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Effects of complex parameters on classical trajectories of Hamiltonian systems

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Abstract. Anderson *et al* have shown that for complex energies, the classical trajectories of real quartic potentials are closed and periodic only on a discrete set of eigencurves. Moreover, recently it was revealed that when time is complex t ($t = t_r e^{i\theta_\tau}$), certain real Hermitian systems possess close periodic trajectories only for a discrete set of values of θ_τ. On the other hand, it is generally true that even for real energies, classical trajectories of non-PT symmetric Hamiltonians with complex parameters are mostly non-periodic and open. In this paper, we show that for given real energy, the classical trajectories of complex quartic Hamiltonians $H = p^2 + ax^4 + bx^k$ (where a is real, b is complex and $k = 1$ or 2) are closed and periodic only for a discrete set of parameter curves in the complex b-plane. It was further found that given complex parameter b, the classical trajectories are periodic for a discrete set of real energies (i.e., classical energy gets discretized or quantized by imposing the condition that trajectories are periodic and closed). Moreover, we show that for real and positive energies (continuous), the classical trajectories of complex Hamiltonian $H = p^2 + \mu x^4$, ($\mu = \mu_r e^{i\theta}$) are periodic when $\theta = 4\tan^{-1}[(n/(2m+n))]$ for $\forall\, n$ and $m \in \mathbb{Z}$.

Keywords. Classical trajectories; non-Hermitian; periodic trajectories.

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1. Introduction

In recent years, classical behaviour of non-Hermitian Hamiltonian systems [1–5] as well as classical motion of Hermitian systems for complex energies [6–16] have attracted much interest. Investigation of classical mechanics in the complex domain is useful for understanding various classical and quantum mechanical phenomena such as barrier tunnelling, dynamical tunnelling [17,18], classical and quantum chaos [7,19], quantum correspondence principle, complex forms of uncertainty relations and the semiclassical limit of complex quantum field theories.

Since the earlier work on classical motion of non-Hermitian systems [6,7], several interesting results have been found on the various aspects of the subject [8–16]. Numerical and analytical investigations have revealed that when energies are real, classical trajectories of complex PT symmetric non-Hermitian systems are closed and periodic. However, when energies of these systems are complex, the periodic trajectories usually become non-periodic and open [15,19]. Recently, it was shown that even though most of the trajectories corresponding to complex energies are open and non-periodic, for some systems, there are special discrete sets of curves in the complex-energy plane for which the trajectories are periodic [20]. On the other hand, in non-Hermitian and non-PT symmetric Hamiltonian systems, even for real energies, almost all trajectories except a few are non-periodic and open. It was also shown recently that when time is taken as a complex quantity with a specific fixed-phase angle or as a specific complex function, non-periodic trajectories of 1D Hamiltonian systems become periodic and closed [21].

In this paper, we investigate the classical trajectories from a different point of view. Here we examine clasical behaviour of the complex Hamiltonian $H = p^2 + ax^4 + bx^k$ (where $k = 1, 2$, and a is real, such that H is not PT symmetric) for complex parameter b and real energy E. The outline of the paper is as follows. In §2, analytic expressions for complex trajectories are derived. Expressions for periods of the periodic trajectories as well as time taken by unbounded trajectories to escape to infinity are found in terms of b and energy E. We shall show that for the given real energy, the classical trajectories of the above quartic Hamiltonian are open except for a discrete set of parameter values in the complex b-plane. In §3, we study how trajectories behave when energy is real and b is a fixed complex parameter. The classical trajectories of complex Hamiltonian $H = p^2 + \mu x^4$ ($\mu = r\mathrm{e}^{i\theta}$) is investigated for real energies in §4 and concluding remarks are given in §5.

2. Classical trajectories of $H = p^2 + ax^4 + bx^k$

In this section, first we study in detail the classical motion of the complex quartic anharmonic oscillator. We assume that the Hamiltonian has the form

$$H = p^2 + ax^4 + bx^k, \tag{1}$$

where a is a real positive constant, b is a complex constant and $k = 1$ or 2. First we derive expressions for $x(t)$ and the period for the above Hamiltonian. When it is needed, value of k is chosen as 1 or 2. Throughout this paper, mass of the particle is taken as half (i.e., $2m = 1$).

