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New exact solutions of the generalized Zakharov–Kuznetsov modified equal-width equation

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Abstract. In this paper, new exact solutions, including soliton, rational and elliptic integral function solutions, for the generalized Zakharov–Kuznetsov modified equal-width equation are obtained using a new approach called the extended trial equation method. In this discussion, a new version of the trial equation method for the generalized nonlinear partial differential equations is offered.

Keywords. The extended trial equation method; generalized Zakharov–Kuznetsov equation; soliton solution; elliptic solutions.

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1. Introduction

One of the main topics of mathematical physics is to find exact solutions of the nonlinear partial differential equations. Equations of these kind, especially in fluid mechanics, plasma physics, optical fibres, solid-state physics and chemical physics, appear in various scientific fields and often play important roles in understanding nonlinear phenomena. It is possible to clarify the physical events by the discovery of exact solutions of nonlinear partial differential equations. To obtain explicit exact solutions of nonlinear evolution equations, some substantial methods such as Hirota method, tanh–coth method, Kudryashov method, F-expansion method, functional variable method, the simplest method, the trial equation method, and so on have been defined [1–9]. In recent years, Liu and other researchers developed the trial equation method and its new versions to classify the travelling wave solutions to nonlinear differential equations [10–19]. However, this paper applies the extended trial equation method to find 1-soliton, singular soliton, elliptic integral function and Jacobi elliptic function solutions. Apart from all these, some new exact solutions are obtained by using the trial equation methods. Some of them are elliptic integral F, E and Π functions, Jacobi elliptic function solutions etc. These types of solutions are very important and encounter in various areas of applied mathematics.

In §2, an extended trial equation method is described for finding exact travelling wave solutions of nonlinear evolution equations with higher-order nonlinearity. In §3, as an application, some exact solutions to nonlinear partial differential equation such as the generalized Zakharov–Kuznetsov modified equal-width equation [20] are obtained:

$$u_t + a(u^n)_x + (bu_{xt} + cu_{yy})_x = 0, \tag{1}$$

where a, b and c are real valued constants. The first term represents the evolution term, while the second term is the nonlinear term, and finally the third and fourth terms, in parentheses, are the dispersion terms. The exponent n, which indicates the power-law nonlinearity parameter, is a positive real number. This equation has already been studied by some authors [21,22]. In §4, a more general trial equation method is proposed.

2. The extended trial equation method

Step 1. We consider a nonlinear partial differential equation

$$P(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{2}$$

and under the general wave transformation

$$u(x_1, x_2, \ldots, x_N, t) = u(\eta), \qquad \eta = \lambda\left(\sum_{j=1}^{N} x_j - ct\right), \tag{3}$$

where $\lambda \neq 0$ and $c \neq 0$. Substituting eq. (3) into eq. (2) yields a nonlinear ordinary differential equation

$$N(u, u', u'', \ldots) = 0. \tag{4}$$

Step 2. Take the finite series and trial equation as follows:

$$u = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \tag{5}$$

where

$$(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \cdots + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + \cdots + \zeta_1 \Gamma + \zeta_0}. \tag{6}$$

Using eqs (5) and (6), we can write

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)}\left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1}\right)^2, \tag{7}$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)}\left(\sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1}\right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)}\left(\sum_{i=0}^{\delta} i(i-1)\tau_i \Gamma^{i-2}\right), \tag{8}$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials. Substituting these relations into eq. (4) yields an equation of polynomial $\Omega(\Gamma)$ of Γ:

$$\Omega(\Gamma) = \varrho_s \Gamma^s + \cdots + \varrho_1 \Gamma + \varrho_0 = 0. \tag{9}$$

According to the balance principle, we can find a relation of θ, ϵ and δ. We can compute some values of θ, ϵ and δ.

Step 3. Let the coefficients of $\Omega(\Gamma)$ all be zero, which will yield an algebraic equation system:

$$\varrho_i = 0, \qquad i = 0, ..., s. \tag{10}$$

Solving this system, we will determine the values of $\xi_0, ..., \xi_\theta, \zeta_0, ..., \zeta_\epsilon$ and $\tau_0, ..., \tau_\delta$.

Step 4. Reduce eq. (6) to the elementary integral form

$$\pm(\eta - \eta_0) = \int \frac{\mathrm{d}\Gamma}{\sqrt{\Lambda(\Gamma)}} = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} \mathrm{d}\Gamma. \tag{11}$$

Using a complete discrimination system for polynomial to classify the roots of $\Phi(\Gamma)$, we solve eq. (11) and obtain the exact solutions to eq. (4). Furthermore, we can write the exact travelling wave solutions to eq. (2).

