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Lie and Noether symmetries of systems of complex ordinary differential equations and their split systems

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Abstract. The Lie and Noether point symmetry analyses of a kth-order system of m complex ordinary differential equations (ODEs) with m dependent variables are performed. The decomposition of complex symmetries of the given system of complex ODEs yields Lie- and Noether-like operators. The system of complex ODEs can be split into $2m$ coupled real partial differential equations (PDEs) and $2m$ Cauchy–Riemann (CR) equations. The classical approach is invoked to compute the symmetries of the $4m$ real PDEs and these are compared with the decomposed Lie- and Noether-like operators of the system of complex ODEs. It is shown that, in general, the Lie- and Noether-like operators of the system of complex ODEs and the symmetries of the decomposed system of real PDEs are not the same. A similar analysis is carried out for restricted systems of complex ODEs that split into $2m$ coupled real ODEs. We summarize our findings on restricted complex ODEs in two propositions.

Keywords. Lie-like operators; Noether-like operators; complex ordinary differential equations; complex Lagrangian.

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1. Introduction

The theory of transformation groups was discovered by Lie [1,2]. The Lie symmetries of differential equations in real domain are discussed widely in the literature (see e.g. [3–6]). Mahomed and Naz [7] performed symmetry analysis for complex PDEs and their decomposed systems. It was proved that, in general, the decomposed Lie-like generators of complex PDEs and symmetries of the decomposed system of real PDEs are different. Ali *et al* [8] studied the Lie symmetries of scalar second-order ODEs in the complex domain and constructed an extended Lie table for restricted complex ODEs with two symmetries.

Subsequently, there has been further work on this topic, as well as on linearization (see [9]). The clarification that arises is that one has Lie- and Noether-like operators which are consequences of the split complex symmetry generators.

The purpose of this paper is to address two issues: (1) to further study systems of complex ODEs with respect to Lie and Noether symmetries and (2) to show that generally the decomposed Lie- and Noether-like operators of systems of complex ODEs and symmetries of the decomposed system are different. A similar result relating to Lie symmetries was obtained for complex PDEs [7].

Complex analyticity is required for symmetry analysis in the complex domain and thus, the constraints of the Cauchy–Riemann (CR) equations hold [8]. The Lie- and Noether-like operators are determined by the decomposition of complex symmetries for the system of complex ODEs into real and imaginary parts. The decomposition of a kth-order system of m complex ODEs yields a coupled system of $4m$ real PDEs. The classical approach is utilized to compute the symmetries of this decomposed system. This analysis is in the real domain. The comparison of these symmetries with the Lie- and Noether-like operators reveal that the Lie- and Noether-like operators and symmetries of the decomposed system are generally different. This result holds true not only for systems of complex ODEs but also for kth-order complex ODEs. This is illustrated by several examples.

If the independent variable is real, then the system of complex ODEs is called a restricted system of complex ODEs. This becomes a system of real ODEs after a complex split. A system of m restricted complex ODEs decomposes into a coupled system of $2m$ real ODEs. For this case, the Lie- and Noether-like operators after decomposition of the complex symmetries of the complex ODEs are not in general, the same as the symmetries of the decomposed system of ODEs. This is encapsulated in two propositions.

This paper is organized as follows: The mathematical concepts of complex Lie and Noether symmetries for a system of complex ODEs are given in §2 and 3. Examples of symmetries for systems of complex ODEs as well as for higher-order complex ODEs are provided. Concluding remarks are presented in §4.

2. Lie symmetries for systems of complex ODEs

Consider a kth-order system of ODEs in the complex domain, viz.,

$$E_\alpha\left(z, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\right) = 0\,, \qquad \alpha = 1, \ldots, m, \tag{1}$$

where $z = x + iy$ and $u = (u^1, \ldots, u^m)$ is a complex valued function of z with $u^\alpha = f^\alpha + ig^\alpha$. The kth derivative of u with respect to z is denoted by $u_{(k)}$.