The equation of motion is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p = 2\sqrt{E - ax^4 - bx^k}. \tag{2}$$

The turning points of this system are taken as x_0, x_1, x_2 and x_3 and by integrating eq. (2) we have

$$\int \frac{\mathrm{d}x}{\sqrt{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}} = 2\sqrt{a}\mathrm{e}^{i\pi/2}t + c, \tag{3}$$

where c is the constant of integration, which depends on initial conditions. The left-hand side of the above equation is an elliptic integral of the first kind and hence eq. (3) becomes

$$\frac{2}{\sqrt{(x_0-x_2)(x_1-x_3)}} F\left(\sin^{-1}\left[\sqrt{\frac{(x-x_1)(x_0-x_2)}{(x-x_0)(x_1-x_2)}}\right], \frac{(x_1-x_2)(x_0-x_3)}{(x_0-x_2)(x_1-x_3)}\right)$$

$$= 2\sqrt{a}\mathrm{e}^{i\pi/2}t + c, \tag{4}$$

where F is an elliptic function. We invert the above equation in terms of Jacobian elliptic function 'sn' as

$$x(t) = \frac{x_1(x_0-x_2) - x_0(x_1-x_2)\mathrm{sn}^2(u)}{(x_0-x_2)-(x_1-x_2)\mathrm{sn}^2(u)}, \tag{5}$$

where

$$u = \sqrt{a(x_0-x_2)(x_1-x_3)}\mathrm{e}^{i\pi/2}t + \alpha$$

and modulus

$$\kappa = \left[\frac{(x_1-x_2)(x_0-x_3)}{(x_0-x_2)(x_1-x_3)}\right]^{1/2}$$

and α is an arbitrary constant, which is determined by the initial conditions. Note that $x(t)$ in the above equation is still a solution of (3), when x_0, x_1, x_2 and x_3 are cyclically changed (e.g., $x_3 \to x_0 \to x_1 \to x_2 \to x_3$). To understand how the trajectories behave, we need to recognize the periodic, bounded and unbounded properties of the function $x(t)$. First, we find complementary modulus κ' and complete elliptic functions K and K'. They are defined by

$$\kappa'^2 = 1-\kappa^2 = \frac{(x_0-x_1)(x_2-x_3)}{(x_0-x_2)(x_1-x_3)} \tag{6}$$

$$K = \int_0^{\pi/2} (1-\kappa^2\sin^2(\phi))^{-1/2}\mathrm{d}\phi \tag{7}$$

$$K' = \int_0^1 \left(1-t^2\right)^{-1/2}(1-\kappa'^2 t^2)^{-1/2}\mathrm{d}t. \tag{8}$$

K and K' are evaluated directly from the above equations and they are independent of phase angle θ.

The trajectory $x(t)$ is given by

$$x(t) = \frac{x_1(x_0-x_2) - x_0(x_1-x_2)\mathrm{sn}^2(u)}{(x_0-x_2)-(x_1-x_2)\mathrm{sn}^2(u)}.$$

The condition for trajectory becomes unbounded and the particle escapes to infinity is

$$(x_0-x_2)-(x_1-x_2)\mathrm{sn}^2(u) = 0 \tag{9}$$

where

$$u = \sqrt{a(x_0-x_2)(x_1-x_3)}\mathrm{e}^{i\pi/2}t + \alpha$$

satisfied for some real positive t and the time taken for the particle to escape to ∞ is given by

$$T_\infty = \frac{\left(\mathrm{sn}^{-1}(z_0) - \alpha\right) \mathrm{e}^{-i\pi/2}}{\sqrt{a\,(x_0 - x_2)\,(x_1 - x_3)}} \tag{10}$$

where

$$z_0 = \sqrt{\frac{(x_0 - x_2)}{(x_1 - x_2)}}.$$

'sn' is doubly periodic with period $4mK + 2niK'$, where m and n are integers. Therefore, the condition for the trajectory to become periodic and particle does not escape to infinity is

$$\sqrt{a\,(x_0 - x_2)\,(x_1 - x_3)}\mathrm{e}^{i\pi/2} t = 4mK + 2niK'; \quad m,\ n \in \mathbb{Z} \tag{11}$$

and $t < T_\infty$.

