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The extended (G'/G)-expansion method and travelling wave solutions for the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity

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Abstract. In this paper, we construct the travelling wave solutions to the perturbed nonlinear Schrödinger's equation (NLSE) with Kerr law non-linearity by the extended (G'/G)-expansion method. Based on this method, we obtain abundant exact travelling wave solutions of NLSE with Kerr law nonlinearity with arbitrary parameters. The travelling wave solutions are expressed by the hyperbolic functions, trigonometric functions and rational functions.

Keywords. Nonlinear Schrödinger's equation with Kerr law nonlinearity; travelling wave solutions; extended (G'/G)-expansion method.

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1. Introduction

During the past decades, the investigation of the exact travelling wave solutions to nonlinear partial differential equations (NLPDEs) plays an important role in the study of complex physical and mechanic phenomena. Several effective methods for obtaining exact solutions of NLPDEs, such as the trigonometric function series method [1,2], the modified mapping method and the extended mapping method [3], the modified trigonometric function series method [4,5], the dynamical system approach and the bifurcation method [6,7], the infinite series method and Jacobi elliptic function expansion method [8], the exp-function method [9], the multiple exp-function method [10], the transformed rational function method [11], the symmetry algebra method (consisting of Lie point symmetries) [12], the Wronskian technique [13], the linear superposition principle [14] and so on have been developed.

Recently, Wang *et al* [15] proposed the (G'/G)-expansion method to construct the travelling wave solutions for NLPDEs. The method is based on the homogeneous balance principle and linear ordinary differential equation (LODE) theory. It is assumed that the travelling wave solutions can be expressed by a polynomial in (G'/G), and that G'' satisfies a second-order LODE $G'' + \lambda G' + \mu G = 0$. The degree of the polynomial can be determined by the homogeneous balance between the highest-order derivative and linear terms appearing in the given NLPDEs. The coefficients of the polynomial can be obtained by solving a set of algebraic equations. Recently, the (G'/G)-expansion method has been successfully applied to obtain exact solutions for a variety of NLPDEs [16–21]. In particular, Miao and Zhang [22] proposed a new method called the modified (G'/G)-expansion method to construct travelling wave solutions of the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity. In their contribution, they obtained new travelling wave solutions by using the modified (G'/G)-expansion method and G'' satisfies a second-order LODE $G'' + \mu G = 0$. In fact, the (G'/G)-expansion method is just a variant of the transformation method that transforms nonlinear partial differential equations into integrable ordinary differential equations to solve, see ref. [2].

In this paper, we investigate the perturbed NLSE with Kerr law nonlinearity given in ref. [3]

$$iu_t + u_{xx} + \alpha|u|^2u + i[\gamma_1 u_{xxx} + \gamma_2|u|^2u_x + \gamma_3(|u|^2)_x u] = 0, \tag{1.1}$$

where γ_1 is the third-order dispersion, γ_2 is the nonlinear dispersion, while γ_3 is also a version of nonlinear dispersion. More details are presented in ref. [3]. It must be very clear that γ_3 is not Raman scattering. When γ_3 is purely imaginary, then only it is Raman scattering. Moreover, Raman scattering is not a Hamiltonian perturbation and therefore it is not an integrable perturbation. More details are presented in ref. [6]. Equation (1.1) describes the propagation of optical solitons in nonlinear optical fibres that exhibit a Kerr law nonlinearity [23,24]. Equation (1.1) has important application in various fields, such as semiconductor materials, optical fibre communications, plasma physics, fluid and solid mechanics etc. More details are presented in ref. [25] and references therein.

The NLSE, which is the ideal Kerr medium, in its original form, is found to be completely integrable by the method of inverse scattering transformation (IST) [26,27]. In the absence of a perturbation term, that is, $\gamma_1 = 0, \gamma_2 = 0, \gamma_3 = 0$, eq. (1.1) reduces to the NLSE with non-Kerr law nonlinearity

$$iu_t + u_{xx} + \alpha|u|^2u = 0. \tag{1.2}$$

Recently, Biswas *et al* [23] investigated the optical solitons of eq. (1.2). It is worth mentioning that Biswas *et al* [24,28] recently investigated the optical soliton perturbation with non-Kerr law media

$$iu_t + u_{xx} + F(|u|^2)u = i\varepsilon R[u, u^*].$$

More details are presented in [24,28].

In the absence of $\gamma_1, \gamma_2, \gamma_3$ (i.e., $\gamma_1 = \gamma_2 = \gamma_3 = 0$), eq. (1.1) reduces to eq. (1.2). It is well known that NLSE (1.2) admits the bright soliton solution (see p. 2835 of [12] and [29]):

$$u(x,t) = k\sqrt{\frac{2}{\alpha}}\operatorname{sech}(k(x - 2\mu t))e^{i[\mu x - (\mu^2 - k^2)t]},$$

where α and k are arbitrary real constants, for the self-focussing case $\alpha > 0$, and the dark soliton solution (p. 2835 of [12] and [30]):

$$u(x,t) = k\sqrt{-\frac{2}{\alpha}}\tanh(k(x-2\mu t))\mathrm{e}^{i[\mu x-(\mu^2+2k^2)t]},$$

where α and k are arbitrary real constants, for the de-focussing case $\alpha < 0$. In ref. [31], Zhang has constructed new types of exact complex travelling wave solutions of NLSE without perturbation effects by using two improved direct algebraic methods.