The first, second, third and fourth derivatives are defined as follows:

$$u_z^\alpha = \frac{1}{2}\left[\left(f_x^\alpha + g_y^\alpha\right) + i\left(g_x^\alpha - f_y^\alpha\right)\right] = \frac{1}{2}[h^\alpha + il^\alpha], \tag{2}$$

$$u_{zz}^\alpha = \frac{1}{4}\left[f_{xx}^\alpha - f_{yy}^\alpha + 2g_{xy}^\alpha\right] + \frac{i}{4}\left[g_{xx}^\alpha - g_{yy}^\alpha - 2f_{xy}^\alpha\right], \tag{3}$$

$$u^\alpha_{zzz} = \frac{1}{8}\left[f^\alpha_{xxx} - g^\alpha_{yyy} + 3g^\alpha_{xxy} - 3f^\alpha_{xyy}\right]$$
$$+\frac{i}{8}\left[g^\alpha_{xxx} + f^\alpha_{yyy} - 3f^\alpha_{xxy} - 3g^\alpha_{xyy}\right], \tag{4}$$

$$u^\alpha_{zzzz} = \frac{1}{16}(f_{xxxx} + f_{yyyy} - 4g_{xyyy} + 4g_{xxxy} - 6f_{xxyy})$$
$$+\frac{i}{16}(g_{xxxx} + g_{yyyy} + 4f_{xyyy} - 4f_{xxxy} - 6g_{xxyy}). \tag{5}$$

One can similarly define the higher-order derivatives. System (1) of m ODEs can be decomposed into a system of $4m$ PDEs in the real domain. Consider for example, the second-order system of ODEs of the form

$$u^\alpha_{(2)} = w^\alpha(z, u, u_{(1)}), \tag{6}$$

where $u^\alpha(z) = f^\alpha + ig^\alpha$. System (6) decomposes into the following system of PDEs:

$$f^\alpha_{xx} - f^\alpha_{yy} + 2g^\alpha_{xy} = 4G^\alpha(x, y, f^\alpha, g^\alpha, h^\alpha, l^\alpha),$$
$$g^\alpha_{xx} - g^\alpha_{yy} - 2f^\alpha_{xy} = 4H^\alpha(x, y, f^\alpha, g^\alpha, h^\alpha, l^\alpha),$$
$$f^\alpha_x = g^\alpha_y, \quad f^\alpha_y = -g^\alpha_x, \tag{7}$$

where $w^\alpha(z, u, u_{(1)}) = G^\alpha + iH^\alpha$. The real Lie symmetries of system (7) can be obtained by the Lie classical approach.

The complex Lie point symmetry generator (see [8]) for the system of complex ODEs (1) is defined as

$$\mathbf{Z} = \xi\frac{\partial}{\partial z} + \eta^\alpha\frac{\partial}{\partial u^\alpha}. \tag{8}$$

The following two real Lie-like operators

$$\mathbf{X} = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \eta^\alpha_1\frac{\partial}{\partial f^\alpha} + \eta^\alpha_2\frac{\partial}{\partial g^\alpha},$$
$$\mathbf{Y} = \xi_2\frac{\partial}{\partial x} - \xi_1\frac{\partial}{\partial y} + \eta^\alpha_2\frac{\partial}{\partial f^\alpha} - \eta^\alpha_1\frac{\partial}{\partial g^\alpha}, \tag{9}$$

are deduced from

$$\mathbf{Z} = \mathbf{X} + i\mathbf{Y}, \qquad \xi = \xi_1 + i\xi_2, \qquad \eta^\alpha = \eta^\alpha_1 + i\eta^\alpha_2. \tag{10}$$

The Lie symmetries of the decomposed system of real PDEs (7) and the Lie-like operators (9) are generally different. This is illustrated by the following example.

Example 1. We consider the following system [10]:

$$q''_1 = -q_1q'_3,$$
$$q''_2 = -q_2q'_3,$$
$$q''_3 = q_1q'_1 + q_2q'_2, \tag{11}$$

in the complex domain, where $q_1(z) = f(x, y) + ig(x, y)$, $q_2 = p(x, y) + iq(x, y)$, $q_3(z) = P(x, y) + iQ(x, y)$ are functions of the complex variable $z = x + iy$. The Lie symmetries of the system (11) are

$$\mathbf{Z}_1 = \frac{\partial}{\partial z}, \qquad \mathbf{Z}_2 = q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2}, \qquad \mathbf{Z}_3 = \frac{\partial}{\partial q_3},$$
$$\mathbf{Z}_4 = z\frac{\partial}{\partial z} - q_1\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial q_2} - q_3\frac{\partial}{\partial q_3}. \tag{12}$$

We set $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$ for the Lie symmetries given in (12) and obtain the following Lie-like operators:

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{Y}_1 = -\frac{\partial}{\partial y}, \qquad \mathbf{X}_2 = p\frac{\partial}{\partial f} + q\frac{\partial}{\partial g} - f\frac{\partial}{\partial p} - g\frac{\partial}{\partial q},$$
$$\mathbf{Y}_2 = q\frac{\partial}{\partial f} - p\frac{\partial}{\partial g} - g\frac{\partial}{\partial p} + f\frac{\partial}{\partial q}, \qquad \mathbf{X}_3 = \frac{\partial}{\partial P}, \qquad \mathbf{Y}_3 = -\frac{\partial}{\partial Q},$$
$$\mathbf{X}_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g} - p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} - P\frac{\partial}{\partial P} - Q\frac{\partial}{\partial Q},$$
$$\mathbf{Y}_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - g\frac{\partial}{\partial f} + f\frac{\partial}{\partial g} - q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q} - Q\frac{\partial}{\partial P} + P\frac{\partial}{\partial Q}. \tag{13}$$

Equation (11), after decomposition, yields the following system of real PDEs:

$$\begin{aligned}
&f_{xx} + 2g_{xy} - f_{yy} + 2f(P_x + Q_y) - 2g(Q_x - P_y) = 0,\\
&g_{xx} - g_{yy} - 2f_{xy} + 2g(P_x + Q_y) + 2f(Q_x - P_y) = 0,\\
&p_{xx} - p_{yy} + 2q_{xy} + 2p(P_x + Q_y) - 2q(Q_x - P_y) = 0,\\
&q_{xx} - q_{yy} - 2p_{xy} + 2q(P_x + Q_y) + 2p(Q_x - P_y) = 0,\\
&P_{xx} - P_{yy} + 2Q_{xy} - 2f(f_x + g_y) + 2g(g_x - f_y)\\
&-2p(p_x + q_y) + 2q(q_x - p_y) = 0,\\
&Q_{xx} - Q_{yy} - 2P_{xy} - 2g(f_x + g_y) - 2f(g_x - f_y)\\
&-2q(p_x + q_y) - 2p(q_x - p_y) = 0,\\
&f_x = g_y, \quad f_y = -g_x, \quad p_x = q_y, \quad p_y = -q_x, \quad P_x = Q_y, \quad P_y = -Q_x.
\end{aligned} \tag{14}$$

The Lie symmetries for the system (14) are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial P}, \quad X_4 = \frac{\partial}{\partial Q}. \tag{15}$$

Only the Lie-like operators $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_3, \mathbf{Y}_3$ are symmetries of the decomposed system (14). Thus, the decomposed Lie-like operators for the system (11) and symmetries of the decomposed system (14) are different.

A few examples of second- and higher-order complex ODEs are now considered.

Example 2. Consider the free particle equation, viz.,

$$u''(z) = 0 \tag{16}$$

in complex domain as discussed by Ali *et al* [8]. The decomposed Lie-like operators as given in [8] are

$$
\begin{aligned}
&\mathbf{X}_1=\frac{\partial}{\partial x}, \quad \mathbf{Y}_1=-\frac{\partial}{\partial y}, \quad \mathbf{X}_2=\frac{\partial}{\partial f}, \quad \mathbf{Y}_2=-\frac{\partial}{\partial g},\\
&\mathbf{X}_3=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}, \quad \mathbf{Y}_3=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y},\\
&\mathbf{X}_4=f\frac{\partial}{\partial f}+g\frac{\partial}{\partial g}, \quad \mathbf{Y}_4=g\frac{\partial}{\partial f}-f\frac{\partial}{\partial g},\\
&\mathbf{X}_5=x\frac{\partial}{\partial f}+y\frac{\partial}{\partial g}, \quad \mathbf{Y}_5=y\frac{\partial}{\partial f}-x\frac{\partial}{\partial g},\\
&\mathbf{X}_6=f\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}, \quad \mathbf{Y}_6=g\frac{\partial}{\partial x}-f\frac{\partial}{\partial y},\\
&\mathbf{X}_7=(xf-yg)\frac{\partial}{\partial x}+(yf+xg)\frac{\partial}{\partial y}+(f^2-g^2)\frac{\partial}{\partial f}+2fg\frac{\partial}{\partial g},\\
&\mathbf{Y}_7=(xg+yf)\frac{\partial}{\partial x}-(xf-yg)\frac{\partial}{\partial y}-(f^2-g^2)\frac{\partial}{\partial g}+2fg\frac{\partial}{\partial f},\\
&\mathbf{X}_8=(xf-yg)\frac{\partial}{\partial f}+(yf+xg)\frac{\partial}{\partial g}+(x^2-y^2)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y},\\
&\mathbf{Y}_8=(yf+xg)\frac{\partial}{\partial f}-(xf-yg)\frac{\partial}{\partial g}-(x^2-y^2)\frac{\partial}{\partial y}+2xy\frac{\partial}{\partial x}.
\end{aligned} \tag{17}
$$