Then the trajectory is periodic with the period.

$$T_p = \frac{\left(4mK + 2niK'\right) \mathrm{e}^{-i\pi/2}}{\sqrt{a\,(x_0 - x_2)\,(x_1 - x_3)}}. \tag{12}$$

Note that if $T_p > T_\infty$, the trajectory is still non-periodic. By imposing the condition that $\mathrm{Im}\left(T_p\right) = 0$, we have

$$r \equiv \frac{n}{m} = \frac{\mathrm{Im}[2iK/z]}{\mathrm{Im}\left[K'/z\right]}; \qquad m,\ n \in \mathbb{Z}, \tag{13}$$

where

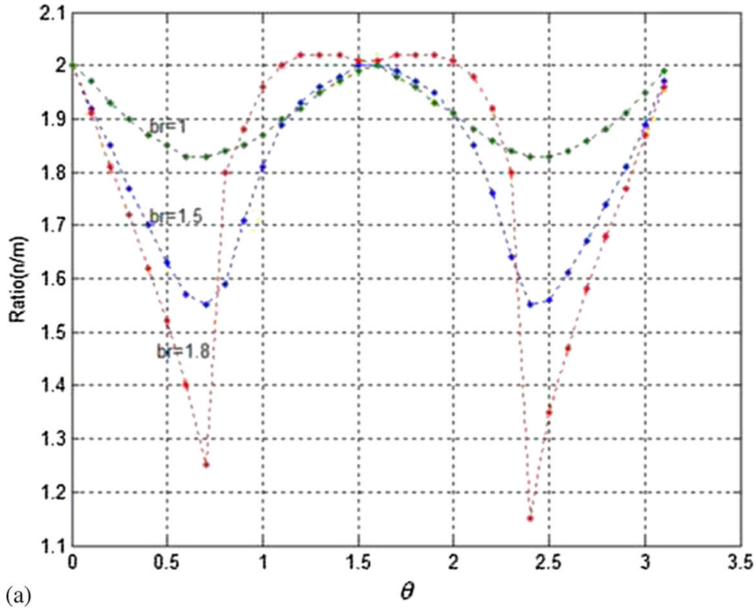
$$z = \sqrt{a\,(x_0 - x_2)\,(x_1 - x_3)}.$$

As n and m are integers and the energy E is fixed, r is rational and eq. (13) provides a discrete set of parameter values in the complex b plane for which classical trajectories are periodic. Let $b = b_r \mathrm{e}^{i\theta}$. Figures 1a and 1b show how the ratio r varies with discrete values of θ for $k = 1$ and $k = 2$, respectively. Without loss of generality, the energy E is taken as unity as it is real. The results can be generalized for any real energy E by simple rescaling of x and t.

