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g_2 Algebra and two-dimensional quasiexactly solvable Hamiltonian related to Poschl–Teller potential

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Abstract. In this article, we write the general form of the quasiexactly solvable Hamiltonian of g_2 algebra via one special representation in the x–y two-dimensional space. Then, by choosing an appropriate set of coefficients and making a gauge rotation, we show that this Hamiltonian leads to the solvable Poschl–Teller model on an open infinite strip. Finally, we assign g_2 hidden algebra to the Poschl–Teller potential and obtain its spectrum by using the representation space of g_2 algebra.

Keywords. g_2 algebra; quasiexactly solvable Hamiltonian; hidden algebra; Poschl–Teller potential.

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1. Introduction

For the past thirty years, the smallest exceptional Lie group G_2, as the isotropy group of a generic 3-form in seven dimensions, has been used in many fields of theoretical and mathematical physics [1–5]. Recently, superstring theory has simulated a deep interest in seven-dimensional manifold with G_2 geometry [6]. A higher-order version of the classical field strength of the Yang–Mills theories is related to g_2 algebra as well [7,8].

The g_2 algebra is also used for studying the spectral problems of some quantum systems. In general, there are two types of problems in such systems: the quasiexactly solvable (QES) models and the exact solvable (ES) models. In the QES models, we cannot find the whole eigenvalues and the corresponding eigenfunctions for a Hamiltonian explicitly. These problems were discovered in the early 1980s by the theoretical physicists Shifman, Turbiner and Ushveridze [9–11]. In these problems, the Schrödinger operator should be written as a bilinear combination of the first-order differential operators spanning a finite-dimensional Lie algebra whose presence underlies partial solvability of the spectral problem. This Lie algebra is called the hidden algebra of the QES Hamiltonian. One way of constructing large families of QES potentials is the Lie algebraic method

[10]. In this method, one uses results from the representation theory of the Lie algebras to ensure that a given Hamiltonian preserves a certain finite-dimensional subspace of polynomials. Then the most general second-order differential operator that preserves this space can be written as a quadratic combination of the first-order differential operators of the Lie algebra generators. This operator is said to be Lie algebraic and its algebra is often referred to as a hidden algebra. On the other hand, this second-order differential operator does not have in general the form of a Schrödinger operator, but it can be transformed into a Schrödinger operator by changing its variables and doing a gauge rotation by a non-vanishing function. As this operator possesses a finite-dimensional representation space of the polynomials, restricting to a linear transformation on this space, the associated eigenvalues can be computed by purely algebraic methods such as matrix eigenvalue calculations [12,13].

There is also another category of spectral problems called ES problems, whose entire spectrum can be determined by algebraic methods. These problems were known much earlier than the QES problems. The simplest examples of such problems are: Calogero potential, Sutherland potential, harmonic oscillator, Poschl–Teller, etc. [14–18].

In this paper and in §2, we use one special representation of g_2 algebra, and write its general QES solvable model on the plane. Then in §3 we show that by choosing an appropriate set of coefficients, this model is transformed to the well-known Poschl–Teller potential on an open infinite strip with hidden algebraic structure $g_2 \supset gl_2 \times R^3$. We calculate its eigenvalues and eigenfunctions by using the representation space of g_2 algebra as well. Finally, in §4, we make some conclusions and future comments.

2. General form of the quasiexactly solvable g_2 algebra

In ref. [20], one can find a complete classification with a list of normal forms of the one-dimensional QES problems where $sl(2)$ and $su(2)$ are their hidden algebras. There are a few known classes of QES problems in two dimensions, with hidden algebra $su(3)$, $su(2) \bigoplus su(2)$ [13]. So, we can conclude that there is a wide array of classification of Lie algebras with differential operators in the plane that should be considered deeply.

The general form of the QES operator is Lie algebraic if it can be written by the linearly independent first-order differential operators of a Lie algebra as

$$T_{\text{general}} = \sum_{i,j=1}^{N} C_{ij} L_i L_j + \sum_{i=1}^{N} C_i L_i, \tag{1}$$

where C_{ij} and C_i are the arbitrary coefficients and L_i is the generator of the algebra. By substituting the realization form of the generators in (1), we can obtain a second-order differential operator, where, according to ref. [20], it has the following form:

$$-T = P(x)\frac{\partial^2}{\partial x^2} + \tilde{Q}(x)\frac{\partial}{\partial x} + \tilde{R}, \tag{2}$$

where P and $\tilde{Q}$ as the arbitrary polynomials with degrees of 4 and 2, respectively, are defined as

$$\tilde{Q}(x) = Q(x) - \frac{n-1}{2}P'(x), \quad \tilde{R} = R - \frac{n}{2}Q'(x) + \frac{n(n-1)}{12}P''(x), \tag{3}$$

and $\tilde{R}$ is a single constant that can be absorbed by only translating the origin. On the other hand, operator (2) is equivalent to the Schrödinger operator:

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x), \tag{4}$$

when there is an appropriate set of coordinates and a suitable gauge rotation. Assume $P(x) > 0$ and define

$$x = \xi(\bar{x}) = \int^{\bar{x}} \frac{\mathrm{d}y}{\sqrt{P(y)}}, \tag{5}$$

$$\mu(\bar{x}) = |P(\bar{x})|^{-n/4}\exp\int^{\bar{x}} \frac{\tilde{Q}(y)}{2P(y)}, \tag{6}$$

then the change of variable $\bar{x} = \zeta(x)$ and the gauge factor $\mu(\bar{x})$ will place the operator T into Schrödinger form (4):

$$-2H = \mu(\bar{x}) \cdot T \cdot \mu^{-1}(\bar{x}). \tag{7}$$

It is also necessary to notice that two differential operators H and T are equivalent if they can be mapped onto each other by a combination of change of independent variable and gauge transformation [21]. Hence this procedure preserves the subspace of the spectrum and eigenfunctions with the same energy, i.e., when we find the eigenvalues of T, they are the same for H too. Therefore, we shall use these values for our problem.

Now, we turn to the exceptional Lie algebra g_2. This algebra can be written as an infinite-dimensional algebra of differential operators on the (x, y) plane. This algebra contains the subalgebra $gl_2 \times R^3$ of the first-order differential operators plus the extra generator L_8 [22]:

$$\begin{aligned} L_1 &= \partial_x, \quad L_2 = \partial_y, \quad L_3 = x\partial_x, \quad L_4 = x\partial_y, \quad L_5 = y\partial_y \\ L_6 &= x^2\partial_x + 2xy\partial_y - nx, \quad L_7 = x^2\partial_y, \quad L_8 = y\partial_x^2, \end{aligned} \tag{8}$$

where n is a non-negative integer number. The operators L_6 and L_7 are the positive root generators and the generator L_8 is the only second-order differential operator, for which we should not consider writing the general form of the QES Hamiltonian of g_2 algebra. If we take the whole generators except L_6, L_7 and L_8, the remaining generators form the Borel subalgebra of $g_2 \supset gl_2 \times R^3$, i.e., its Hamiltonian is solvable; otherwise we have a QES Hamiltonian [14].

The general form of the QES Hamiltonian (1) related to the first-order differential generators (8) is written as

$$T_{\text{general}} = \sum_{i,j=1}^{7} C_{ij} L_i L_j + \sum_{i=1}^{7} C_i L_i. \tag{9}$$

By replacing the generators of g_2 algebra (of course, except L_8), the resultant operator has a finite-dimensional representation of the following polynomials [22]:

$$P(x, y) = \{x^i y^j | 0 \le i + 2j \le N\}, \qquad i, j \ge 0. \tag{10}$$

The invariant space (10) can be mapped according to the linear fractional or Mobius transformation to itself, which is isomorphic to the standard representation of this space [23]. This means that the canonical form (2) is not unique. By doing this work the QES

Hamiltonians can be extended greatly. This work was performed by Turbiner in [22] for obtaining the generators of g_2 algebra initially in the x–y plane.

3. Two-dimensional Hamiltonian and Poschl–Teller potential

Consider the Lie algebra g_2 spanned by the first-order differential operators (8) and by choosing the following particular choice of Lie algebraic coefficients

$$C_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

$$C_i = (0, 0, -4n-3, 0, -8n-6, 0, 0), \qquad C_0 = (n+1)(4n+2), \tag{12}$$

we obtain a quasiexactly solvable two-dimensional Hamiltonian from (9) as

$$\begin{aligned} T \,=\, & (1+x^2)\frac{\partial^2}{\partial x^2} + 4xy\frac{\partial^2}{\partial x \partial y} + 4(1+x^2+y^2)\frac{\partial^2}{\partial y^2} \\ & -2(2n+1)x\frac{\partial}{\partial x} - 2(4n+1)y\frac{\partial}{\partial y} - (n+1)(4n+2), \end{aligned} \tag{13}$$

which has the $gl_2 \times R^3$ hidden symmetry algebra.

As the generators of g_2, i.e. (8), are independent and separable from each other, they have particular representations too, that is, we have written relation (9) with all of g_2 generators, while the whole coefficients of L_8 are zero in it for all terms. Therefore, relation (13) has g_2 hidden algebra and its Riemannian metric components are

$$g^{11} = (1+x^2), \qquad g^{12} = 2xy, \qquad g^{22} = 4(1+x^2+y^2). \tag{14}$$

Let us now consider the gauge factor and new variables from eqs (5) and (6) as

$$\mu(\bar{x}, \bar{y}) = \frac{\cos^{2n+2}\bar{x}}{\cosh^{n+1} 2\bar{y}}, \qquad \bar{x} = \tan^{-1} x, \qquad \bar{y} = \frac{1}{2}\sinh^{-1}\frac{y}{1+x^2}. \tag{15}$$