Equation (16), after decomposition, yields

$$
\begin{aligned}
&f_{xx}+2g_{xy}-f_{yy}=0,\\
&g_{xx}-2f_{xy}-g_{yy}=0,\\
&f_x=g_y, \;\; f_y=-g_x.
\end{aligned} \tag{18}
$$

The classical Lie approach yields the following Lie symmetries for the decomposed system (18) given by

$$
\begin{aligned}
&X_1=\frac{\partial}{\partial x}, \quad X_2=\frac{\partial}{\partial y}, \quad X_3=\frac{\partial}{\partial f}, \quad X_4=\frac{\partial}{\partial g},\\
&X_5=x\frac{\partial}{\partial f}+y\frac{\partial}{\partial g}, \quad X_6=y\frac{\partial}{\partial f}-x\frac{\partial}{\partial g},\\
&X_7=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2f\frac{\partial}{\partial f}+2g\frac{\partial}{\partial g},\\
&X_8=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}+2g\frac{\partial}{\partial f}-2f\frac{\partial}{\partial g}.
\end{aligned} \tag{19}
$$

The Lie-like operators $\mathbf{X}_6, \mathbf{Y}_6, \mathbf{X}_7, \mathbf{Y}_7, \mathbf{X}_8, \mathbf{Y}_8$ are not symmetries of the decomposed system (18). We observe that $X_7=\mathbf{X}_3+2\mathbf{X}_4$, $X_8=\mathbf{Y}_3+2\mathbf{Y}_4$ and $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2, \mathbf{Y}_2, \mathbf{X}_5, \mathbf{Y}_5$ are the Lie symmetries of the decomposed system. Hence, we may conclude that not all the decomposed Lie-like operators for the system (16) are symmetries of the decomposed system (18).

Example 3. Consider the non-linear third-order ODE [11],

$$2u'u''' - 3u''^2 = 0, \tag{20}$$

now taken in the complex domain, where $u = f + ig$. The complex Lie symmetries of eq. (20) are [11]

$$\mathbf{Z}_1 = \frac{\partial}{\partial z}, \quad \mathbf{Z}_2 = \frac{\partial}{\partial u}, \quad \mathbf{Z}_3 = z\frac{\partial}{\partial z},$$
$$\mathbf{Z}_4 = z^2\frac{\partial}{\partial z}, \qquad \mathbf{Z}_5 = u\frac{\partial}{\partial u}, \qquad \mathbf{Z}_6 = u^2\frac{\partial}{\partial u}. \tag{21}$$

We set $\mathbf{Z}_j = \mathbf{X}_j + i\mathbf{Y}_j,\ \ j = 1, 2, \ldots, 6$ in (21) in order to obtain the following Lie-like operators

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{Y}_1 = -\frac{\partial}{\partial y}, \quad \mathbf{X}_2 = \frac{\partial}{\partial f}, \quad \mathbf{Y}_2 = -\frac{\partial}{\partial g},$$
$$\mathbf{X}_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad \mathbf{Y}_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$\mathbf{X}_4 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \qquad \mathbf{Y}_4 = 2xy\frac{\partial}{\partial x} - (x^2 - y^2)\frac{\partial}{\partial y},$$
$$\mathbf{X}_5 = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \quad \mathbf{Y}_5 = g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g},$$
$$\mathbf{X}_6 = (f^2 - g^2)\frac{\partial}{\partial f} + 2fg\frac{\partial}{\partial g}, \quad \mathbf{Y}_6 = 2fg\frac{\partial}{\partial f} - (f^2 - g^2)\frac{\partial}{\partial g}. \tag{22}$$

The decomposition of (20) yields

$$2(f_x + g_y)(f_{xxx} - g_{yyy} + 3g_{xxy} - 3f_{xyy})$$
$$- 2(g_x - f_y)(g_{xxx} + f_{yyy} - 3f_{xxy} - 3g_{xyy})$$
$$- 3(f_{xx} - f_{yy} + 2g_{xy})^2 + 3(g_{xx} - g_{yy} - 2f_{xy})^2 = 0,$$
$$2(f_x + g_y)(g_{xxx} + f_{yyy} - 3f_{xxy} - 3g_{xyy})$$
$$+ 2(g_x - f_y)(f_{xxx} - g_{yyy} + 3g_{xxy} - 3f_{xyy})$$
$$- 6(f_{xx} - f_{yy} + 2g_{xy})(g_{xx} - 2f_{xy} - g_{yy}) = 0,$$
$$f_x = g_y, \ \ f_y = -g_x. \tag{23}$$