3. Application to the generalized Zakharov–Kuznetsov equation

To look for travelling wave solutions of eq. (1), we make the transformation $u(x, y, t) = u(\eta)$, $\eta = B_1 x + B_2 y - wt$, where B_1 and B_2 represent the inverse widths in the x and y directions, respectively. Also, w represents the velocity of the soliton. Then, integrating this equation with respect to η twice and setting the integration constant to zero, we obtain

$$-wu + aB_1 u^n + \left(cB_1 B_2^2 - wbB_1^2\right) cu'' = 0. \tag{12}$$

We use the following transformation:

$$u = v^{2/(n-1)}. \tag{13}$$

Equation (12) turns into the equation

$$2\left(cB_1 B_2^2 - wbB_1^2\right)(n-1)vv'' + \left(cB_1 B_2^2 - wbB_1^2\right)(6-2n)(v')^2 - w(n-1)^2 v^2 + aB_1(n-1)^2 v^4 = 0. \tag{14}$$

Substituting eqs (7) and (8) into eq. (14) and using balance principle yield

$$\theta = \epsilon + 2\delta + 2.$$

After this solution procedure, we obtain the results as follows:

Case 1

If we take $\epsilon = 0$, $\delta = 1$ and $\theta = 4$, then

$$(v')^2 = \frac{(\tau_1)^2(\xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0}, \tag{15}$$

$$v'' = \frac{\tau_1(4\xi_4\Gamma^3 + 3\xi_3\Gamma^2 + 2\xi_2\Gamma + \xi_1)}{2\zeta_0}, \tag{16}$$

where $\xi_4 \neq 0$, $\zeta_0 \neq 0$. Solving the algebraic equation system (10) yields

$$\begin{aligned} \xi_0 &= \frac{\tau_0^2(-5\xi_3\tau_0 + 4\xi_2\tau_1)}{4\tau_1^3}, \qquad \xi_1 = \frac{2\tau_0(-\xi_3\tau_0 + \xi_2\tau_1)}{\tau_1^2}, \\ \xi_4 &= \frac{\xi_3\tau_1}{4\tau_0}, \quad \xi_2 = \xi_2, \quad \xi_3 = \xi_3, \\ \tau_0 &= \tau_0, \qquad \tau_1 = \tau_1, \\ \zeta_0 &= \frac{-c(1+n)B_2^2\xi_3 + 4abB_1^2\tau_0(3\xi_3\tau_0 - 2\xi_2\tau_1)}{2a(-1+n)^2\tau_0\tau_1}, \end{aligned} \tag{17}$$

$$w = \frac{4aB_1\tau_0(3\xi_3\tau_0 - 2\xi_2\tau_1)}{(1+n)\xi_3}. \tag{18}$$

Substituting these results into eqs (6) and (11), we have

$$\pm(\eta - \eta_0) = 2\sqrt{\frac{-c(1+n)B_2^2\xi_3 + 4abB_1^2\tau_0(3\xi_3\tau_0 - 2\xi_2\tau_1)}{2a(-1+n)^2\xi_3\tau_1^2}} \times \int \frac{\mathrm{d}\Gamma}{\sqrt{\Gamma^4 + \frac{4\tau_0}{\xi_3\tau_1}\Gamma^3 + \frac{4\tau_0}{\xi_3\tau_1}\Gamma^2 + \frac{8\tau_0^2(-\xi_3\tau_0 + \xi_2\tau_1)}{\xi_3\tau_1^3}\Gamma + \frac{4\tau_0^3(-5\xi_3\tau_0 + 4\xi_2\tau_1)}{4\xi_3\tau_1^4}}}. \tag{19}$$

Integrating eq. (19), we obtain the solutions to eq. (1) as follows:

$$\pm(\eta - \eta_0) = \frac{-2A}{\Gamma - \alpha_1}, \tag{20}$$

$$\pm(\eta - \eta_0) = \frac{4A}{\alpha_2 - \alpha_1}\sqrt{\frac{\Gamma - \alpha_2}{\Gamma - \alpha_1}}, \quad \alpha_2 > \alpha_1, \tag{21}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\alpha_1 - \alpha_2}\ln\left|\frac{\Gamma - \alpha_1}{\Gamma - \alpha_2}\right|, \quad \alpha_1 > \alpha_2, \tag{22}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}} \times \ln\left|\frac{\sqrt{(\Gamma - \alpha_2)(\alpha_1 - \alpha_3)} - \sqrt{(\Gamma - \alpha_3)(\alpha_1 - \alpha_2)}}{\sqrt{(\Gamma - \alpha_2)(\alpha_1 - \alpha_3)} + \sqrt{(\Gamma - \alpha_3)(\alpha_1 - \alpha_2)}}\right|, \quad \alpha_1 > \alpha_2 > \alpha_3, \tag{23}$$

$$\pm(\eta-\eta_0)=\frac{4A}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}F(\varphi,l),\quad \alpha_1>\alpha_2>\alpha_3>\alpha_4, \tag{24}$$

where

$$A=\sqrt{\frac{-c(1+n)B_2^2\xi_3+4abB_1^2\tau_0(3\xi_3\tau_0-2\xi_2\tau_1)}{2a(-1+n)^2\xi_3\tau_1^2}},$$

$$F(\varphi,l)=\int_0^{\varphi}\frac{\mathrm{d}\psi}{\sqrt{1-l^2\sin^2\psi}}, \tag{25}$$

and

$$\varphi=\arcsin\sqrt{\frac{(\Gamma-\alpha_1)(\alpha_2-\alpha_4)}{(\Gamma-\alpha_2)(\alpha_1-\alpha_4)}},\qquad l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}. \tag{26}$$