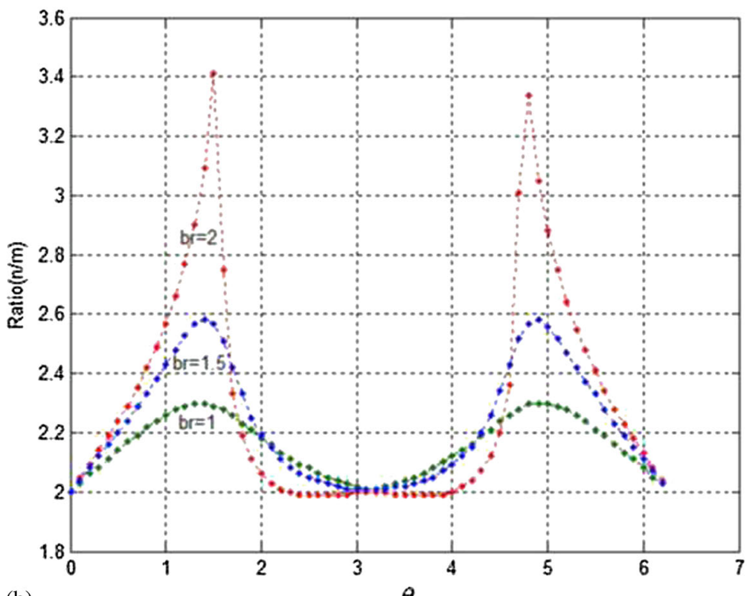
3. Discretization of classical energy

Next we consider the case when parameter b is a fixed complex number and E is a variable (assume $a = 1$ and $b = 1 + i$). As a result, eq. (13) allows only a discrete set of values of E for which trajectories are periodic. It was found that these discrete values of E can be either real or complex, satisfying the condition (13). Tables 1 and 2 show some real discrete values of E, which make trajectories periodic when $k = 1$ and $k = 2$, respectively. Figures 2 and 3 show the periodic trajectories of systems $k = 1$ and $k = 2$ for two values of real energies.

Moreover, it was found that if energy E corresponds to the periodic trajectories of $p^2 + ax^4 + bx$ then $-E$ will be the energy that makes trajectories of $p^2 - ax^4 + i\bar{b}x$ ($\bar{b}$ is



(a)



(b)

Figure 1. By applying the condition that classical trajectories are periodic, we obtain discrete values of θ for a fixed value of b_r. **(a)** Shows how the ratio r varies with complex phase angle θ of the potential $V(x) = x^4 + b_r e^{i\theta} x$. In order to have periodic trajectories, r ($r \equiv n/m$) has to be rational and hence only discrete values of θ satisfy the condition (13). Each point in the graph represents such a value. **(b)** Same as (**a**) but for the potential $V(x) = x^4 + b_r e^{i\theta} x^2$.

Table 1. Classical energy spectrum corresponding to periodic trajectories of $V(x) = x^4 + (1+i)x$ for various (m, n).

m	n	E
1	1	0.27499
1	2	0.71624
1	3	0.78605
2	3	0.60480
2	5	0.74280
2	1	−0.28103
3	1	−0.53968
3	2	−0.07449
5	2	−0.42562

Table 2. Classical energy spectrum corresponding to periodic trajectories of $V(x) = x^4 + (1+i)x^2$ for various (m, n).

m	n	E
1	1	−0.02143
1	2	−0.16951
1	3	−0.32417
2	3	−0.08940
2	5	−0.24827
2	1	1.458020
3	1	2.99725
3	2	0.81963
5	2	2.17849

the complex conjugate of b) periodic. Further, E and $-E$ are solutions corresponding to the same n and m in the periodic condition (13) for these two Hamiltonians, respectively. In other words, if S_E is the discrete set of energies for which classical trajectories of $p^2 + ax^4 + bx$ are periodic then S_{-E} is the set of energies for which trajectories of $p^2 - ax^4 + i\bar{b}x$ are periodic. Figures 4a and 4b show two periodic trajectories illustrating the above claim.

4. Periodic classical trajectories of $H = p^2 + \mu_r e^{i\theta} x^4$

Next we assume that $a = \mu$ and $b = 0$ in the Hamiltonian (1). Then new Hamiltonian has the form

$$H = p^2 + \mu x^4, \tag{14}$$

where μ is complex and $\mu = \mu_r e^{i\theta}$.