The Lie symmetry generators for the decomposed system (23) are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial f}, \qquad X_4 = \frac{\partial}{\partial g},$$
$$X_5 = \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g},$$
$$X_6 = \frac{1}{2}y\frac{\partial}{\partial x} - \frac{1}{2}x\frac{\partial}{\partial y} + g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g}. \tag{24}$$

The Lie-like operators $\mathbf{X}_4, \mathbf{Y}_4, \mathbf{X}_6, \mathbf{Y}_6$ are not symmetries of the decomposed system (23). We observe that $X_5 = \frac{1}{2}\mathbf{X}_3 + \mathbf{X}_5$, $X_6 = \frac{1}{2}\mathbf{Y}_3 + \mathbf{Y}_5$ and $\mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2, \mathbf{Y}_2$ are the Lie symmetries of the decomposed system. The conclusion here is that, not all decomposed Lie-like operators are Lie symmetries of the decomposed system.

Example 4. We study the non-linear fourth-order complex ODE

$$uu'''' = 2(1-u')u''' \tag{25}$$

possessing the following complex Lie symmetries:

$$\mathbf{Z}_1 = \frac{\partial}{\partial z}, \quad \mathbf{Z}_2 = z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u},$$
$$\mathbf{Z}_3 = {}^2\frac{\partial}{\partial z} + 2zu\frac{\partial}{\partial u}. \tag{26}$$

The Lie-like operators of (25) are

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \ \mathbf{Y}_1 = -\frac{\partial}{\partial y},$$
$$\mathbf{X}_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \ \mathbf{Y}_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g},$$
$$\mathbf{X}_3 = (x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2(xf-yg)\frac{\partial}{\partial f} + 2(yf+xg)\frac{\partial}{\partial g},$$
$$\mathbf{Y}_3 = 2xy\frac{\partial}{\partial x} - (x^2-y^2)\frac{\partial}{\partial y} + 2(yf+xg)\frac{\partial}{\partial f} - 2(xf-yg)\frac{\partial}{\partial g}. \tag{27}$$

Equation (25) separates into four coupled PDEs in the real domain which are

$$\begin{aligned}
&f(f_{xxxx} + f_{yyyy} - 4g_{xyyy} + 4g_{xxxy} - 6f_{xxyy})\\
&- g(g_{xxxx} + g_{yyyy} + 4f_{xyyy} - 4f_{xxxy} - 6g_{xxyy})\\
&- 2(2 - f_x - g_y)(f_{xxx} - g_{yyy} + 3g_{xxy} - 3f_{xyy})\\
&- 2(g_x - f_y)(g_{xxx} + f_{yyy} - 3f_{xxy} - 3g_{xyy}) = 0,\\
&g(f_{xxxx} + f_{yyyy} - 4g_{xyyy} + 4g_{xxxy} - 6f_{xxyy})\\
&+ f(g_{xxxx} + g_{yyyy} + 4f_{xyyy} - 4f_{xxxy} - 6g_{xxyy})\\
&- 2(g_x - f_y)(f_{xxx} - g_{yyy} + 3g_{xxy} - 3f_{xyy})\\
&- 2(2 - f_x - g_y)(g_{xxx} + f_{yyy} - 3f_{xxy} - 3g_{xyy}) = 0,\\
&f_x = g_y, \qquad f_y = -g_x.
\end{aligned} \tag{28}$$

The Lie classical approach gives the following four Lie symmetries of system (28):

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial f}, \qquad X_4 = \frac{\partial}{\partial g}. \tag{29}$$

Only the Lie-like operators $\mathbf{X}_1$, $\mathbf{Y}_1$ are symmetries of the decomposed system (28). Therefore, the decomposed Lie-like operators of the system (25) and the symmetries of the decomposed system (28) are different.

We now focus on the symmetry for a system of restricted complex ODEs. Consider a kth-order system of restricted complex ODEs

$$E_\alpha\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\right) = 0\,, \qquad \alpha = 1, \ldots, m, \tag{30}$$

where $z = x$ is a real variable and $u = (u^1, \ldots, u^m)$ with coordinates $u^\alpha(x) = f^\alpha(x) + ig^\alpha(x)$. The kth derivative of u with respect to x is denoted by $u_{(k)}$.