Also α_1, α_2, α_3 and α_4 are the roots of the polynomial equation

$$\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}=0. \tag{27}$$

Substituting the solutions (20)–(24) into (5) and (13), we have

$$u(x,y,t)=\left[\tau_0+\tau_1\alpha_1\pm\frac{2\tau_1A}{B_1\left(x+\frac{B_2}{B_1}y-\frac{4a\tau_0(3\xi_3\tau_0-2\xi_2\tau_1)}{(1+n)\xi_3}t-\frac{\eta_0}{B_1}\right)}\right]^{2/(n-1)}, \tag{28}$$

$$u(x,y,t)=\left[\tau_0+\tau_1\alpha_1+\frac{16A^2(\alpha_2-\alpha_1)\tau_1}{16A^2-\left[B_1(\alpha_1-\alpha_2)\left(x+\frac{B_2}{B_1}y-\frac{4a\tau_0(3\xi_3\tau_0-2\xi_2\tau_1)}{(1+n)\xi_3}t-\frac{\eta_0}{B_1}\right)\right]^2}\right]^{2/(n-1)}, \tag{29}$$

$$u(x,y,t)=\left[\tau_0+\tau_1\alpha_2+\frac{(\alpha_2-\alpha_1)\tau_1}{\exp\left[\frac{B_1(\alpha_1-\alpha_2)}{2A}\left(x+\frac{B_2}{B_1}y-\frac{4a\tau_0(3\xi_3\tau_0-2\xi_2\tau_1)}{(1+n)\xi_3}t-\frac{\eta_0}{B_1}\right)\right]-1}\right]^{2/(n-1)}, \tag{30}$$

$$u(x,y,t)=\left[\tau_0+\tau_1\alpha_1+\frac{(\alpha_1-\alpha_2)\tau_1}{\exp\left[\frac{B_1(\alpha_1-\alpha_2)}{2A}\left(x+\frac{B_2}{B_1}y-\frac{4a\tau_0(3\xi_3\tau_0-2\xi_2\tau_1)}{(1+n)\xi_3}t-\frac{\eta_0}{B_1}\right)\right]-1}\right]^{2/(n-1)}, \tag{31}$$

$$u(x,y,t) = \left[\tau_0+\tau_1\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\frac{B_1\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{2A}\left(x+\frac{B_2}{B_1}y-\frac{4a\tau_0(3\xi_3\tau_0-2\xi_2\tau_1)}{(1+n)\xi_3}t\right)-\frac{\eta_0}{B_1}\right]}\right]^{2/(n-1)}, \tag{32}$$

$$u(x,y,t) = \left[\tau_0+\tau_1\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)\tau_1}{(\alpha_1-\alpha_4)\mathrm{sn}^2\left(\frac{B_1\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{4A}\left(x+\frac{B_2}{B_1}y-\frac{4a\tau_0(3\xi_3\tau_0-2\xi_2\tau_1)}{(1+n)\xi_3}t-\frac{\eta_0}{B_1}\right),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right)+\alpha_4-\alpha_2}\right]^{2/(n-1)}, \tag{33}$$

If we take $\tau_0 = -\tau_1\alpha_1$ and $\eta_0 = 0$, then the solutions (28) and (29) and (31) and (32) can reduce to rational function solution

$$u(x,y,t) = \left(\pm\frac{\tilde{A}}{x+y-vt}\right)^{2/(n-1)}, \tag{34}$$

$$u(x,y,t) = \left[\frac{16A^2(\alpha_2-\alpha_1)\tau_1}{16A^2-\left[B_1(\alpha_1-\alpha_2)\,(x+y-vt)\right]^2}\right]^{2/(n-1)}, \tag{35}$$

travelling wave solutions

$$u(x,y,t)=\left\{\frac{(\alpha_2-\alpha_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{B_1(\alpha_1-\alpha_2)}{2A}\,(x+y-vt)\right]\right\}\right\}^{2/(n-1)} \tag{36}$$

and soliton solution

$$u(x,y,t) = \frac{A_1}{\left(D+\cosh\left[B(x+y-vt)\right]\right)^{2/(n-1)}}, \tag{37}$$

where $\tilde{A} = 2A\tau_1/B_1$, $B_2 = B_1$,

$$A_1 = \left(\frac{2\tau_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{(\alpha_3-\alpha_2)}\right)^{2/(n-1)},$$

$$B = \frac{B_1\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{2A}, \quad D = \frac{2\alpha_1-\alpha_2-\alpha_3}{\alpha_3-\alpha_2}$$

and

$$v = \frac{4a\tau_1^2\alpha_1(2\xi_2-3\xi_3\alpha_1)}{(1+n)\xi_3}.$$

Here, A_1 is the amplitude of the soliton, while v is the velocity and B is the inverse width of the solitons. Thus, we can say that the solitons exist for $\tau_1 > 0$.

If we take $\tau_0 = -\tau_1\alpha_2$ and $\eta_0 = 0$, then the solution (33) can reduce to elliptic soliton solution

$$u(x,y,t) = \frac{A_2}{\left(M+N\mathrm{sn}^2(\varphi,l)\right)^{2/(n-1)}}, \tag{38}$$

where

$$A_2 = ((\alpha_1 - \alpha_2)(\alpha_4 - \alpha_2)\tau_1)^{2/(n-1)}, \qquad B_2 = B_1, \qquad M = \alpha_4 - \alpha_2,$$

$$N = \alpha_1 - \alpha_4, \quad \varphi = \frac{\sqrt{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}}{4A} B_1 (x + y - vt),$$

$$l^2 = \frac{(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)}$$

and

$$v = \frac{4a\tau_1^2\alpha_1(2\xi_2 - 3\xi_3\alpha_1)}{(1+n)\xi_3}.$$

Thus, we can say that the solitons exist for $\tau_1 > 0$.