Recently, as for perturbation effects, there are many contributions regarding eq. (1.1) (see for instance [32–42] and references therein). Their contributions are related to finding various types of solutions, including fronts (kinks), bright solitary waves and dark solitary waves in various media, such as power law (or dual-power law), parabolic law and Kerr law. In ref. [32], Zhang and Si investigated NLSE with dual-power law nonlinearity. Because of the dual-power law nonlinearity, the equation cannot be directly dealt with by the method and requires some kinds of techniques. By means of two proper transformations and the new generalized algebraic method, they transformed NLSE to an ordinary differential equation that is easy to solve and find a rich variety of new exact solutions for the equation, which include soliton solutions, combined soliton solutions, triangular periodic solutions and rational function solutions. In ref. [33], Taghizadeh and Mirzazadeh obtained the exact solutions of the perturbed NLSE (1.1) with Kerr law nonlinearity by using the simplest equation method. In ref. [34], Biswas and Milovic studied the NLSE in a non-Kerr law medium and obtained doubly periodic wave solutions by using travelling wave ansatz. In ref. [35], Biswas and Porsezian considered the solitons of the modified NLSE by using the soliton perturbation theory. In particular, the nonlinear gain (damping) and filters or the coefficient of finite conductivity are treated as perturbation terms for the solitons. In ref. [36], Khalique and Biswas investigated the NLSE in non-Kerr law media and obtained the stationary 1-soliton solution by using the Lie symmetry analysis technique. The types of nonlinearity that are considered are: Kerr law, power law, parabolic law and the dual-power law. In ref. [37], Biswas investigated the 1-soliton solution of the NLSE in $1 + 2$ dimensions for parabolic law nonlinearity by means of the solitary wave ansatze. In ref. [38], Biswas studied the topological 1-soliton solution of the NLSE with Kerr nonlinearity in $1 + 2$ dimensions by the solitary wave ansatze method. Also, they studied these topological solitons in the context of dark optical solitons. In ref. [39], Biswas and Milovic considered the generalized NLSE including Kerr law, power law, parabolic law and the dual-power law. Also, they obtained the bright and dark solitons and the adiabatic parameter dynamics of the solitons due to perturbation terms. In ref. [40], Topkara *et al* studied optical solitons with non-Kerr law nonlinearity and intermodal dispersion with time-dependent coefficients. The coefficients of group velocity dispersion, nonlinearity and intermodal dispersion terms have time-dependent coefficients. The types of nonlinearity that are considered are Kerr, power, parabolic and dual-power laws. The solitary wave ansatz is used to carry out the integration of the governing NLSE with time-dependent coefficients. Moreover, they obtained the bright and dark optical solitons and showed that the only necessary condition for these solitons to exist is that these time-dependent coefficients of group velocity dispersion and inter-modal dispersion are Riemann integrable. As for the perturbed NLSE with periodic boundary conditions, Guo and Chen [41,42] established the existence of homoclinic orbits for a perturbed cubic–quintic NLSE with even periodic boundary conditions under the generalized parameter conditions and proved the persistence of homoclinic orbits by using geometric singular perturbation theory, Melnikov analysis and integrable theory.

However, in our novel contribution, we propose the extended (G'/G)-expansion method, which can be thought of as an extension of the (G'/G)-expansion method. The key idea of this method is that the travelling wave solutions of NLPDEs can be expressed by a polynomial in two variables (G'/G) and $(1/G)$, in which $G = G(\xi)$ satisfies a second-order LODE. The degree of the polynomial can be determined by the homogeneous balance between the highest-order derivative and nonlinear terms appearing in the given NLPDEs, and the coefficients of the polynomial can be obtained by solving a set of algebraic equations. More details are presented in §2.

Remark 1.1. It is worth mentioning that Zhang *et al* [3,4,6,8,22] considered the NLSE with Kerr law nonlinearity and obtained some new exact travelling wave solutions of eq. (1.1). In ref. [3], by using the modified mapping method and the extended mapping method, Zhang *et al* derived some new exact solutions of eq. (1.1), which are the linear combination of two different Jacobi elliptic functions and investigated the solutions in the limit cases. In ref. [4], by using the modified trigonometric function series method, Zhang *et al* studied some new exact travelling wave solutions. In ref. [6], by using qualitative theory of dynamical systems, Zhang *et al* obtained the travelling wave solutions in terms of bright and dark optical solitons and the cnoidal waves. The authors found that NLSE with Kerr law nonlinearity has only three types of bounded travelling wave solutions, namely, bell-shaped solitary wave solutions, kink-shaped solitary wave solutions and Jacobi elliptic function periodic solutions. Moreover, we pointed out the region that these periodic wave solutions lie in. We showed the relation between the bounded travelling wave solution and the energy level h. We observed that these periodic wave solutions tend to the corresponding solitary wave solutions as h increases or decreases. Finally, for some special selections of the energy level h, it was shown that the exact periodic solutions evolute into solitary wave solution. In ref. [7], by using the dynamical system approach, Zhang *et al* investigated the dynamic behaviour of travelling wave solutions to eq. (1.1). Under the given parametric conditions, all possible representations of explicit exact solitary wave solutions and periodic wave solutions were obtained. In ref. [8], Zhang *et al* investigated the perturbed NLSE (1.1) given in ref. [3] (*Appl. Math. Comput.* **216**, 3064 (2010)) and obtained exact travelling solutions by using infinite series method (ISM) and cosine-function method (CFM). We showed that the solutions by using ISM and CFM are equal. Finally, we obtained abundant exact travelling wave solutions of NLSE (1.1) by using Jacobi elliptic function expansion method (JEFEM). In ref. [22], by using the modified (G'/G)-expansion method, Miao and Zhang obtained the travelling wave solutions of eq. (1.1), which were expressed by the hyperbolic functions, trigonometric functions and rational functions.

2. Description of the extended (G'/G)-expansion method

In this section, we shall describe the main idea of our present method for constructing travelling wave solutions of NLPDEs.

Assume that a second-order LODE

$$G'' + \lambda G = \mu, \tag{2.1}$$

where

$$\phi = \frac{G'}{G}, \qquad \psi = \frac{1}{G}. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\phi' = -\phi^2 + \mu\psi - \lambda, \qquad \psi' = -\phi\psi. \tag{2.3}$$

To facilitate further on our analysis, we discuss the general solutions of the LODE (2.1) as follows:

Case 1. When $\lambda < 0$, the general solutions of the LODE (2.1)

$$G(\xi) = A_1 \sinh\sqrt{-\lambda}\xi + A_2 \cosh\sqrt{-\lambda}\xi + \frac{\mu}{\lambda}, \tag{2.4}$$

and we get

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}(\phi^2 - 2\mu\psi + \lambda),$$

where A_1 and A_2 are two constants and $\sigma = A_1^2 - A_2^2$.