System (30) which consists of m complex ODEs decomposes into a system of $2m$ real ODEs. The two real Lie-like operators

$$\mathbf{X} = \xi_1 \frac{\partial}{\partial x} + \frac{1}{2}\eta_1^\alpha \frac{\partial}{\partial f^\alpha} + \frac{1}{2}\eta_2^\alpha \frac{\partial}{\partial g^\alpha},$$
$$\mathbf{Y} = \xi_2 \frac{\partial}{\partial x} + \frac{1}{2}\eta_2^\alpha \frac{\partial}{\partial f^\alpha} - \frac{1}{2}\eta_1^\alpha \frac{\partial}{\partial g^\alpha}, \tag{31}$$

are obtained from the complex generator given in (8) by the substitution of

$$z = x, \qquad \mathbf{Z} = \mathbf{X} + i\mathbf{Y}, \qquad \xi = \xi_1 + i\xi_2, \qquad \eta^\alpha = \eta_1^\alpha + i\eta_2^\alpha. \tag{32}$$

The Lie approach is used to derive the Lie symmetries of the decomposed system of the $2m$ real ODEs. These symmetries and the Lie-like operators (31) are in general different.

Example 5. The 2D Ermakov system in complex domain is [12]

$$u''u^3 = 1,$$
$$v''v^3 = 1, \tag{33}$$

where $u(x) = f(x) + ig(x),\ v(x) = p(x) + iq(x)$ are functions of the real variable $z = x$. The complex Lie symmetries for the system (33) are

$$\mathbf{Z}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{Z}_2 = x\frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u} + \frac{v}{2}\frac{\partial}{\partial v},$$
$$\mathbf{Z}_3 = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u} + xv\frac{\partial}{\partial v}. \tag{34}$$

The decomposition of the Lie symmetries (34) yields the following Lie-like operators:

$$\mathbf{X}_1 = 2\frac{\partial}{\partial x},\ \mathbf{Y}_1 = 0,$$
$$\mathbf{X}_2 = 2x\frac{\partial}{\partial x} + \frac{f}{2}\frac{\partial}{\partial f} + \frac{g}{2}\frac{\partial}{\partial g} + \frac{p}{2}\frac{\partial}{\partial p} + \frac{q}{2}\frac{\partial}{\partial q},$$
$$\mathbf{Y}_2 = \frac{g}{2}\frac{\partial}{\partial f} - \frac{f}{2}\frac{\partial}{\partial g} + \frac{q}{2}\frac{\partial}{\partial p} - \frac{p}{2}\frac{\partial}{\partial q},$$
$$\mathbf{X}_3 = 2x^2\frac{\partial}{\partial x} + xf\frac{\partial}{\partial f} + xg\frac{\partial}{\partial g} + xp\frac{\partial}{\partial p} + xq\frac{\partial}{\partial q},$$
$$\mathbf{Y}_3 = xg\frac{\partial}{\partial f} - xf\frac{\partial}{\partial g} + xq\frac{\partial}{\partial p} - xp\frac{\partial}{\partial q}. \tag{35}$$

The system of ODEs corresponding to the system of complex ODEs (33) in the real domain is

$$f''(f^3 - 3fg^2) + g''(g^3 - 3f^2g) = 1,$$
$$g''(f^3 - 3fg^2) - f''(g^3 - 3f^2g) = 0,$$
$$p''(p^3 - 3pq^2) + q''(q^3 - 3p^2q) = 1,$$
$$q''(p^3 - 3pq^2) - p''(q^3 - 3p^2q) = 0. \tag{36}$$

The Lie classical approach yields the following Lie symmetries for system (36):

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x},\ X_2 = 2x\frac{\partial}{\partial x} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}, \\ X_3 &= x^2\frac{\partial}{\partial x} + xf\frac{\partial}{\partial f} + xg\frac{\partial}{\partial g} + xp\frac{\partial}{\partial p} + xq\frac{\partial}{\partial q}. \end{aligned} \tag{37}$$

Only the Lie-like operator $\mathbf{X}_1$ is a symmetry of the decomposed system (36). Hence, we may conclude that not all the decomposed Lie-like operators are symmetries of the decomposed system for systems of restricted complex ODEs.