Remark 1. The solutions (34)–(37) and (38) obtained by using the extended trial equation method for eq. (1) have been checked by *Mathematica*. To our knowledge, the rational function solution, the hyperbolic function solution and the Jacobi doubly periodic wave solutions, that we find in this paper, are not shown in the previous literature. These results are new travelling wave solutions of eq. (1).

Remark 2. If we choose the corresponding values for some parameters, the solution (37) is in full agreement with the solution (12) mentioned in [20].

Remark 3. When the modulus $l \to 1$, then the solution (38) can be converted into the hyperbolic function solution of the generalized KP equation

$$u(x, y, t) = \frac{A_2}{\left(M + N \tanh^2\left[\frac{B_1\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}{4A}(x + y - wt)\right]\right)^{2/(1-n)}}, \tag{39}$$

where $\alpha_3 = \alpha_4$. If we choose the corresponding values for some parameters, the solution (39) is in full agreement with the solution (30) mentioned in [21].

Remark 4. When the modulus $l \to 0$, then the solution (38) can be reduced to the periodic wave solution

$$u(x, y, t) = \frac{A_2}{\left(M + N \sin^2\left[\frac{B_1\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_4)}}{4A}(x + y - wt)\right]\right)^{2/(1-n)}}, \tag{40}$$

where $\alpha_2 = \alpha_3$.

Remark 5. The convergence analysis of the obtained elliptic integral function solution F was examined and the solution (38) converges when $0 < l < 1$. This convergence analysis is the same as in [16].

Case 2

If we take $\epsilon = 0$, $\delta = 2$ and $\theta = 6$, then

$$(v')^2 = \frac{(\tau_1 + 2\tau_2\Gamma)^2(\xi_6\Gamma^6 + \xi_5\Gamma^5 + \xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0},$$

$$v'' = \frac{4\tau_2\left(\xi_6\Gamma^6+\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0\right) + (\tau_1+2\tau_1\Gamma)\left(6\xi_6\Gamma^5+5\xi_5\Gamma^4+4\xi_4\Gamma^3+3\xi_3\Gamma^2+2\xi_2\Gamma+\xi_1\Gamma\right)}{2\zeta_0}, \tag{41}$$

where $\xi_6 \neq 0$, $\zeta_0 \neq 0$. Solving the algebraic equation system (10) yields

$$\xi_0 = -\frac{\left(\xi_5^3 - 54\xi_3\xi_6^2\right)^2\left(\xi_5^3 - 18\xi_3\xi_6^2\right)}{497664\xi_5^3\xi_6^5}, \quad \xi_1 = \frac{14580\xi_3^2\xi_6^4 + 13\xi_5^6 - 972\xi_3\xi_5^3\xi_6^2}{62208\xi_5\xi_6^4},$$

$$\xi_4 = \frac{5\xi_5^3 + 27\xi_3\xi_6^2}{18\xi_5\xi_6}, \qquad \xi_3 = \xi_3,$$

$$\xi_2 = \frac{-115\xi_5^6 + 2484\xi_3\xi_5^3\xi_6^2 + 14580\xi_3^2\xi_6^4}{20736\xi_5^2\xi_6^3}, \quad \xi_5 = \xi_5, \quad \xi_6 = \xi_6,$$

$$\tau_0 = \frac{-\xi_5^3\tau_1 + 54\xi_3\xi_6^2\tau_1}{48\xi_5^2\xi_6}, \quad \tau_1 = \tau_1,$$

$$\tau_2 = \frac{3\xi_6\tau_1}{\xi_5}, \quad \zeta_0 = \frac{-1152c(n+1)B_2^2\xi_5^4\xi_6^2 + abB_1^2\tau_1^2\left(5\xi_5^3 - 54\xi_3\xi_6^2\right)}{1296a(n-1)^2\xi_5^2\xi_6^3\tau_1^2},$$

$$w = \frac{aB_1\tau_1^2\left(5\xi_5^3 - 54\xi_3\xi_6^2\right)^2}{1152(n+1)\xi_5^4\xi_6^2}. \tag{42}$$

Substituting these results into eqs (6) and (11), we get

$$\pm(\eta-\eta_0) = A_3\int \frac{\mathrm{d}\Gamma}{\sqrt{\Gamma^6+\frac{\xi_5}{\xi_6}\Gamma^5+\frac{5\xi_5^3+27\xi_3\xi_6^2}{18\xi_5\xi_6^2}\Gamma^4+\frac{\xi_3}{\xi_6}\Gamma^3+\frac{-115\xi_5^6+2484\xi_3\xi_5^3\xi_6^2+14580\xi_3^2\xi_6^4}{20736\xi_5^2\xi_6^4}\Gamma^2+\frac{14580\xi_3^2\xi_6^4+13\xi_5^6-972\xi_3\xi_5^3\xi_6^2}{62208\xi_5\xi_6^5}\Gamma-\frac{\left(\xi_5^3-54\xi_3\xi_6^2\right)^2\left(\xi_5^3-18\xi_3\xi_6^2\right)}{497664\xi_5^3\xi_6^6}}}, \tag{43}$$

where

$$A_3 = \sqrt{\frac{-1152c(n+1)B_2^2\xi_5^4\xi_6^2 + abB_1^2\tau_1^2(5\xi_5^3 - 54\xi_3\xi_6^2)}{1296a(n-1)^2\xi_5^2\xi_6^4\tau_1^2}}.$$

