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Quantum restoration of broken symmetry in one-dimensional loop space

PINAKI PATRA*, TANMAY MANDAL and JYOTI PRASAD SAHA
Department of Physics, University of Kalyani, Kalyani 741 235, India
*Corresponding author. E-mail: monk.ju@gmail.com

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Abstract. For one-dimensional loop space, a nonlinear nonlocal transformation of fields is given to make the action of the self-interacting quantum field to the free one. A specific type of classically broken symmetry is restored in quantum theory. One-dimensional sine-Gordon system and sech interactions are treated as the explicit examples.

Keywords. Non-local transformation; broken symmetry; sine-Gordon; sech interaction.

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1. Introduction

Though quantum theory and classical theory are completely different from the philosophical point of view, it is a long-time belief among many people (including the present authors) that the classical property of a system and the classical limit of the corresponding quantum system will be identical [1]. In the quantum field theory, one has to estimate the probability of occurrence of some particular event. So, it is necessary to know about the probability measure of the concerned scenario. Sometimes, for the description of a system, the knowledge of the probability measure is sufficient. But, for the interacting theory, it is difficult (most of the cases impossible) to have exact knowledge about the system. Recently, Belokurov *et al* [2–4] gave an idea on how to handle the interacting system for a one-dimensional case. With the help of a nonlinear nonlocal transformation of the fields, they explicitly studied the ϕ^4 interaction theory and were able to transform the theory as a free field theory. But, they landed in a strange situation. The classical theory and the classical limit of the corresponding quantum theory are completely different before and after the transformation. As a result of the transformation, singularity property of the two fields (the original and the transformed ones) are completely different. In this article, we have proposed a nonlinear nonlocal transformation of the fields for the general functional form of the interaction term and, as a result, we got a stand point similar to that of Belokurov *et al*. Also, it can be observed that sometimes the original classical system is symmetric with respect to the transformation $T(\phi) = -\phi$ while the symmetry is not preserved for the transformed classical system. But, as for the quantum system, we have to deal only with the functional Weiner measure; from the discussion of this article we shall see that the functional measure is invariant with respect to the transformation. In this sense, the classically broken symmetry is restored in the quantum case.

In the next section, we have described the general formalism and then subsequently two special cases are studied.

2. General construction

If $\mathcal{B}$ is the Borel σ-field on $\Omega = \mathcal{C}[0,1]$ and $\mathcal{B}_t$ is the σ-field generated by $\phi(s)$ for $0 \le s \le t$ and if we consider the Wiener measure P on $(\Omega, \mathcal{B})$, then one can identify that $\phi(t)$ is a martingale with respect to $(\Omega, \mathcal{B}_t, P)$. So, for a meromorphic function $f(\phi(t))$, one can consider the Wiener measure [5]

$$P = \exp\left\{\int_{t=0}^{t=1}\left[-\frac{1}{2}\dot{\phi}^2(t) - \frac{\lambda^2}{4}f^2(\phi(t))\right]\mathrm{d}t\right\}\mathrm{d}\phi, \tag{1}$$

where λ is the parameter, the dimension of which depends on the form of $f(\phi)$. Now, for our concerned loop space Ω, we can write the following proposition.

PROPOSITION 1

For the nonlocal transformation,

$$\chi(t) = \phi(t) + \frac{\lambda}{\sqrt{2}}\int_{\tau=0}^{\tau=t} f(\phi)\mathrm{d}\tau \tag{2}$$

the following equality holds:

$$\begin{aligned}\int \exp\left(-\frac{1}{2}\int_{t=0}^{t=1}\dot{\chi}^2\mathrm{d}t\right)\mathrm{d}\chi \\ = \int \exp\left\{\int_{t=0}^{t=1}\left[-\frac{1}{2}\dot{\phi}^2(t) - \frac{\lambda^2}{4}f^2(\phi(t)) + \frac{\lambda}{2\sqrt{2}}\frac{\partial^2 h}{\partial\phi^2}\right]\mathrm{d}t \\ -\frac{\lambda}{\sqrt{2}}(h(\phi(1)) - h(\phi(0)))\right\}\mathrm{d}\phi\end{aligned} \tag{3}$$

if

$$h(\phi) = \int f(\phi)\mathrm{d}\phi. \tag{4}$$

Proof

To prove the proposition it is sufficient to note that with the help of the well-known procedure of the Ito stochastic integration [5], we can write

$$\int_0^1 \dot{\phi} f(\phi)\mathrm{d}t = h(\phi(1)) - h(\phi(0)) - \frac{1}{2}\int_0^1 \frac{\partial^2 h}{\partial\phi^2}\mathrm{d}t; \quad h(\phi) = \int f(\phi)\mathrm{d}\phi. \tag{5}$$