Case 2. When $\lambda > 0$, the general solutions of the LODE (2.1)

$$G(\xi) = A_1 \sinh\sqrt{\lambda}\xi + A_2 \cosh\sqrt{\lambda}\xi + \frac{\mu}{\lambda}, \tag{2.5}$$

and we get

$$\psi^2 = \frac{\lambda}{\lambda^2\rho - \mu^2}(\phi^2 - 2\mu\psi + \lambda),$$

where A_1 and A_2 are two constants and $\rho = A_1^2 + A_2^2$.

Case 3. When $\lambda = 0$, the general solutions of the LODE (2.1)

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{2.6}$$

and we get

$$\psi^2 = \frac{\lambda}{A_1^2 - 2\mu A_2}(\phi^2 - 2\mu\psi),$$

where A_1 and A_2 are two constants.

Suppose that an NLPDE is given by

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.7}$$

where $u = u(x, t)$ is an unknown function and F is a polynomial. Now, we are in a position to show the main steps of the extended (G'/G)-expansion method.

Step 1. To construct the travelling wave solutions of (2.7), we introduce the wave transformation

$$u(x, t) = u(\xi), \qquad \xi = x - \eta t + \xi_1, \tag{2.8}$$

where ξ_1, η are constants. Substituting (2.8) into (2.7), we obtain the following ODE:

$$P(u, u', u'', u''', \ldots) = 0. \tag{2.9}$$

Step 2. Assume that the solution of eq. (2.9) can be expressed by a polynomial in ϕ and ψ as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i \phi^i + \sum_{j=1}^{N} b_j \phi^{j-1} \psi, \tag{2.10}$$

where $G = G(\xi)$ satisfies the LODE (2.1), $a_i (i = 0, 1, \ldots, N)$, $b_j (j = 1, \ldots, N)$, λ, μ are constants to be determined later, and the positive integer N can be determined by considering the homogeneous balance between the highest-order derivatives and the non-linear terms in ODE (2.9).

Step 3. Substituting the solution (2.10) together with (2.8) into (2.9) yields an algebraic equation including powers of $\phi^i \psi^j$. Equating the coefficients of each power of $\phi^i \psi^j$ to zero gives a system of algebraic equation for a_i, b_j, η λ, μ, A_1 and A_2.

Step 4. Solve the algebraic equations in Step 3 with the aid of *Mathematica*. Then substituting the values of parameters, one can obtain the travelling wave solutions of (2.7).

3. Travelling wave solutions of NLSE (1.1)

In this section, we shall illustrate the extended (G'/G)-expansion method in detail by constructing the travelling wave solutions of NLSE (1.1).

Assume that eq. (1.1) has travelling wave solutions in the form [3]

$$u(x, t) = \Phi(\xi) \exp(i(Kx - \Omega t)), \qquad \xi = k(x - ct), \tag{3.1}$$

where c is the propagation speed of a wave.

Substituting (3.1) into eq. (1.1) yields

$$\begin{aligned} &i(\gamma_1 k^3 \Phi''' - 3\gamma_1 K^2 k \Phi' + \gamma_2 k \Phi^2 \Phi' + 2\gamma_3 k \Phi^2 \Phi' - ck\Phi' + 2Kk\Phi') \\ &+(\Omega \Phi + k^2 \Phi'' - K^2 \Phi + \alpha \Phi^3 + 3\gamma_1 K k^2 \Phi'' + \gamma_1 K^3 \Phi - \gamma_2 K \Phi^3) = 0, \end{aligned}$$

where $\gamma_i (i = 1, 2, 3)$, α, k are positive constants and the prime denotes differentiation with respect to ξ.

By virtue of p. 3065 of ref. [3] we have

$$(\gamma_1 k^2)\Phi''(\xi)+(2K-c-3\gamma_1 K^2)\Phi(\xi)+\left(-\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3\right)\Phi^3(\xi)=0(\gamma_1 k^2\neq 0).$$

That is,

$$\Phi''(\xi)+\frac{2K-c-3\gamma_1 K^2}{\gamma_1 k^2}\Phi(\xi)+\frac{-\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3}{\gamma_1 k^2}\Phi^3(\xi)=0. \tag{3.2}$$

For simplicity, we assume

$$A=\frac{2K-c-3\gamma_1 K^2}{\gamma_1 k^2},\qquad B=\frac{-\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3}{\gamma_1 k^2}.$$

Thus (3.2) leads to ordinary differential equation (ODE)

$$\Phi''(\xi)+A\Phi(\xi)+B\Phi^3(\xi)=0. \tag{3.3}$$

In what follows, we shall discuss the travelling wave solutions to (3.3).

By balancing the highest-order derivative term Φ'' and the nonlinear term Φ^3 in (3.3), we obtain $N=1$ in (2.10). So, we assume that (3.3) has a solution in the form

$$\Phi(\xi)=a_1\phi+b_1\psi, \tag{3.4}$$

where a_1 and b_1 are constants to be determined later and satisfy $a_1^2+b_1^2\neq 0$. Next, there are three cases to be investigated and we give the corresponding travelling wave solutions.