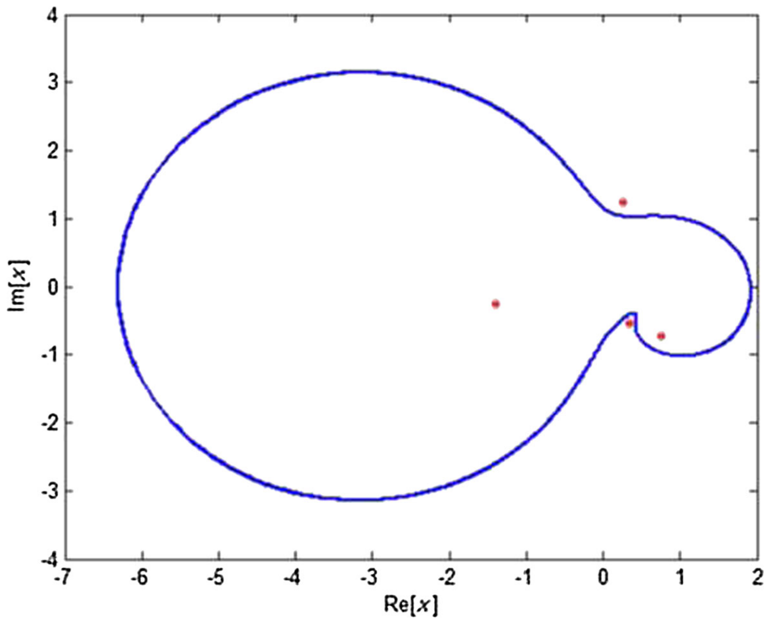


Figure 2. A periodic trajectory corresponding to $(m, n) = (1, 2)$ for the quartic potential $V(x) = x^4 + (1 + i)x$ with real energy $E = 0.71624$.

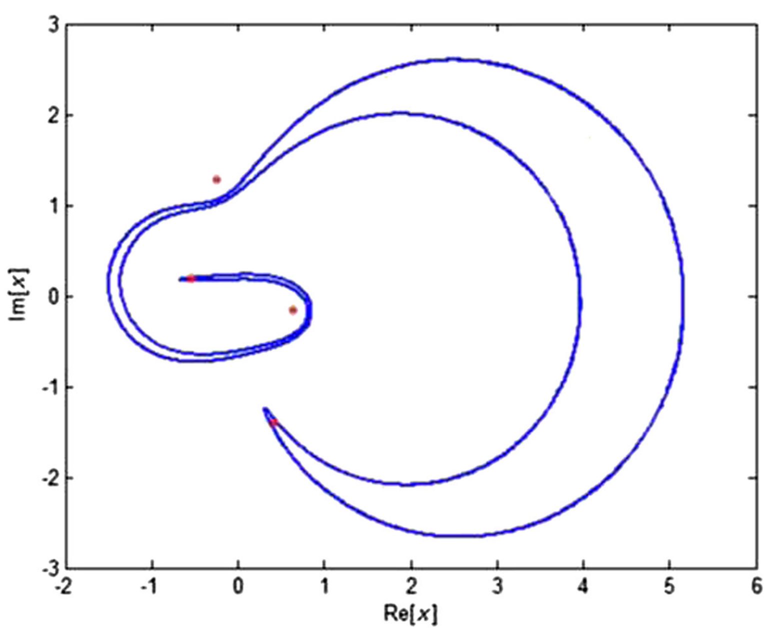
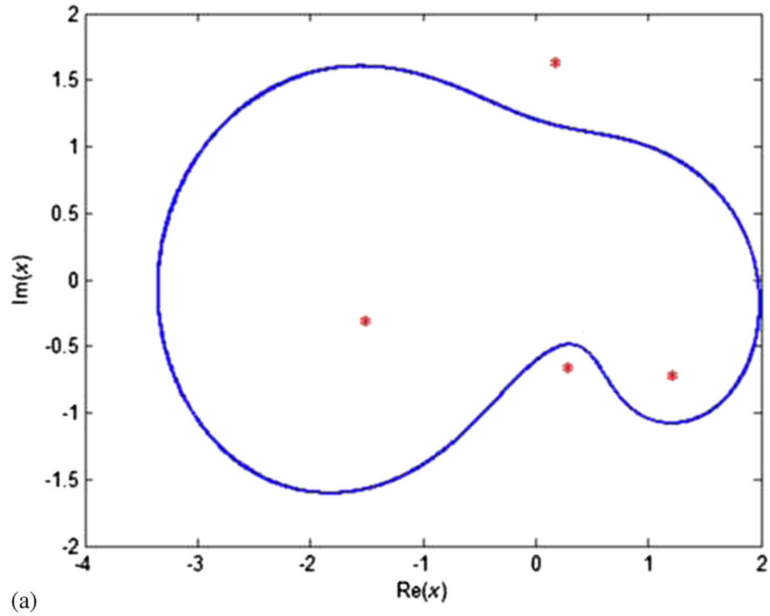
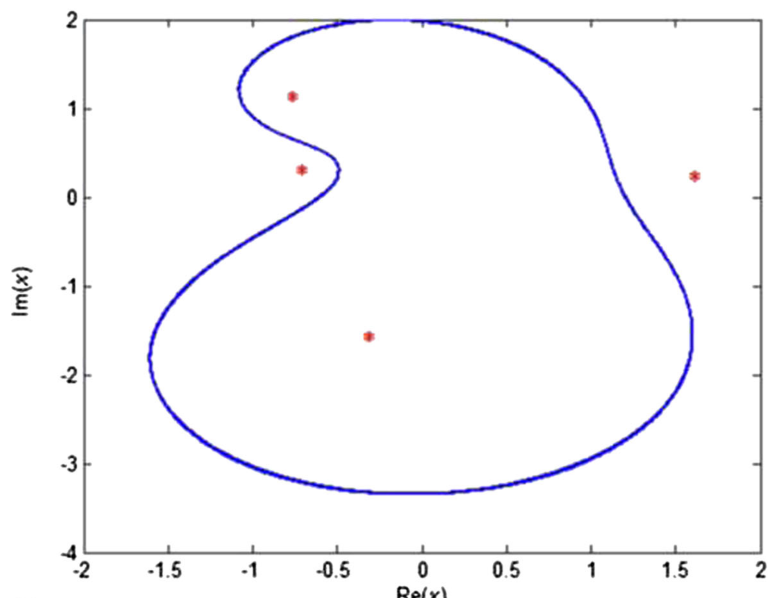