After noting the above-mentioned result, the proof is rather straightforward. We use the above-mentioned transformation of fields and write

$$\dot{\chi} = \dot{\phi} + \frac{\lambda}{\sqrt{2}} f(\phi)$$

and put the expression in the Weiner measure P.

Note that, though the above-mentioned equality holds, $\phi(t)$ and $\chi(t)$ belong to different functional space and the normalization of the spaces of ϕ and χ is different. That is, if $\chi(t) \in \mathcal{C}[0, 1]$ and $\phi(t) \in X$, then $X \neq \mathcal{C}[0, 1]$; in fact, the space of ϕ is more singular than that of χ. In [3], Belokurov *et al* have constructed the detailed singularity property for $f(\phi) = \phi^4$. From that discussion we can intuitively argue that, the space of χ is more singular than that of ϕ. But the exact study of singularity for the general case $f(\phi)$ is beyond the knowledge and capability of the present authors.

If we set $\exp\{-\frac{1}{2}\dot{\chi}^2\}\,\mathrm{d}\chi$ as the measure on $\mathcal{C}[0, 1]$, then obviously, $\exp\{-\frac{1}{2}\dot{\phi}^2\}\,\mathrm{d}\phi$ is not the measure on X. But we can set

$$\exp\left\{\int_{t=0}^{t=1}\left[-\frac{1}{2}\dot{\phi}^2(t) - \frac{\lambda^2}{4} f^2(\phi(t)) + \frac{\lambda}{2\sqrt{2}}\frac{\partial^2 h}{\partial \phi^2}\right]\mathrm{d}t - \frac{\lambda}{\sqrt{2}}(h(\phi(1)) - h(\phi(0)))\right\}\mathrm{d}\phi$$

as the measure on X. The above-mentioned identification leads to the following interesting proposition. □

PROPOSITION 2

If the function $f(\phi)$ is an even function of ϕ, then the classical action and the classical limit of the corresponding quantum action show different symmetry properties under the symmetry transformation $T(\phi) = -\phi$. Moreover, if $f(\phi)$ is an odd function of ϕ, then the classical action and the classical limit of the corresponding quantum action show similar symmetry property under the symmetry transformation $T(\phi) = -\phi$.

Proof
For the classical action

$$A_{\text{old}} = \int_{t=0}^{t=1}\left[\frac{1}{2}\dot{\phi}^2(t) + \frac{\lambda^2}{4} f^2(\phi(t))\right]\mathrm{d}t \tag{6}$$

the equation of motion takes the form

$$\ddot{\phi} = \frac{\lambda^2}{2} f(\phi)\frac{\partial f(\phi)}{\partial \phi}, \tag{7}$$

which is symmetric under the transformation $T(\phi) = -\phi$ for any of the two cases of odd and even functional forms of $f(\phi)$.

But, for the classical action $A_{\text{new}} = A_{\text{old}} - A_{\text{extra}}$, where

$$A_{\text{extra}} = \frac{\lambda}{2\sqrt{2}}\int_{t=0}^{t=1}\frac{\partial f(\phi)}{\partial \phi}\mathrm{d}t, \tag{8}$$

the equation of motion

$$\ddot{\phi} = \frac{\lambda^2}{2} f(\phi) \frac{\partial f(\phi)}{\partial \phi} - \frac{\lambda}{2\sqrt{2}} \frac{\partial^2 f(\phi)}{\partial \phi^2} \tag{9}$$

is not symmetric for the case of even function $f(\phi)$ under the transformation T due to the presence of a second derivative term (as the second derivative of the even function remains even function), whereas for the case of odd function $f(\phi)$, the symmetry is not broken (as the second derivative of the odd function remains odd function).

Now, the corresponding quantum theory deals with the functional measure $\int \exp\{-A_{\text{new}}(\phi)\}\text{d}\phi$. Because of the equality of the functional measure as stated in Proposition 1, the symmetry is restored in quantum case even after the nonlinear nonlocal transformation (2).