Case 1. When $\lambda<0$. Substituting (3.4) into (3.3), the left-hand side of (3.3) becomes a polynomial in ϕ and ψ. Setting its coefficients to zero yields a system of algebraic equations as follows:

$$\phi^3\colon\ 2a_1+B\left(a_1^3-\frac{3a_1b_1^2\lambda}{\lambda^2\sigma+\mu^2}\right)=0,$$

$$\phi^2\psi\colon\ 2b_1+B\left(3a_1^2b_1-\frac{b_1^3\lambda}{\lambda^2\sigma+\mu^2}\right)=0,$$

$$\phi^2\colon\ b_1\mu\lambda-\frac{2Bb_1^3\mu\lambda^2}{\lambda^2\sigma+\mu^2}=0,$$

$$\phi\psi\colon\ -a_1\mu+\frac{2Ba_1^2b_1\mu\lambda}{\lambda^2\sigma+\mu^2}=0,$$

$$\phi\colon\ 2a_1\lambda+Aa_1-\frac{3Ba_1b_1^2\lambda^2}{\lambda^2\sigma+\mu^2}=0,$$

$$\psi\colon\ b_1\lambda(\lambda^2\sigma-\mu^2)+Ab_1(\lambda^2\sigma+\mu^2)+\frac{Bb_1^3\lambda^2(3\mu^2-\lambda^2\sigma)}{\lambda^2\sigma+\mu^2}=0,$$

$$\psi^0\colon\ b_1\mu\lambda^2-\frac{2Bb_1^3\mu\lambda^2}{(\lambda^2\sigma+\mu^2)^2}=0.$$

Solving the above system by *Mathematica*, we have

Case i. If $A < 0$ and $B > 0$ or (< 0), then

$$a_1 = 0, \;\; b_1 = \pm\sqrt{\frac{2A\sigma}{B}}, \qquad \sigma > 0 \qquad \text{or} \qquad < 0, \lambda = A, \mu = 0.$$

Case ii. If $A > 0$ and $B < 0$, then

$$a_1 = \pm\sqrt{\frac{-2}{B}}, \;\; b_1 = 0, \sigma = \text{an arbitrary constant}, \lambda = -\frac{A}{2}, \mu = 0.$$

Case iii. If $A > 0$ and $B < 0$, then

$$a_1 = \pm\sqrt{\frac{-1}{2B}}, \qquad b_1 = \pm\sqrt{-\frac{4A\sigma+\mu^2}{4AB}}, \qquad \sigma \geq -\frac{\mu^2}{4A^2}, \qquad \lambda = -2A,$$
$$\mu = \text{an arbitrary constant}.$$

From the above cases, we obtain the hyperbolic function solutions of (1.2) and (1.3) as follows:

Case 1.1. From eqs (3.1), (3.4) and Case i, we get

$$\Phi = b_1\psi = b_1\frac{1}{G}$$

and

$$u_1 = \pm\sqrt{\frac{2A(A_1^2 - A_2^2)}{B}} \times \frac{1}{A_1 \sinh\sqrt{-A}(x - \eta t + \xi_1) + A_2 \cosh\sqrt{-A}(x - \eta t + \xi_1)} \times \exp(i(kx + \omega t + \xi_0)).$$

That is,

$$|u_1| = \sqrt{\frac{2A(A_1^2 - A_2^2)}{B}} \times \left|\frac{1}{A_1 \sinh\sqrt{-A}(x - \eta t + \xi_1) + A_2 \cosh\sqrt{-A}(x - \eta t + \xi_1)}\right|. \quad (3.5)$$

Setting $A = -1, B = 6, A_1 = 1, A_2 = 2, \eta = 1, \xi_1 = 0$, we obtain the solution of eq. (3.5) (see figure 1), where $|u|$ is the norm of u.

Case 1.2. From eqs (3.1), (3.4) and Case ii, we get

$$\Phi = a_1\phi = a_1\frac{G'}{G}$$

and

$$u_2 = \pm\sqrt{-\frac{2}{B}}\frac{A_1 \cosh\sqrt{\frac{A}{2}}(x - \eta t + \xi_1) + A_2 \sinh\sqrt{\frac{A}{2}}(x - \eta t + \xi_1)}{A_1 \sinh\sqrt{\frac{A}{2}}(x - \eta t + \xi_1) + A_2 \cosh\sqrt{\frac{A}{2}}(x - \eta t + \xi_1)} \times \exp(i(kx + \omega t + \xi_0)).$$

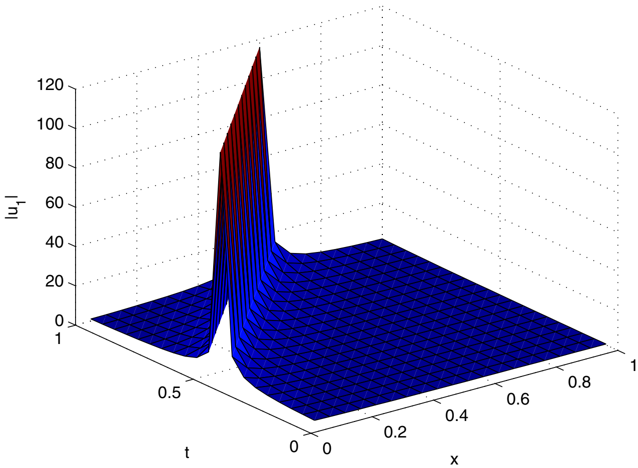

Figure 1. The graphics of trigonometric function solution (3.5).

That is,

$$|u_2| = \sqrt{-\frac{2}{B}}\left|\frac{A_1\cosh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)+A_2\sinh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)+A_2\cosh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)}\right|. \tag{3.6}$$

Setting $A = 2, B = -2, A_1 = 1, A_2 = 2, \eta = 1, \xi_1 = 0$, we obtain the solution of eq. (3.6) (see figure 2).

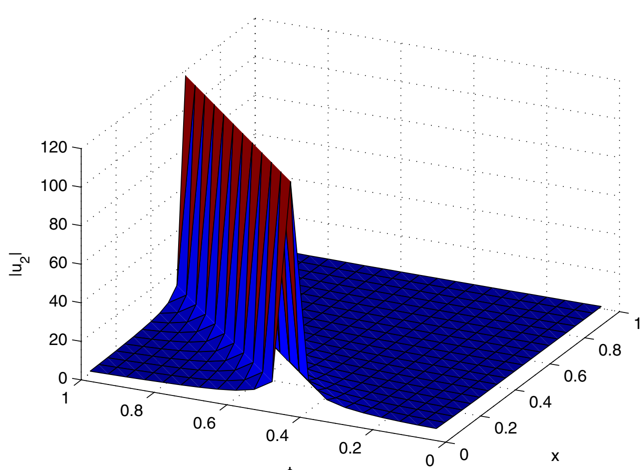

Figure 2. The graphics of trigonometric function solution (3.6).