We have the following proposition which is similar to that given by Mahomed and Naz [7] for PDEs. As shown in the following proposition, this holds for complex ordinary differential equations too.

PROPOSITION 1

The decomposed Lie-like operators corresponding to the complex symmetries of a system of restricted complex ODEs are the Lie symmetries of the decomposed system if the symmetries are real or pure imaginary.

The proof is immediate if one invokes the complex Lie symmetry and its real decomposed symmetry conditions.

Note that we do not have a similar result for the more general system (1) as the Lie-like operators (9) have all four coefficient functions in each of the operators **X** and **Y**.

3. Noether symmetries for systems of complex ODEs

We now pursue the Noether case. It is known from [13] that if $L = L_1 + iL_2$ is a complex Lagrangian of a restricted second-order complex ODE, then both L_1 and L_2 are two Lagrangians of the corresponding system of two second-order ODEs. An analogous result holds in the general case. If we have that $L = L_1 + iL_2$ is a complex Lagrangian for the system (1) of m ODEs which is variational, then L_1 and L_2 are two Lagrangians of the corresponding system of DEs.~For example, consider the complex free particle equation of Example 2, viz. (16) which has the usual Lagrangian $L = u'^2/2$. The system of PDEs corresponding to the complex ODE (16) is (18). This real system of PDEs (18) has the two Lagrangians [14]

$$\begin{aligned} L_1 &= \frac{1}{8}(h^2 - l^2), \\ L_2 &= \frac{1}{4}hl, \end{aligned} \tag{38}$$

where $h = f_x + g_y$ and $l = g_x - f_y$.

We now review the meaning of a Noether-like operator.

The operators $\mathbf{X} = \xi_1\partial/\partial x + \xi_2\partial/\partial y + \eta_1^\alpha\partial/\partial f^\alpha + \eta_2^\alpha\partial/\partial g^\alpha$ and $\mathbf{Y} = \xi_2\partial/\partial x - \xi_1\partial/\partial y + \eta_2^\alpha\partial/\partial f^\alpha - \eta_1^\alpha\partial/\partial g^\alpha$ are said to be Noether-like operators of the system (1) with respect to the pth-order ($p \leq k$) Lagrangians L_1 and L_2 if they satisfy (see [13] in which this was stated for first-order Lagrangians)

$$
\begin{aligned}
2\mathbf{X}^{[p]}L_1 - 2\mathbf{Y}^{[p]}L_2 + (D_x(\xi_1) + D_y(\xi_2))L_1 + (D_y(\xi_1) - D_x(\xi_2))L_2 \\
= D_x(A_1) + D_y(A_2), \qquad D_x = \frac{\mathrm{d}}{\mathrm{d}x}, \qquad D_y = \frac{\mathrm{d}}{\mathrm{d}y}, \\
2\mathbf{X}^{[p]}L_2 + 2\mathbf{Y}^{[p]}L_1 + (D_x(\xi_1) + D_y(\xi_2))L_2 + (D_x(\xi_2) - D_y(\xi_1)L_1 \\
= D_x(A_2) - D_y(A_1)
\end{aligned}
\tag{39}
$$

for suitable functions A_1 and A_2. These conditions arise from the split of the Noether condition on the complex Lagrangian $L = L_1 + iL_2$ given by

$$
\mathbf{Z}^{[p]}L + D_x(\xi) = D_x(A) \tag{40}
$$

for some complex function $A = A_1 + iA_2$. In the case of restricted complex ODEs (30) the Noether-like condition is

$$
\begin{aligned}
\mathbf{X}^{[p]}L_1 - \mathbf{Y}^{[p]}L_2 + D_x(\xi_1)L_1 - D_x(\xi_2)L_2 = D_x(A_1), \qquad D_x = \frac{\mathrm{d}}{\mathrm{d}x} \\
\mathbf{X}^{[p]}L_2 + \mathbf{Y}^{[p]}L_1 + D_x(\xi_1)L_2 + D_x(\xi_2)L_1 = D_x(A_2).
\end{aligned}
\tag{41}
$$

The formulas for the Noether complex integral $I = I_1 + iI_2$ and its split integrals can be deduced from Noether's theorem (see [13] for second-order variational problems).

Some examples are presented below. These illustrate that the Noether-like operators are not, in general, the same as the Noether symmetries of the Lagrangians of the decomposed system.