The above-mentioned argument completes the proof. From this perspective, we get another interesting property. □

The equation of motion for the action of the left side of eq. (3), i.e., $A_\chi = -\frac{1}{2}\int \dot{\chi}^2 \text{d}t$, is

$$\ddot{\chi} = 0 \Longrightarrow \dot{\chi} = \text{constant} = \alpha(\text{let}) \tag{10}$$

which implies (by virtue of the transformation (2))

$$\dot{\phi} = \alpha - \frac{\lambda}{\sqrt{2}} f(\phi). \tag{11}$$

Also, from the equation of motion in (6) we get

$$\dot{\phi} = \pm\beta f(\phi) \qquad \text{where } \beta = \text{some constant} \tag{12}$$

which can be identified with eq. (10), if we set $\alpha = 0$ (which we can always choose, as this satisfies the boundary condition) and $\beta = \lambda/\sqrt{2}$. The $\pm$ corresponds to two different branches of the solutions. In fact, the appearance of the branches are quite natural as we are taking the square root of $f^2(\phi)$ in eq. (11). The explicit form of the solution is

$$\phi(t) = u^{-1}\left(-\frac{\lambda}{\sqrt{2}}t + \delta\right), \qquad \delta = \text{some constant}, \qquad \int \frac{\text{d}\phi}{f(\phi)} = u(\phi). \tag{13}$$

But, eq. (9) cannot be reduced in the form of eq. (11). So, a solution of the form of (12) satisfies (7), but not (9). But they both correspond to the classical limit of the same quantum system (because of the equality of the functional measure (3)). So, in the classical case, the symmetry is broken, whereas in quantum case, the symmetry is preserved.

In the next section, some examples are discussed with the help of this construction.

3. Examples

Now we shall discuss the situation for the sine-Gordon system and for the sech interaction.

3.1 *Sine-Gordon system* (*odd form of* $f(\phi)$)

As our discussion is confined to a loop space, it is quite natural to think of the analysis for the case of sine-Gordon system [6,7]. It is well known that for specific values of the parameter of sine-Gordon system it becomes equivalent to the Thirring model [10]. One of the most important observations for the Thirring model is that though the sine-Gordon equation is the theory of massless scalar field, the sine-Gordon soliton [10] can be identified with the fundamental fermion of the Thirring model.

To obtain the sine-Gordon equation on $\mathcal{C}[0,1]$, we can consider the action

$$A_{sg} = -\frac{1}{2}\int_{t=0}^{t=1}\left[\dot{\phi}^2 + \frac{\lambda^2}{4}\sin^2\frac{\phi}{2}\right]\mathrm{d}t. \tag{14}$$

Then the nonlinear nonlocal transformation

$$\chi(t) = \phi(t) + \frac{\lambda}{\sqrt{2}}\int_0^t \sin\frac{\phi(\tau)}{2}\mathrm{d}\tau \tag{15}$$

leads to the equality of the functional measure

$$\int \mathrm{e}^{-\frac{1}{2}\int_0^1 \dot{\chi}^2\mathrm{d}t}\mathrm{d}\chi = \int \exp\left\{-\frac{1}{2}\int_0^1 \dot{\phi}^2\mathrm{d}t - \frac{\lambda^2}{4}\int_0^1 \sin^2\frac{\phi}{2}\mathrm{d}t + \frac{\lambda}{4\sqrt{2}}\int_0^1 \cos\frac{\phi}{2}\mathrm{d}t \right.$$
$$\left. +2\frac{\lambda}{\sqrt{2}}\left(\cos\frac{\phi(1)}{2} - \cos\frac{\phi(0)}{2}\right)\right\}\mathrm{d}\phi. \tag{16}$$

Though the symmetry properties of the old system and the new system (after nonlocal transformation) are similar with respect to the symmetry transformation $T(\phi) = -\phi$, the two systems show different properties with respect to their solutions.

In particular, the equation of motion obtained from A_{sg} admits a solution of the form

$$\phi(t) = 4\cot^{-1}(i\mathrm{e}^{-it/2}). \tag{17}$$

But the solution of that form will not be the solution of

$$\ddot{\phi} = \frac{\lambda^2}{8}\sin\phi - \frac{\lambda}{8\sqrt{2}}\sin\frac{\phi}{2}, \tag{18}$$

which is obtained from the new action after the nonlocal transformation.