Case 1.3. From eqs (3.1), (3.11) and Case iii, we get

$$\Phi = a_1\phi + b_1\psi = a_1\frac{G'}{G} + b_1\frac{1}{G}$$

and

$$u_3 = \pm\sqrt{-\frac{1}{2B}}\frac{A_1\cosh\sqrt{2A}(x-\eta t+\xi_1) + A_2\sinh\sqrt{2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{2A}(x-\eta t+\xi_1) + A_2\cosh\sqrt{2A}(x-\eta t+\xi_1)}$$

$$\times\exp(i(kx+\omega t+\xi_0))$$

$$\pm\sqrt{-\frac{4A(A_1^2-A_2^2)+\mu^2}{4AB}}\frac{1}{A_1\sinh\sqrt{2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{2A}(x-\eta t+\xi_1)}$$

$$\times\exp(i(kx+\omega t+\xi_0))$$

$$= \pm\sqrt{-\frac{1}{2B}}\frac{\sqrt{\frac{4A(A_1^2-A_2^2)+\mu^2}{2A}}+A_1\cosh\sqrt{2A}(x-\eta t+\xi_1)+A_2\sinh\sqrt{2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{2A}(x-\eta t+\xi_1)}$$

$$\times\exp(i(kx+\omega t+\xi_0)).$$

That is,

$$|u_3| = = \sqrt{-\frac{1}{2B}}$$

$$\times\left|\frac{\sqrt{\frac{4A(A_1^2-A_2^2)+\mu^2}{2A}}+A_1\cosh\sqrt{2A}(x-\eta t+\xi_1)+A_2\sinh\sqrt{2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{2A}(x-\eta t+\xi_1)}\right|. \tag{3.7}$$

Setting $A = 2$, $B = -2$, $A_1 = 1$, $A_2 = 2$, $\mu = \sqrt{7}$, $\eta = 1$, $\xi_1 = 0$, we obtain the solution of eq. (3.7) (see figure 3).

Remark 3.1. Taking $A_1 = 0$ and $A_2 > 0$, eq. (3.5) becomes

$$u_4 = \pm\sqrt{\frac{-2AA_2^2}{B}}\text{sech}\sqrt{-A}(x-\eta t+\xi_1)\exp(i(kx+\omega t+\xi_0)).$$

That is,

$$|u_4| = \sqrt{\frac{-2AA_2^2}{B}}|\text{sech}\sqrt{-A}(x-\eta t+\xi_1)|. \tag{3.8}$$

Setting $A = -1$, $B = 6$, $A_1 = 0$, $A_2 = 1$, $\eta = 1$, $\xi_1 = 0$, we obtain the solution of eq. (3.8) (see figure 4).

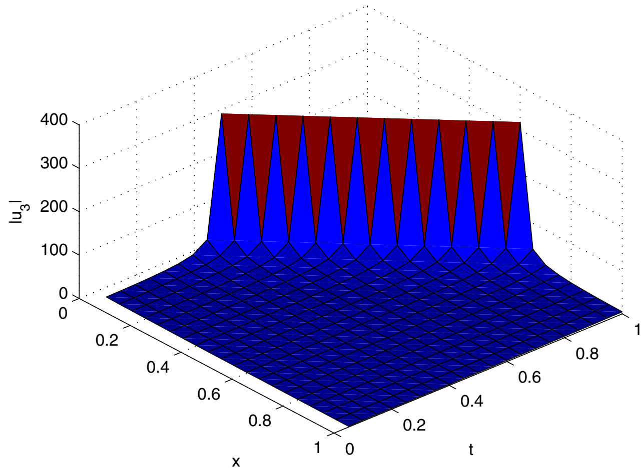

Figure 3. The graphics of trigonometric function solution (3.7).

Taking $A_1 > 0$ and $A_2 = 0$, eq. (3.5) becomes

$$u_5 = \pm\sqrt{\frac{-2AA_1^2}{B}}\operatorname{csch}\sqrt{-A}(x - \eta t + \xi_1)\exp(i(kx + \omega t + \xi_0)).$$

That is,

$$|u_5| = \sqrt{\frac{-2AA_1^2}{B}}|\operatorname{csch}\sqrt{-A}(x - \eta t + \xi_1)|. \tag{3.9}$$

Setting $A = -1, B = 2, A_1 = 1, A_2 = 0, \eta = 1, \xi_1 = 0$, we obtain the solution of eq. (3.9) (see figure 5).

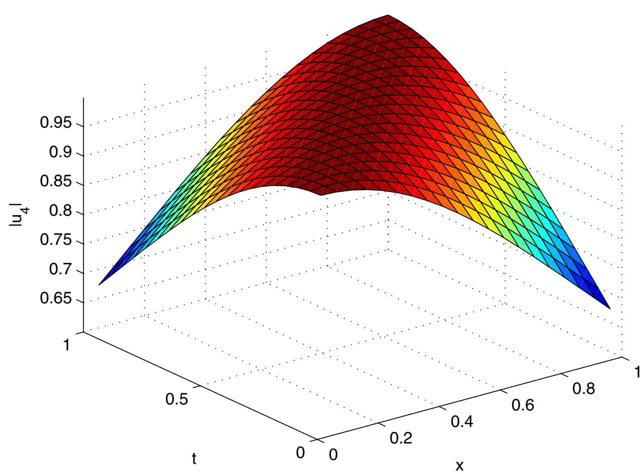

Figure 4. The graphics of hyperbolic function solution (3.8).

