2}}{h^3} + \frac{2\beta\phi_{n-1}\,(\rho_{n-1})^{1/2}}{h^3} - i\,\frac{2\phi_{n+1}\,(\rho_{n+1})^{1/2}}{h^2} \\
& + \frac{2\beta\phi_n\,(\rho_{n+1})^{1/2}}{h^3} + i\,\frac{\beta\,(\rho_{n-2})^{1/2}\,(\phi_{n-1})^2}{h^3} + 4\,(\rho_n)^{3/2} + i\,\frac{2\beta\,(\rho_{n+1})^{1/2}}{h^3} \\
& + i\,\frac{2\phi_n\,(\rho_{n-1})^{1/2}}{h^2} - i\,\frac{2\phi_{n-1}\,(\rho_{n-1})^{1/2}}{h^2} - 2\dot{\phi}_n\,(\rho_n)^{1/2} + \frac{4\,(\rho_n)^{1/2}}{h^2} \\
& - \frac{2\,(\rho_{n-1})^{1/2}}{h^2} - \frac{2\,(\rho_{n+1})^{1/2}}{h^2} - i\,\frac{\beta\,(\rho_{n+2})^{1/2}\,(\phi_{n+1})^2}{h^3} + i\,\frac{2\phi_n\,(\rho_{n+1})^{1/2}}{h^2} \\
& + i\,\dot{\rho}_n\,(\rho_n)^{-1/2} = 0. \qquad (4.6)
\end{aligned}
$$

Imaginary and real parts of eq. (4.6) are

$$
\begin{aligned}
& \frac{\beta\phi_n\phi_{n+1}\,(\rho_{n+2})^{1/2}}{h^3} + \frac{\beta\phi_n\phi_{n-2}\,(\rho_{n-2})^{1/2}}{h^3} + \frac{\beta\phi_{n+2}\phi_{n+1}\,(\rho_{n+2})^{1/2}}{h^3} \\
& - \frac{\beta\phi_n\phi_{n+2}\,(\rho_{n+2})^{1/2}}{h^3} - \frac{\beta\phi_{n-2}\phi_{n-1}\,(\rho_{n-2})^{1/2}}{h^3} - \frac{\beta\phi_n\phi_{n-1}\,(\rho_{n-2})^{1/2}}{h^3} \\
& - \frac{\beta\,(\rho_{n+2})^{1/2}}{h^3} + \frac{2\beta\,(\rho_{n+1})^{1/2}}{h^3} + \frac{\beta\,(\rho_{n-2})^{1/2}\,(\phi_{n-1})^2}{h^3} + \frac{\beta\,(\rho_{n-2})^{1/2}}{h^3} \\
& + \frac{2\phi_n\,(\rho_{n-1})^{1/2}}{h^2} - \frac{2\phi_{n+1}\,(\rho_{n+1})^{1/2}}{h^2} - \frac{2\phi_{n-1}\,(\rho_{n-1})^{1/2}}{h^2} - \frac{2\beta\,(\rho_{n-1})^{1/2}}{h^3} \\
& - \frac{\beta\,(\rho_{n+2})^{1/2}\,(\phi_{n+1})^2}{h^3} + \frac{2\phi_n\,(\rho_{n+1})^{1/2}}{h^2} + \dot{\rho}_n\,(\rho_n)^{-1/2} = 0 \qquad (4.7)
\end{aligned}
$$

and

$$
\begin{aligned}
& - \frac{\beta\phi_n\,(\rho_{n+2})^{1/2}}{h^3} + \frac{\beta\phi_{n+2}\,(\rho_{n+2})^{1/2}}{h^3} - \frac{2\beta\phi_{n+1}\,(\rho_{n+1})^{1/2}}{h^3} - \frac{2\beta\phi_n\,(\rho_{n-1})^{1/2}}{h^3} \\
& + \frac{\beta\phi_n\,(\rho_{n-2})^{1/2}}{h^3} - \frac{\beta\phi_{n-2}\,(\rho_{n-2})^{1/2}}{h^3} + \frac{2\beta\phi_{n-1}\,(\rho_{n-1})^{1/2}}{h^3} + 4\,(\rho_n)^{3/2} \\
& 2\frac{\beta\phi_n\,(\rho_{n+1})^{1/2}}{h^3} + \frac{4\,(\rho_n)^{1/2}}{h^2} - \frac{2\,(\rho_{n-1})^{1/2}}{h^2} - \frac{2\,(\rho_{n+1})^{1/2}}{h^2} \\
& - 2\dot{\phi}_n\,(\rho_n)^{1/2} = 0, \qquad (4.8)
\end{aligned}
$$

respectively.