So in that sense, the symmetry is broken for the classical case, whereas for the quantum case, by virtue of the equality of the functional measure, the symmetry is preserved.

3.2 *Sech interaction* (*even form of* $f(\phi)$)

Owning to the appearance as a soliton solution of a large class of integrable, nonlinear, partial, differential equation [6,7], sech interaction is much interesting among physicists.

One of the interesting features of this type of interaction is that it may be used as a model of reflectionless potential [8] and also for the interaction of the form of compactly supported function (for example, the alternate deposition of thin layers of GaAs and $Al_xGa_{1-x}As$ [9] may be modelled with sech interaction, as in this case the lower bound of the interaction is present).

Now, for this case we can consider the action

$$A_{\text{sec}} = \int_0^1 \left(\frac{1}{2}\dot{\phi}^2 + \frac{\lambda^2}{4}\text{sech}^2\phi\right) \mathrm{d}t. \tag{19}$$

The corresponding Weiner measure is

$$P = \exp\left\{-\frac{1}{2}\int_0^1 \dot{\phi}^2(t) - \frac{\lambda^2}{4}\int_0^1 \text{sech}^2\phi(t)\mathrm{d}t\right\} \mathrm{d}\phi. \tag{20}$$

Therefore, according to Proposition 1, for the nonlinear nonlocal transformation

$$\chi(t) = \phi(t) + \frac{\lambda}{2}\int_0^t \text{sech}\,\phi(\tau)\mathrm{d}\tau \tag{21}$$

the following equality holds:

$$\begin{aligned}\int & \mathrm{e}^{-1/2\int_0^1 \dot{\chi}^2 \mathrm{d}t}\mathrm{d}\chi \\ &= \int \exp\left\{\int_0^1 \left[-\frac{1}{2}\dot{\phi}^2 - \frac{\lambda^2}{4}\text{sech}^2\phi - \frac{\lambda}{2\sqrt{2}}\text{sech}\,\phi\tanh\phi\right]\mathrm{d}t\right. \\ &\qquad \left. -\frac{2\lambda}{\sqrt{2}}(\tan^{-1}\mathrm{e}^{\phi(1)} - \tan^{-1}\mathrm{e}^{\phi(0)})\right\}\mathrm{d}\phi.\end{aligned} \tag{22}$$

Now the equation of motion obtained from A_{sec}

$$\ddot{\phi} = -\frac{\lambda^2}{2}\text{sech}^2\phi\tanh\phi \tag{23}$$

admits a solution of the form (if we set the integration constant with $\lambda^2/2$)

$$\phi(t) = \sinh^{-1}\left(\frac{\lambda}{\sqrt{2}}t + \beta\right); \qquad \beta = \text{constant}. \tag{24}$$

If we look at the equation of motion after transformation, this gives

$$\ddot{\phi} = -\frac{\lambda^2}{2}\text{sech}^2\phi\tanh\phi + \frac{\lambda}{2\sqrt{2}}\text{sech}\,\phi(1 - 2\tanh^2\phi) \tag{25}$$

which is not symmetric with respect to $T(\phi)$ (contrary to the original case).

But as already noted in the previous section that the quantum system deals with the functional measure by virtue of the above-mentioned equality of the functional measure, we can conclude that symmetry is preserved in the quantum case, whereas symmetry is broken in the classical case.

4. Discussion

Following the above construction, we can say that the nonlocal transformation of fields may lead to some completely different systems. In the article of Belokurov *et al*, they point out the well-known Haag's theorem that may be helpful to understand this scenario. Even if we completely forget the Haag's theorem, we must admit the fact that the inclusion of nonlocality in the field theory will give additional interesting information. Nowadays, it is well known that, in some cases, the inclusion of nonlocality can be modelled as the Lorentz invariant CPT violating theories [11,12]. These theories give some additional interesting phenomena of mass splitting between particle and the corresponding antiparticle and may be helpful to justify the recent data analysis results that speculates the mass difference between neutrino and antineutrino [13,14]. Therefore, we can conclude that behind the simple mathematical operations presented in this article, a deeper physical phenomenon may be latent, that may be interesting for future study.

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