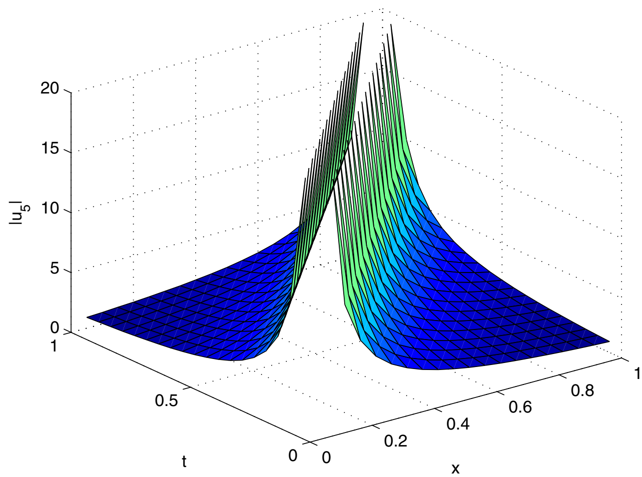

Figure 5. The graphics of rational function solution (3.9).

Remark 3.2. Taking $A_1 = 0$ and $A_2 \neq 0$, eq. (3.6) becomes

$$u_6 = \pm\sqrt{-\frac{2}{B}}\tanh\sqrt{\frac{A}{2}}(x - \eta t + \xi_1)\exp(i(kx + \omega t + \xi_0)).$$

That is,

$$|u_6| = \sqrt{-\frac{2}{B}}\left|\tanh\sqrt{\frac{A}{2}}(x - \eta t + \xi_1)\right|, \tag{3.10}$$

Setting $A = 2$, $B = -2$, $A_1 = 0$, $A_2 = 2$, $\eta = 1$, $\xi_1 = 0$, we obtain the solution of eq. (3.6) (see figure 6).

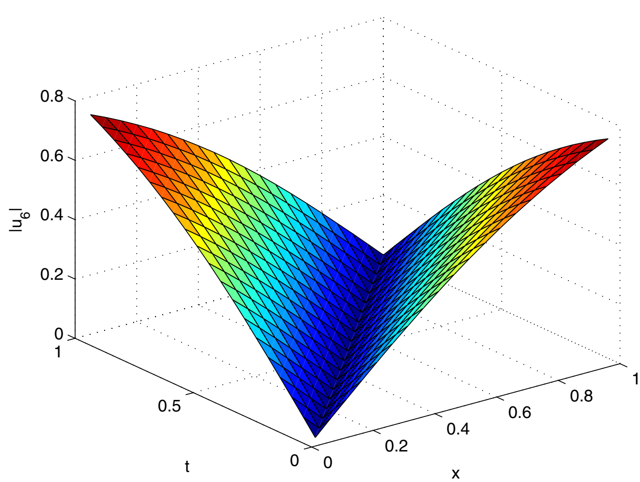

Figure 6. The graphics of hyperbolic function solution (3.10).

Taking $A_1 \neq 0$ and $A_2 = 0$, respectively, eq. (3.7) becomes

$$u_7 = \pm\sqrt{\frac{-2}{B}}\coth\sqrt{\frac{A}{2}}(x - \eta t + \xi_1)\exp(i(kx + \omega t + \xi_0)).$$

That is,

$$|u_7| = \left|\sqrt{\frac{-2}{B}}\coth\sqrt{\frac{A}{2}}(x - \eta t + \xi_1)\right|. \tag{3.11}$$

Setting $A = 2, B = -2, A_1 = 1, A_2 = 0, \eta = 1, \xi_1 = 0$, we obtain the solution of eq. (3.7) (see figure 7).

Case 2. When $\lambda > 0$. Similar to Case 1, after solving the system of algebraic equations, we obtain

Case iv. If $A > 0$ and $B < 0$, then

$$a_1 = 0, \qquad b_1 = \pm\sqrt{\frac{-2A(A_1^2 + A_2^2)}{B}}, \qquad \lambda = A,\ \mu = 0.$$

Case v. If $A < 0$ and $B < 0$, then

$$a_1 = \pm\sqrt{\frac{-2}{B}}, \qquad b_1 = 0, \qquad \lambda = -\frac{A}{2}, \qquad \mu = 0.$$

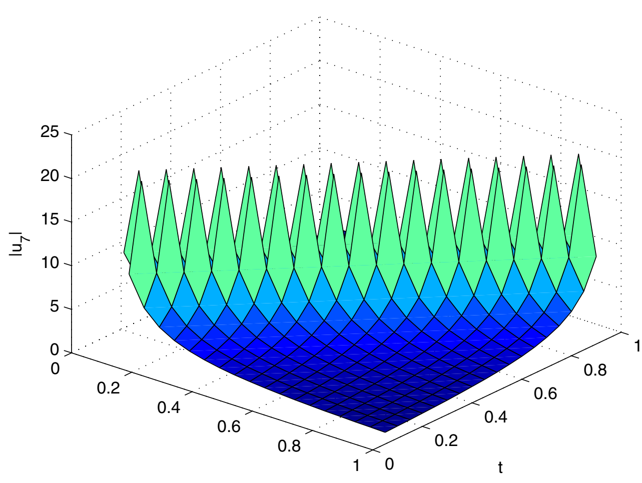

Figure 7. The graphics of rational function solution (3.11).

Case vi. If $A < 0$ and $B < 0$, then

$$a_1 = \pm\sqrt{\frac{-1}{2B}}, \qquad b_1 = \pm\sqrt{\frac{4A(A_1^2 + A_2^2) - \mu^2}{4AB}},$$

$$A_1^2 + A_2^2 \geq \frac{\mu^2}{4A}, \qquad \lambda = -2A, \qquad \mu = \text{an arbitrary constant}.$$

From the above cases, we obtain the trigonometric function solutions of NLSE (1.1) as follows:

Case 2.1. From eqs (3.1), (3.4) and Case iv, we get

$$\Phi = b_1\psi = b_1\frac{1}{G}$$

and

$$u_8 = \pm\sqrt{\frac{-2A(A_1^2 + A_2^2)}{B}} \times \frac{1}{A_1 \sinh\sqrt{A}(x - \eta t + \xi_1) + A_2 \cosh\sqrt{A}(x - \eta t + \xi_1)} \times \exp(i(kx + \omega t + \xi_0)).$$

That is,

$$|u_8| = \left|\sqrt{\frac{-2A(A_1^2 + A_2^2)}{B}} \times \frac{1}{A_1 \sinh\sqrt{A}(x - \eta t + \xi_1) + A_2 \cosh\sqrt{A}(x - \eta t + \xi_1)}\right|. \qquad (3.12)$$

Setting $A = 2, B = -4, A_1 = 1, A_2 = 1, \eta = 1, \xi_1 = 0$, we obtain the solution of eq. (3.12) (see figure 8).