Figure 3. A periodic trajectory for the quartic potential $V(x) = x^4 + (1 + i)x^2$ corresponding to $(m, n) = (3, 2)$ with real energy $E = 0.81963$.



(a)



(b)

Figure 4. Typical periodic classical trajectories of the potentials (**a**) $V(x) = x^4 + (2+3i)x$ and (**b**) $V(x) = -x^4 + (3+2i)x$. Energies of trajectories corresponding to figures (**a**) and (**b**) are $E = 2.42227$ and $E = -2.42227$, respectively. The four turning points are marked as dots.

The equation of motion is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\sqrt{E - \mu x^4}, \tag{15}$$

where E is the total energy. Following the same procedure as in §2, we obtain required equations. By integrating (15) we have

$$\int \frac{\mathrm{d}x}{\sqrt{E - \mu x^4}} = 2t + c, \tag{16}$$

where c is the constant of integration, which depends on initial conditions. The left-hand side of the above equation is an elliptic integral of the first kind and hence eq. (16) becomes

$$F\left(\sin^{-1}\left[\left(\frac{\mu}{E}\right)^{1/4} x(t)\right], -1\right) = 2\,(\mu E)^{1/4}\, t + \alpha, \tag{17}$$

where $\alpha = (\mu E)^{1/4}\, c$ and F is an elliptic function. We invert the above equation in terms of Jacobian elliptic function 'sn' as

$$x\,(t) = \left(\frac{E}{\mu}\right)^{1/4} \mathrm{sn}\left(2\,(\mu E)^{1/4}\, t + \alpha;\ -1\right). \tag{18}$$

Note that modulus $\kappa^2 = -1$ for the above problem. K and K' are defined in (7) and (8) and

$$\kappa'^2 = 1 - \kappa^2 = 2. \tag{19}$$

Then K and K' are obtained as

$$K = \frac{\sqrt{\pi}\,\Gamma\,(1/4)}{4\Gamma\,(3/4)} \tag{20}$$

$$K' = \frac{\sqrt{\pi}\,\Gamma\,(1/4)}{4\Gamma\,(3/4)}\,(1 - i)\,. \tag{21}$$

As in the previous sections, the condition of trajectory become periodic and particle does not escape to infinity is

$$2\,(\mu E)^{1/4}\, t = 4mK + 2niK'; \quad m,\ n \in \mathbb{Z}. \tag{22}$$

Then the trajectory is periodic with the period

$$T_p(\mu_r) = \frac{2mK + niK'}{(\mu E)^{1/4}}. \tag{23}$$

As $\mu = \mu_r \mathrm{e}^{i\theta}$

$$T_p(\mu_r) = \frac{K}{(\mu_r E)^{1/4}}\,[(2m + n) + in]\,(\cos\,(\theta/4) - i\,\sin\,(\theta/4))\,. \tag{24}$$

$$T_p(\mu_r) = \frac{K}{(\mu_r E)^{1/4}}\Big[((2m + n)\cos\,(\theta/4) + n\sin\,(\theta/4)) + i\,(n\cos\,(\theta/4) - (2m + n)\sin\,(\theta/4))\Big]\,. \tag{25}$$