Using expansions (3.13)–(3.15) in eqs (4.7) and (4.8), we derive that

$$\begin{aligned}
&\cdots+\Big[(\beta\left(\phi_{1,n}-\phi_{1,n-1}\right)\left(\phi_{1,n-2}-\phi_{1,n-1}\right)+\beta\left(\phi_{1,n+1}-\phi_{1,n}\right)\left(\phi_{1,n+2}-\phi_{1,n+1}\right)\\
&-\beta\rho_{1,n-1}+\frac{1}{2}\beta\rho_{1,n-2}+\beta\rho_{1,n+1}+\left(\rho_{1,n}\right)_{t'}+\frac{1}{4}c\rho_{1,n}\left(\rho_{1,n+1}-\rho_{1,n-1}\right)+4\phi_{2,n}\\
&-\frac{1}{2}\beta\rho_{1,n+2}+\rho_{1,n+1}\left(\phi_{1,n}-\phi_{1,n+1}\right)+\rho_{1,n-1}\left(\phi_{1,n}-\phi_{1,n-1}\right)-2\phi_{2,n-1}-2\phi_{2,n+1}\\
&-\frac{c}{2}\left(\rho_{2,n+1}-\rho_{2,n-1}\right)\Big]\epsilon^{5/2}+\Big[4\phi_{1,n}-2\phi_{1,n+1}-2\phi_{1,n-1}\\
&-\frac{1}{2}c\left(\rho_{1,n+1}-\rho_{1,n-1}\right)\Big]\epsilon^{3/2}=0\\
&\cdots+\Big[-\rho_{1,n+1}-2\left(\phi_{1,n}\right)_{t'}+\frac{1}{2}c\rho_{1,n}\left(\phi_{1,n+1}-\phi_{1,n-1}\right)+\frac{3}{2}\left(\rho_{1,n}\right)^2-\rho_{1,n-1}\\
&+2\beta\left(\phi_{1,n-1}-\phi_{1,n}\right)+\beta\left(\phi_{1,n}-\phi_{1,n-2}\right)+\beta\left(\phi_{1,n+2}-\phi_{1,n}\right)+2\beta\left(\phi_{1,n}-\phi_{1,n+1}\right)\\
&+2\rho_{1,n}+6\rho_{2,n}+c\left(\phi_{2,n+1}-\phi_{2,n-1}\right)\Big]\epsilon^2+[6\rho_{1,n}+c\left(\phi_{1,n+1}-\phi_{1,n-1}\right)]\epsilon=0.
\end{aligned} \tag{4.9}$$

After the required operations, we get

$$-4(\phi_1)_{t'x'}+\left(4\beta+\frac{\sqrt{2}}{\sqrt{3}}\right)(\phi_1)_{x'x'x'x'}+4(\phi_1)_{x'}(\phi_1)_{x'x'}=0, \tag{4.10}$$

which is a kind of KdV equation. Integrating (4.10) with respect to x' once and setting the integration constant to zero we find that equation

$$(\phi_1)_{t'}=\left(\beta+\frac{1}{2\sqrt{6}}\right)(\phi_1)_{x'x'x'}+\frac{1}{2}\left((\phi_1)_{x'}\right)^2. \tag{4.11}$$

5. Conclusion

In this paper, multiple scaling method is employed to derive KdV equation from some types of nonlinear Schrödinger equations in discrete world. Thus, we conclude that the proposed method can be extended to get another discrete evolution equation from any discrete evolution equation, which represents nonlinear problems in the theory of solitons and other areas. Besides, KdV to Schrödinger expansions can be naturally studied. We foresee that our results can be found potentially useful for applications in mathematical physics and engineering problems, including series expansions.

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