Case 2.2. From eqs (3.1), (3.4) and Case v, we get

$$\Phi = a_1\phi = a_1\frac{G'}{G}$$

and

$$u_9 = \pm\sqrt{-\frac{2}{B}} \times \frac{A_1 \cosh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1) - A_2 \sinh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1)}{A_1 \sinh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1) + A_2 \cosh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1)} \times \exp(i(kx + \omega t + \xi_0)).$$

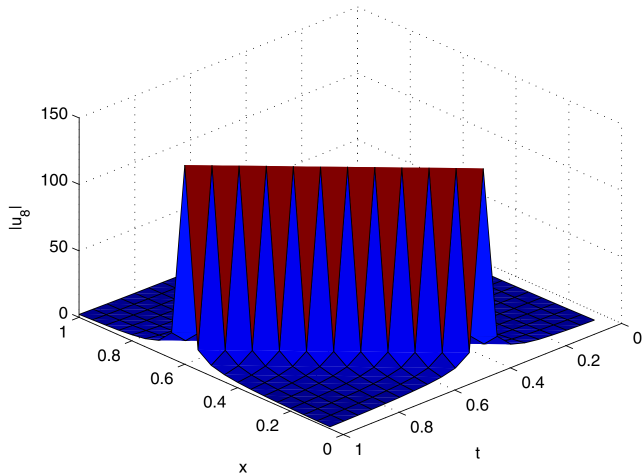

Figure 8. The graphics of trigonometric function solution (3.12).

That is,

$$|u_9| = \left| \sqrt{-\frac{2}{B}} \frac{A_1 \cosh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1) - A_2 \sinh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1)}{A_1 \sinh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1) + A_2 \cosh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1)} \right|. \tag{3.13}$$

Setting $A = -2$, $B = -2$, $A_1 = 1$, $A_2 = 1$, $\eta = 1$, $\xi_1 = 0$, we obtain the solution of eq. (3.13) (see figure 9).

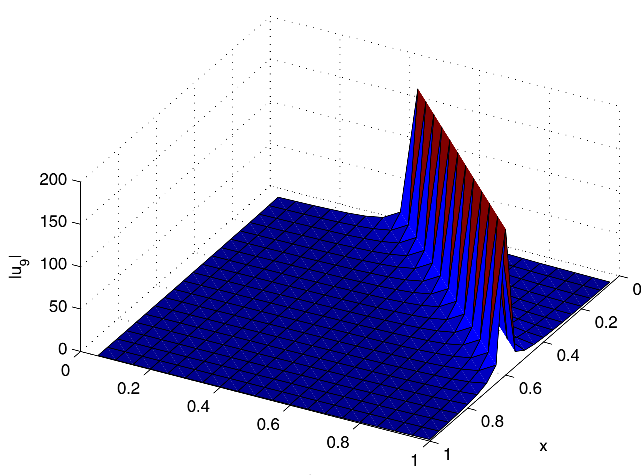

Figure 9. The graphics of trigonometric function solution (3.13).

Case 2.3. From eqs (3.1), (3.4) and Case vi, we get

$$\Phi = a_1\phi + b_1\psi = a_1\frac{G'}{G} + b_1\frac{1}{G}$$

and

$$u_{10} = \pm\sqrt{-\frac{1}{2B}}\frac{A_1\cosh\sqrt{-2A}(x-\eta t+\xi_1) - A_2\sinh\sqrt{-2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{-2A}(x-\eta t+\xi_1) + A_2\cosh\sqrt{-2A}(x-\eta t+\xi_1)}$$
$$\times\exp(i(kx+\omega t+\xi_0))$$
$$\pm\sqrt{-\frac{4A(A_1^2+A_2^2)-\mu^2}{4AB}}\frac{1}{A_1\sinh\sqrt{-2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{-2A}(x-\eta t+\xi_1)}$$
$$\times\exp(i(kx+\omega t+\xi_0)).$$

That is,

$$|u_{10}| = \left|\pm\sqrt{-\frac{1}{2B}}\frac{A_1\cosh\sqrt{-2A}(x-\eta t+\xi_1) - A_2\sinh\sqrt{-2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{-2A}(x-\eta t+\xi_1) + A_2\cosh\sqrt{-2A}(x-\eta t+\xi_1)}\right.$$
$$\times\exp(i(kx+\omega t+\xi_0))$$
$$\left.\pm\sqrt{-\frac{4A(A_1^2+A_2^2)-\mu^2}{4AB}}\frac{1}{A_1\sinh\sqrt{-2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{-2A}(x-\eta t+\xi_1)}\right|.$$
$$\times\exp(i(kx+\omega t+\xi_0)). \tag{3.14}$$

Setting $A = -\frac{1}{2}$, $B = -\frac{1}{2}$, $A_1 = 1$, $A_2 = 1$, $\mu = 1$, $\eta = 1$, $\xi_1 = 0$, and taking symbol $+$, we obtain the solution of eq. (3.14) (see figure 10).

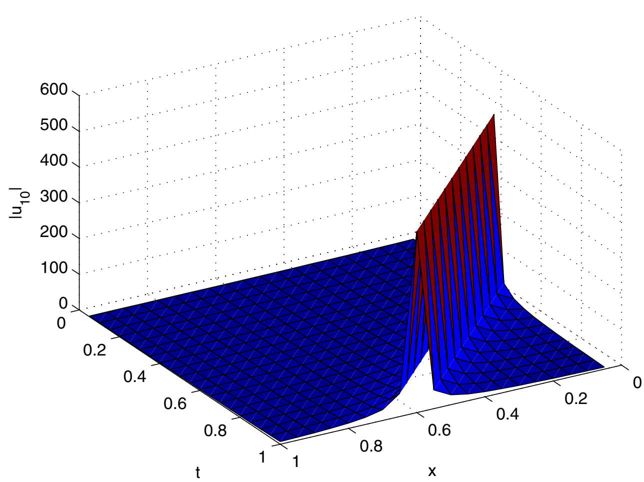

Figure 10. The graphics of trigonometric function solution (3.14).