Integrating eq. (43), we obtain the solutions to eq. (1) as follows:

$$\pm(\eta - \eta_0) = -\frac{A_3}{2(\Gamma - \alpha_1)^2}, \tag{44}$$

$$\pm(\eta - \eta_0) = \frac{2A_3(2\Gamma - 3\alpha_1 + \alpha_2)}{3(\alpha_1 - \alpha_2)^2}\sqrt{\frac{\Gamma - \alpha_2}{(\Gamma - \alpha_1)^3}}, \quad \alpha_1 > \alpha_2, \tag{45}$$

$$\pm(\eta-\eta_0)=\frac{A_3\left(\alpha_2-\alpha_1\left(\ln\left|\frac{\Gamma-\alpha_2}{\Gamma-\alpha_1}\right|+1\right)-\Gamma\ln\left|\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}\right|\right)}{(\Gamma-\alpha_1)(\alpha_1-\alpha_2)^2}, \tag{46}$$

$$\pm(\eta-\eta_0)=\frac{-2A_3(2\Gamma-\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)^2\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_2)}},\quad \alpha_1>\alpha_2>\alpha_3, \tag{47}$$

$$\pm(\eta-\eta_0)=\frac{-A_3\left(\alpha_1\ln\left|\frac{\Gamma-\alpha_2}{\Gamma-\alpha_3}\right|+\alpha_2\ln\left|\frac{\Gamma-\alpha_3}{\Gamma-\alpha_1}\right|+\alpha_3\ln\left|\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}\right|\right)}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_1-\alpha_3)},\quad \alpha_1>\alpha_2>\alpha_3, \tag{48}$$

$$\pm(\eta-\eta_0)=A_3\left(\frac{\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_3)}}{(\Gamma-\alpha_2)(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)}+\frac{(\alpha_1-2\alpha_2+\alpha_3)i\log[V(\Gamma)]}{2(\alpha_1-\alpha_2)^{\frac{3}{2}}(\alpha_1-\alpha_2)^{\frac{3}{2}}}\right),\quad \alpha_1>\alpha_2>\alpha_3, \tag{49}$$

$$V(\Gamma)=\frac{-4(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_3)}+2i\sqrt{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)}(\alpha_1(\Gamma+\alpha_2-2\alpha_3)-2\Gamma\alpha_2+\alpha_3(\Gamma+\alpha_2))}{(\Gamma-\alpha_2)(\alpha_1-2\alpha_2+\alpha_3)}, \tag{50}$$

$$\pm(\eta-\eta_0)=A_3\left(\frac{2}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)}\sqrt{\frac{\Gamma-\alpha_1}{\Gamma-\alpha_2}}-\frac{2\arctan\left[\frac{(\Gamma-\alpha_1)(\alpha_3-\alpha_2)}{(\Gamma-\alpha_2)(\alpha_1-\alpha_3)}\right]}{(\alpha_2-\alpha_3)\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_2)}}\right),\quad \alpha_1>\alpha_2>\alpha_3, \tag{51}$$

$$\pm(\eta-\eta_0)=\frac{2A_3}{(\alpha_2-\alpha_3)(\alpha_3-\alpha_4)}\times\left(\sqrt{\frac{(\Gamma-\alpha_2)(\Gamma-\alpha_4)}{(\Gamma-\alpha_1)(\Gamma-\alpha_3)}}+M_1F(\varphi_1,l_1)+N_1[(\alpha_1-\alpha_4)F(\varphi_1,l_1)-(\alpha_2-\alpha_4)E(\varphi_1,l_1)]\right), \tag{52}$$

where

$$E(\varphi_1,l_1)=\int_0^{\varphi_1}\sqrt{1-l_1^2\sin^2\psi}\,\mathrm{d}\psi,\quad \varphi_1=\arcsin\sqrt{\frac{(\Gamma-\alpha_2)(\alpha_1-\alpha_4)}{(\Gamma-\alpha_1)(\alpha_2-\alpha_4)}},$$
$$l_1^2=\frac{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}, \tag{53}$$

$$M_1=\frac{\alpha_1-\alpha_2+\alpha_3-\alpha_4}{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)}},\qquad N_1=\frac{1}{(\alpha_1-\alpha_3)\sqrt{(\alpha_2-\alpha_4)}}, \tag{54}$$

$$\pm(\eta-\eta_0)=\frac{-A_3(\Gamma-\alpha_3)}{\Gamma-\alpha_4}\left(\frac{1}{\sqrt{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}}\log(G(\Gamma))\right.$$
$$\left.+\frac{i}{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_2)}}\log(H(\Gamma))\right), \quad (55)$$