Then, according to (7), the operator (13) will map to the normal Schrödinger form by changing the coordinates and making a gauge rotation. These transformations map the plane to an open infinite strip $(-\pi/2, \pi/2) \times R$ in a locally diffeomorphic way and the potential is

$$V(\bar{x}, \bar{y}) = \frac{-2(n+1)(n+2)\cos^2\bar{x}}{\cosh^2 2\bar{y}}. \tag{16}$$

So our Hamiltonian will be as

$$H = -\frac{1}{2}\cos^2\bar{x}\left(\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2}\right) + V(\bar{x}, \bar{y}). \tag{17}$$

Hence, operator (13), which is verified in the x–y plane, after making a gauge rotation, is transformed to the Hamiltonian (17) on an open infinite strip and it can be separated into two equations as follows [23]:

$$-\frac{1}{2}\bar{X}''(\bar{x}) - \frac{E}{\cos^2(\bar{x})}\bar{X}(\bar{x}) = -\alpha\bar{X}(\bar{x}), \tag{18}$$

$$-\frac{1}{2}\bar{Y}''(2\bar{y}) - \frac{(n+1)(n+2)}{2\cosh^2(2\bar{y})}\bar{Y}(2\bar{y}) = \alpha\bar{Y}(2\bar{y}), \tag{19}$$

where α is the separation constant. Both of these equations are exactly solvable and the second equation has the Poschl–Teller potential form. Now, instead of solving the above equations, we use the standard approach of representation theory for the QES models, that is, we write the eigenfunction of the system in the following algebraic form:

$$\Psi(\bar{x}, \bar{y}) = \frac{\cos^{2n+2}\bar{x}}{\cosh^{n+1}2\bar{y}}P(\tan\bar{x}, (1+x^2)\sinh(2\bar{y})), \tag{20}$$

where $P(x, y)$ is calculated from (10) in terms of the new variables (15). Thus, for a given number N of (10), we first obtain the matrix representation of T in standard basis of $P(x, y)$, and then we exactly compute the eigenvalues and the corresponding eigenfunctions of (13) by diagonalizing the obtained matrix.

For example, if we take $N = 2$, the standard basis of $P(x, y)$ is $\{1, x, y, x^2\}$ and the matrix representation T is

$$T = \begin{pmatrix} -(n+1)(4n+2) & 0 & 0 & 1 \\ 0 & -2(2n+1) & 0 & 0 \\ 0 & 0 & -2(4n+1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

Hence its eigenvalues and eigenfunctions can be easily calculated as

$$\begin{aligned} \lambda_1 &= -4n^2 - 6n - 2, \qquad \lambda_2 = -4n - 2, \\ \lambda_3 &= -8n - 2, \qquad \lambda_4 = 1, \end{aligned} \tag{22}$$

$$\begin{aligned} v_1 &= [1, 0, 0, 0], \quad v_2 = [0, 1, 0, 0], \quad v_3 = [0, 0, 1, 0], \\ v_4 &= [1, 0, 0, 4n^2 + 6n + 3]. \end{aligned} \tag{23}$$

The corresponding values of $H = -\frac{1}{2}T$ are also as

$$E_1 = 2n^2 + 3n + 1, \quad E_2 = 2n + 1, \quad E_3 = 4n + 1, \quad E_4 = -\frac{1}{2}, \tag{24}$$

$$\begin{aligned} \Psi_1(\bar{x}, \bar{y}) &= \mu(\bar{x}, \bar{y}), \quad \Psi_2(\bar{x}, \bar{y}) = \mu(\bar{x}, \bar{y})x, \\ \Psi_3(\bar{x}, \bar{y}) &= \mu(\bar{x}, \bar{y})y, \quad \Psi_4(\bar{x}, \bar{y}) = \mu(\bar{x}, \bar{y})(1 + x^2(4n^2 + 6n + 3)). \end{aligned} \tag{25}$$

Therefore, we have obtained the eigenvalues and eigenfunctions of (17) without solving it. Also we have seen that by using some algebraic methods and one special representation of g_2 algebra, we can obtain the Poschl–Teller exactly solvable model on an open infinite strip with hidden algebra g_2.

4. Conclusion

We have shown that by taking an appropriate set of variables and coefficients, the Hamiltonian of the QES $g_2 \supset gl_2 \times R^3$ algebra in one special representation concludes the solvable Poschl–Teller potential on an open infinite strip. So, we can put the solvable Poschl–Teller potential in the list of the Schrödinger operators possessing g_2 hidden algebraic structure. Also, we could get its eigenvalues and eigenfunctions by representating the polynomial space of g_2 algebra. As a future work, one can take the other terms and appropriate coefficients in operator (9) and obtain other models on the plane that possess the g_2 hidden algebra.

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