Case 3. When $\lambda = 0$, by analogous computations, we obtain that if $A = 0$ and $B < 0$, then

$$a_1 = 0, \qquad b_1 = \pm\sqrt{\frac{-2}{B}}, \qquad \mu = 0.$$

From the above case, we obtain the rational function solutions of (1.1). From eqs (2.6), (3.1) and (3.4), we get

$$\Phi = b_1\psi = b_1\frac{1}{G}$$

and

$$u_{11} = \pm\sqrt{\frac{-2}{B}}\frac{A_1}{A_1(x-\eta t+\xi_1)+A_2}\exp(i(kx+\omega t+\xi_0)).$$

That is,

$$|u_{11}| = \left|\sqrt{\frac{-2}{B}}\frac{A_1}{A_1(x-\eta t+\xi_1)+A_2}\right|. \tag{3.15}$$

Setting $A = 0, B = -2, A_1 = 1, A_2 = 1, \eta = 1, \xi_1 = 0$, we obtain the solution of eq. (3.15) (see figure 11).

Remark 3.3. Indeed, the solutions in figures 5, 7 and 11 are called smooth kink or antikink solutions. From the book of Li and Dai [43], we have the following facts: suppose that $\phi(x-ct) = \phi(\xi)$ is a smooth solution of a travelling wave equation with smoothness for $\xi \in (-\infty, \infty)$ and $\phi(\xi) \to \alpha$ (as $\xi \to \infty$) and $\phi(\xi) \to \beta$ (as $\xi \to -\infty$). It is well known that: (i) $\phi(x-ct)$ is called a smooth solitary wave solution if $\alpha = \beta$, (ii) $\phi(x-ct)$ is called a smooth kink or antikink solution if $\alpha \neq \beta$. Usually, a smooth solitary wave solution of NLPDEs corresponds to a smooth homoclinic orbit of travelling wave solution. A smooth kink or antikink solution corresponds to a smooth heteroclinic orbit of travelling wave solution. We can also see refs [6,7].

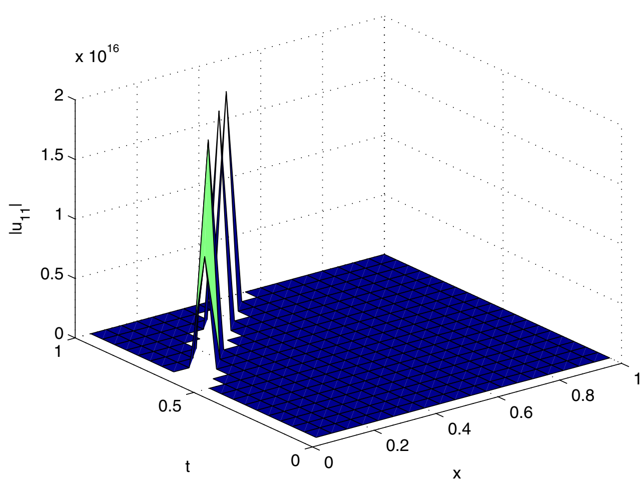

Figure 11. The graphics of rational function solution (3.15).

4. Conclusion and discussion

In this paper, we propose the extended (G'/G)-expansion method for finding multiple exact solutions involving arbitrary parameters for the perturbed NLSE with Kerr law nonlinearity. By using this method and symbolic computation, we have found new types of exact solutions for the NLSE (1.1). To our knowledge, these results have not been reported in the literature. When $\mu = 0$ in (2.1) and $b_i = 0$ in expansion (2.10), our proposed method is the (G'/G)-expansion method. It is easy to check that the solutions (3.6), (3.13) and (3.15) are in full agreement with the results obtained by using the (G'/G)-expansion method, see ref. [22].

We plot all the figures to describe the propagations of all kinds of travelling wave solutions expressed by the hyperbolic functions, trigonometric functions and rational functions. These travelling solutions include existing and some new ones. Especially, some travelling wave solutions have not been reported in the literature, such as refs [3,4,6,8,22]. More precisely, in our paper, the following travelling wave solutions have been reported: $|u_2|$ (see refs [22], p. 4262) – the modified (G'/G)-expansion method), $|u_4|$ (see ref. [6], p. 1279) – the dynamical system approach and the bifurcation method), $|u_5|$ (see ref. [8], p. 769) – the infinite series method and Jacobi elliptic function expansion method), $|u_6|$ ((see ref. [6], p. 1278, ref. [4], p. 3101) – trigonometric function series method and ref. [8], p. 767), $|u_{11}|$ (see ref. [22], p. 4262). But, solutions $|u_1|$, $|u_3|$, $|u_7|$, $|u_8|$, $|u_9|$, $|u_{10}|$ are new travelling wave solutions by using the extension of (G'/G)-expansion method.

We can also compare our proposed method with other methods such as the extended hyperbolic function method. In the latter method, the projective Raccati equations are chosen as its subsidiary ODE to construct the solutions of NLPDEs. Although in our contribution, we have seen that two variable $\phi = G'/G$ and $\psi = 1/G$ given in (2.2) also satisfy the projective Raccati equations $\phi' = -\phi^2 + \mu\psi - \lambda$, $\psi' = -\phi\psi$ given in (2.3), it is worthy of note that we do not use the special solutions of eqs (2.3) at all. Instead, we directly use the general solutions of the second-order LODE (2.1), which is well known to researchers, to construct the solutions of NLPDEs. Thus, our proposed method has its own advantages: direct, concise and elementary. More importantly, we believe that this method can be used for many other NLPDEs in mathematical physics.

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