where

$$G(\Gamma)=\frac{-\Gamma+\alpha_4}{2\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)}+2\Gamma\alpha_4-\alpha_2(\Gamma+\alpha_4)-\alpha_1(\Gamma-2\alpha_2+\alpha_4)} \quad (56)$$

and

$$H(\Gamma)$$
$$=\frac{-(\alpha_3-\alpha_4)\left(2\sqrt{(\Gamma-\alpha_1)(\Gamma-\alpha_2)(\alpha_1-\alpha_3)(\alpha_3-\alpha_2)}-i(-2\Gamma\alpha_3+\alpha_2(\Gamma+\alpha_3)+\alpha_1(\Gamma-2\alpha_2+\alpha_3))\right)}{\sqrt{(\alpha_1-\alpha_3)(\alpha_3-\alpha_2)}(-\Gamma+\alpha_4)}, \quad (57)$$

$$\pm(\eta-\eta_0)=\frac{2A_3}{\sqrt{(\alpha_2-\alpha_3)(\alpha_1-\alpha_5)}}\left(\frac{\pi(\varphi_2,n,l_2)}{(\alpha_2-\alpha_4)}+\frac{F(\varphi_2,l_2)}{(\alpha_1-\alpha_2)}\right),$$
$$\alpha_1>\alpha_2>\alpha_3>\alpha_4>\alpha_5, \quad (58)$$

where

$$\pi(\varphi_2,n,l_2)=\int_0^{\varphi}\frac{\mathrm{d}\psi}{(1+n\sin^2\psi)\sqrt{1-l^2\sin^2\psi}},$$
$$\varphi_2=\arcsin\sqrt{\frac{(\Gamma-\alpha_2)(\alpha_1-\alpha_5)}{(\Gamma-\alpha_1)(\alpha_2-\alpha_5)}} \quad (59)$$

and

$$l_2^2=\frac{(\alpha_1-\alpha_3)(\alpha_2-\alpha_5)}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_5)}, \qquad n=\frac{(\alpha_1-\alpha_4)(\alpha_2-\alpha_5)}{(\alpha_2-\alpha_4)(\alpha_1-\alpha_5)}. \quad (60)$$

Also, α_1, α_2, α_3, α_4, α_5 and α_6 are the roots of the polynomial equation

$$\Gamma^6+\frac{\xi_5}{\xi_6}\Gamma^5+\frac{\xi_4}{\xi_6}\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}=0. \quad (61)$$

Substituting the solutions (44), (45) and (47) into (5) and (13), we have

$$u(x,y,t)=\left[\tau_0+\tau_1\alpha_1\pm\tau_1\sqrt{\pm\frac{A_3}{2B_1\left(x+\frac{B_2}{B_1}y-\frac{a\tau_1^2\left(5\xi_5^3-54\xi_3\xi_6^2\right)^2}{1152(n+1)\xi_5^4\xi_6^2}\right)t-\eta_0}}\right.$$

$$+\tau_2\left(\alpha_1 \pm \sqrt{\pm\frac{A_3}{2B_1\left(x+\frac{B_2}{B_1}y-\frac{a\tau_1^2\left(5\xi_5^3-54\xi_3\xi_6^2\right)^2}{1152(n+1)\xi_5^4\xi_6^2}t\right)-\eta_0}}\right)^2\Bigg]^{2/(1-n)}, \qquad (62)$$

$$u(x,y,t)=\left[\tau_0+\tau_1\left(\alpha_1-W-2T\right)+\tau_2\left(\alpha_1-W-2T\right)^2\right]^{2/(1-n)}, \qquad (63)$$

$$u(x,y,t)=\Bigg[\tau_0+\tau_1\left(\alpha_1+\frac{1+i\sqrt{3}}{2}W+(1-i\sqrt{3})T\right)$$
$$+\tau_2\left(\alpha_1+\frac{1+i\sqrt{3}}{2}W+(1-i\sqrt{3})T\right)^2\Bigg]^{2/(1-n)}, \qquad (64)$$

$$u(x,y,t)=\Bigg[\tau_0+\tau_1\left(\alpha_1-W+(1+i\sqrt{3})T\right)$$
$$+\tau_2\left(\alpha_1-W+(1+i\sqrt{3})T\right)^2\Bigg]^{2/(1-n)}, \qquad (65)$$

where

$$W=\frac{4^{1/3}2A_3^2(\alpha_1-\alpha_1)^2}{\left(A_3^2(\alpha_1-\alpha_1)^3(16A_3^2-9(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2)^2+3A_3^2(\eta-\eta_0)(\alpha_1-\alpha_2)^5\sqrt{(16A_3^2-9(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2)^3}\right)^{1/3}},$$

$$T=\frac{\left(A_3^2(\alpha_1-\alpha_1)^3\left(16A_3^2-9(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2\right)^2+3A_3^2(\eta-\eta_0)(\alpha_1-\alpha_2)^5\sqrt{(16A_3^2-9(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2)^3}\right)^{1/3}}{16A_3^2-9(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2}, \qquad (66)$$

$$u(x,y,t)=\Bigg[\tau_0+\tau_1\left(\frac{(\alpha_1+\alpha_2)\left(16A_3^2-(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2\right)\pm(\alpha_1-\alpha_3)^3(\eta-\eta_0)\sqrt{(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2-16A_3^2}}{2\left(16A_3^2-(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2\right)}\right)$$
$$+\tau_2\left(\frac{(\alpha_1+\alpha_2)\left(16A_3^2-(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2\right)\pm(\alpha_1-\alpha_3)^3(\eta-\eta_0)\sqrt{(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2-16A_3^2}}{2\left(16A_3^2-(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2\right)}\right)^2\Bigg]^{2/(1-n)}. \qquad (67)$$