Example 6. There are five complex Noether symmetries of the usual Lagrangian of the complex free particle equation [14]. We see that

$$
\mathbf{Z}_3 = 2z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} \tag{42}
$$

when split into [14]

$$
\begin{aligned}
\mathbf{X}_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2}f\frac{\partial}{\partial f} + \frac{1}{2}g\frac{\partial}{\partial g}, \\
\mathbf{Y}_3 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{1}{2}g\frac{\partial}{\partial f} - \frac{1}{2}f\frac{\partial}{\partial g},
\end{aligned}
\tag{43}
$$

are not Noether symmetries corresponding to the Lagrangians (38). In fact, only the translation of Noether symmetries $\mathbf{Z}_1 = \partial/\partial z$ and $\mathbf{Z}_2 = \partial/\partial u$ give rise to the four translation Noether-like operators in x, y, f and g which are Noether symmetries of the real Lagrangians (38).

If one considers the restricted complex free particle equation, then the complex usual Lagrangian splits into the two Lagrangians

$$
\begin{aligned}
L_1 &= \frac{1}{2}(f'^2 - g'^2), \\
L_2 &= f'g'.
\end{aligned}
\tag{44}
$$

These Lagrangians correspond to the free particle system $f'' = 0$, $g'' = 0$. There are nine Noether-like operators which arise from the complex split of the five complex Noether symmetries. Only the translation symmetries in x, f and g are Noether symmetries of the two Lagrangians (44).

Example 7. The restricted complex Emden–Fowler equation [14]

$$u'' + \frac{2}{z}u' = 3u^5 \tag{45}$$

has the complex Lagrangian

$$L = \frac{1}{2}z^2(u'^2 + u^6) \tag{46}$$

with a single complex Noether symmetry

$$\mathbf{Z} = 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}. \tag{47}$$

The decomposed system (in [14] the system of PDEs were obtained) corresponding to (45) is

$$\begin{aligned} f'' + \frac{2}{x}f' &= 3(f^5 - 10f^3g^2 + 5fg^4), \\ g'' + \frac{2}{x}g' &= 3(5f^4g - 10f^2g^3 + g^5). \end{aligned} \tag{48}$$

This system (48) has the two Lagrangians

$$\begin{aligned} L_1 &= \frac{1}{2}x^2(f'^2 - g'^2 + f^6 - 15f^4g^2 + 15f^2g^4 - g^6), \\ L_2 &= \frac{1}{2}x^2(2f'g' + 6f^5g - 20f^3g^3 + 6fg^5). \end{aligned} \tag{49}$$

The Noether-like operators deduced from (48) are

$$\begin{aligned} \mathbf{X} &= 2x\frac{\partial}{\partial x} - \frac{1}{2}f\frac{\partial}{\partial f} - \frac{1}{2}g\frac{\partial}{\partial g}, \\ \mathbf{Y} &= -g\frac{\partial}{\partial f} + f\frac{\partial}{\partial g}. \end{aligned} \tag{50}$$

These operators (50) are not Noether symmetries of the system (49). The complex Emden–Fowler equation can be split into a system of four PDEs as in [14] (herein integrability was considered). Then too the Noether-like operators which arise from (47) are not Noether symmetries of the system of PDEs.

We have the following proposition.

PROPOSITION 2

The decomposed Noether-like operators corresponding to the complex Noether symmetries for a Lagrangian of a system of variational restricted complex ODEs are the Noether symmetries of the decomposed variational system if the symmetries are real or pure imaginary.

The proof of this is a consequence of the symmetry conditions (41) and (40).

Again it is important to remark that we do not have a similar result for the more general system (1) as a variational system because the Noether-like operators of the form (9) have all four coefficient functions in each of the operators **X** and **Y**.

4. Concluding remarks

The Lie and Noether point symmetry analysis for a kth-order system of m complex ODEs with m dependent variables was performed. The Lie- and Noether-like operators were obtained by decomposing the Lie and Noether symmetries of a complex ODE into real and imaginary parts. The system of complex ODEs was decomposed into a system of coupled real PDEs. The Lie and Noether classical approach was invoked to determine the symmetries of the decomposed system of DEs in real domain and a comparison was made with the Lie- and Noether-like operators for several examples. We concluded that the Lie- and Noether-like operators and symmetries of the decomposed system were, in general, different.

We have shown that the decomposed Lie- and Noether-like operators which correspond to the complex symmetries of a system of restricted complex ODEs are the symmetries of the decomposed system if, the symmetries are purely imaginary or real. A similar result for more general systems such as (1) is an open question and requires further investigation.

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