As K is real and E is real and positive, by imposing the condition that $\mathrm{Im}\left(T_p\right) = 0$, we have

$$\frac{m}{n} = \frac{\cot(\theta/4) - 1}{2}; \quad m,\ n \in \mathbb{Z} \tag{26}$$

or

$$\theta = 4\tan^{-1}\left[\frac{n}{2m+n}\right]; \quad m,\ n \in \mathbb{Z}. \tag{27}$$

When $n = 0$ and $m \neq 0$, $H = p^2 + \mu_r x^4$ and it is Hermitian. Then H possesses periodic trajectories and the period $T_p(\mu_r)$ becomes

$$T_p(\mu_r) = \frac{2mK}{(\mu_r E)^{1/4}}$$

but the period corresponds to the minimum non-zero m and the resulting period is

$$T_{p+}(\mu_r) = \frac{\sqrt{\pi}\,\Gamma(1/4)}{2\,(\mu_r E)^{1/4}\,\Gamma(3/4)}. \tag{28}$$

On the other hand, when $n \neq 0$ and $m = 0$, $H = p^2 - \mu_r x^4$ and it is the non-Hermitian 'wrong sign' potential, which also possesses periodic trajectories. The period is

$$T_{p-}(\mu_r) = \frac{\sqrt{\pi}\,\Gamma(1/4)}{2\sqrt{2}\,(\mu_r E)^{1/4}\,\Gamma(3/4)} = \frac{\sqrt{\pi}\,\Gamma(1/4)}{2\,(4\mu_r E)^{1/4}\,\Gamma(3/4)}. \tag{29}$$

It is evident from eqs (28) and (29) that the Hamiltonians $p^2 + 4\mu_r x^4$ and $p^2 - \mu_r x^4$ have the same classical period (i.e., $T_{p+}(4\mu_r) = T_{p-}(\mu_r)$). Note that these two Hamiltonians are the classical limits of the quantum mechanical isospectral Hamiltonians as shown in [22–26].

5. Concluding remarks

In this paper, we have presented three main results. First, for the given real energy, the classical trajectories of quartic Hamiltonians $H = p^2 + ax^4 + bx^k$ (where a is real, b is complex and $k = 1$ or 2) are closed and periodic only for a discrete set of parameter curves in the complex b-plane.

The second result is that given complex parameter b, the classical trajectories are found to be periodic only for a discrete set of real energies. As a result, real classical energies get discretized or quantized by the condition that trajectories are periodic and closed. This result is analogous to what was obtained by Anderson *et al* in [20] for real potential parameters with complex E (here it is for complex potential parameters with real energies). Further, we showed that if $S(E)$ is the discrete set of energies for which classical trajectories of $p^2 + ax^4 + bx$ are periodic, then $S(-E)$ is the set of energies for which trajectories of $p^2 - ax^4 + i\bar{b}x$ are periodic. We presented our results with illustrations. It is important to note that when b is complex and not pure imaginary, the entire quantum eigenspectrum corresponding to the Hamiltonian H is complex and eigenenergies do not come as complex conjugate pairs. Therefore, H cannot be pseudo-Hermitian and cannot have any antilinear symmetry.

As the third result, we showed that for real energies, the classical trajectories of complex Hamiltonian $H = p^2 + \mu x^4$ ($\mu = re^{i\theta}$) are periodic only for discrete values of θ satisfying the condition $\theta = 4 \tan^{-1}[(n/(2m + n))]$ for n and $m \in \mathbb{Z}$. Further, it was found that Hamiltonians $p^2 + 4\mu_r x^4$ and $p^2 - \mu_r x^4$, which are the classical limits of the quantum mechanical isospectral Hamiltonians introduced in [22–26], have the same classical period.

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