For simplicity, we can write the solutions (62)–(65) and (67) as follows:

$$u(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(B_k(x,t)\right)^i\right]^{2/(1-n)}, \quad k=1,\ldots,5, \qquad (68)$$

where

$$B_1(x,t) = \alpha_1 \pm \tau_1 \sqrt{\pm \frac{A_3}{2B_1\left(x + \frac{B_2}{B_1}y - \frac{a\tau_1^2\left(5\xi_5^3 - 54\xi_3\xi_6^2\right)^2}{1152(n+1)\xi_5^4\xi_6^2}\right)t - \eta_0}}, \tag{69}$$

$$B_2(x,t) = \frac{(\alpha_1+\alpha_2)\left(16A_3^2 - (\alpha_1-\alpha_2)^4(\eta-\eta_0)^2\right) \pm (\alpha_1-\alpha_3)^3(\eta-\eta_0)\sqrt{(\alpha_1-\alpha_2)^4(\eta-\eta_0)^2 - 16A_3^2}}{2\left(16A^2 - (\alpha_1-\alpha_2)^4(\eta-\eta_0)^2\right)}, \tag{70}$$

$$B_3(x,t) = \alpha_1 - W - 2T, \quad B_4(x,t) = \alpha_1 + \frac{1+i\sqrt{3}}{2}W + (1-i\sqrt{3})T,$$
$$B_5(x,t) = \alpha_1 - W + (1+i\sqrt{3})T. \tag{71}$$

Remark 6. To the author's knowledge, the solutions (68) found in this paper are not shown in previous literature, and also these are new exact solutions of eq. (1).

Case 3

If we take $\epsilon = 1$, $\delta = 1$ and $\theta = 5$, then

$$(v')^2 = \frac{\tau_1^2(\xi_5\Gamma^5 + \xi_4\Gamma^4 + \xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0 + \zeta_1\Gamma}, \tag{72}$$

$$v'' = \frac{\tau_1\left[(\zeta_0+\zeta_1\Gamma)\left(5\xi_5\Gamma^4+4\xi_4\Gamma^3+3\xi_3\Gamma^2+2\xi_2\Gamma+\xi_1\right) - \zeta_1\left(\xi_5\Gamma^5+\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0\right)\right]}{2\left(\zeta_0+\zeta_1\Gamma\right)^2}, \tag{73}$$

where $\xi_5 \neq 0$, $\zeta_1 \neq 0$. Solving the algebraic equation system (10) yields

$$\xi_0 = \xi_0,$$

$$\xi_1 = \frac{32\zeta_1^6\xi_0\xi_4\xi_5^2 + 224\zeta_0\zeta_1^5\xi_0\xi_5^3 + 3\zeta_0^4\zeta_1^2\xi_4^2\xi_5^2(1+\xi_4) - 4\zeta_0^5\zeta_1\xi_4\xi_5^3 + \zeta_0^2\zeta_1^4\xi_4^4 - 4\zeta_0^3\zeta_1^3\xi_4^3\xi_5 + \zeta_0^6\xi_5^4}{32\zeta_0\zeta_1^4\xi_5^2(\zeta_1\xi_4 - \zeta_0\xi_5)},$$

$$\xi_2 = \frac{(\zeta_1\xi_4 - \zeta_0\xi_5)^2\left(-17\zeta_0^3\zeta_1\xi_4\xi_5^2 + \zeta_0\zeta_1^3\xi_4^3 + 7\zeta_0^2\zeta_1^2\xi_4^2\xi_5 + 9\zeta_0^4\xi_5^3\right) + 256\zeta_1^5\xi_0\xi_5^3(\zeta_0\xi_5 + \zeta_1\xi_4)}{32\zeta_0\zeta_1^3\xi_5^2\left(\zeta_1\xi_4 - \zeta_0\xi_5\right)^2},$$

$$\xi_4 = \xi_4,$$

$$\xi_3 = \frac{5\zeta_0\zeta_1^4\xi_4^4 - 4\zeta_0^2\zeta_1^3\xi_4^3\xi_5 - 18\zeta_0^3\zeta_1^2\xi_4^2\xi_5^2 + 4\zeta_1\xi_5^3\left(7\zeta_0^4\xi_4 + 64\zeta_1^4\xi_0\right) - 11\zeta_0^5\xi_5^4}{16\zeta_0\zeta_1^2\xi_5(\zeta_1\xi_4 - \zeta_0\xi_5)^2},$$

$$\xi_5 = \xi_5, \quad \zeta_0 = \zeta_0, \quad \zeta_1 = \zeta_1,$$

$$\tau_0 = -\frac{iB_2\sqrt{2c(1+n)\zeta_0}(\zeta_1\xi_4-\zeta_0\xi_5)^2}{2\sqrt{-a\left(bB_1^2\left(\zeta_0\zeta_1^4\xi_4^4-4\zeta_0^2\zeta_1^3\xi_4^3\xi_5+6\zeta_0^3\zeta_1^2\xi_4^2\xi_5^2-4\zeta_1\xi_5^3\left(64\zeta_1^4\xi_0+\zeta_0^4\xi_4\right)+\zeta_0^5\xi_5^4\right)-4(n-1)^2\zeta_0\zeta_1^3\xi_5\left(\zeta_1\xi_4-\zeta_0\xi_5\right)^2\right)}},$$

$$\tau_1 = \frac{2iB_2\zeta_1\xi_5\sqrt{2c(1+n)\zeta_0}(\zeta_0\xi_5-\zeta_1\xi_4)}{\sqrt{-a\left(bB_1^2\left(\zeta_0\zeta_1^4\xi_4^4-4\zeta_0^2\zeta_1^3\xi_4^3\xi_5+6\zeta_0^3\zeta_1^2\xi_4^2\xi_5^2-4\zeta_1\xi_5^3(64\zeta_1^4\xi_0+\zeta_0^4\xi_4)+\zeta_0^5\xi_5^4\right)-4(n-1)^2\zeta_0\zeta_1^3\xi_5(\zeta_1\xi_4-\zeta_0\xi_5)^2\right)}},$$

$$w = \frac{cB_1B_2^2\left(\zeta_0\zeta_1^4\xi_4^4-4\zeta_0^2\zeta_1^3\xi_4^3\xi_5+6\zeta_0^3\zeta_1^2\xi_4^2\xi_5^2-4\zeta_1\xi_5^3\left(64\zeta_1^4\xi_0+\zeta_0^4\xi_4\right)+\zeta_0^5\xi_5^4\right)}{bB_1^2\left(\zeta_0\zeta_1^4\xi_4^4-4\zeta_0^2\zeta_1^3\xi_4^3\xi_5+6\zeta_0^3\zeta_1^2\xi_4^2\xi_5^2-4\zeta_1\xi_5^3\left(64\zeta_1^4\xi_0+\zeta_0^4\xi_4\right)+\zeta_0^5\xi_5^4\right)-4(n-1)^2\zeta_0\zeta_1^3\xi_5\left(\zeta_1\xi_4-\zeta_0\xi_5\right)^2}. \tag{74}$$

Substituting these results into eqs (6) and (11), we get

$$\pm(\eta-\eta_0) = \sqrt{\frac{\zeta_1}{\xi_5}}\int\sqrt{\frac{\Gamma+\frac{\zeta_0}{\zeta_1}}{\Gamma^5+\frac{\xi_4}{\xi_5}\Gamma^4+\frac{\xi_3}{\xi_5}\Gamma^3+\frac{\xi_2}{\xi_5}\Gamma^2+\frac{\xi_1}{\xi_5}\Gamma+\frac{\xi_0}{\xi_5}}}\mathrm{d}\Gamma. \tag{75}$$

Integrating eq. (75), we obtain the solutions to eq. (1) as follows:

$$\pm(\eta-\eta_0) = -\frac{2A_4}{3\sqrt{\zeta_1}(\zeta_0+\zeta_1\alpha_1)}\left(\frac{\zeta_0+\zeta_1\Gamma}{\Gamma-\alpha_1}\right)^{3/2}, \tag{76}$$

$$\pm(\eta-\eta_0) = -\frac{A_4}{\alpha_1-\alpha_2}\left\{\frac{\zeta_0+\zeta_1\alpha_2}{2\sqrt{\zeta_1(\zeta_0+\zeta_1\alpha_1)(\alpha_1-\alpha_2)}}\times\ln|P(\Gamma)| + \frac{1}{\Gamma-\alpha_1}\sqrt{\frac{(\zeta_0+\zeta_1\Gamma)(\Gamma-\alpha_2)}{\zeta_1}}\right\},\ \alpha_1>\alpha_2, \tag{77}$$

where

$$P(\Gamma) = \frac{\Gamma-\alpha_1}{\zeta_0(\Gamma+\alpha_1-2\alpha_2)+2\sqrt{(\zeta_0+\zeta_1\Gamma)(\zeta_0+\zeta_1\alpha_1)(\Gamma-\alpha_2)(\alpha_1-\alpha_2)}+\zeta_1(2\Gamma\alpha_1-\alpha_2(\Gamma+\alpha_1))}, \tag{78}$$

$$\pm(\eta-\eta_0) = -\frac{2A_4}{\alpha_1-\alpha_2}\left\{\sqrt{\frac{\zeta_0+\zeta_1\Gamma}{\zeta_1(\Gamma-\alpha_1)}}+\sqrt{\frac{\zeta_0+\zeta_1\alpha_2}{\zeta_1(\alpha_1-\alpha_2)}}\arctan\left[\sqrt{\frac{(\Gamma-\alpha_1)(\zeta_0+\zeta_1\alpha_2)}{(\alpha_1-\alpha_2)(\zeta_0+\zeta_1\Gamma)}}\right]\right\}, \tag{79}$$

$$\pm(\eta-\eta_0) = -\frac{A_4}{\alpha_1-\alpha_3}\left\{\sqrt{\frac{\zeta_0+\zeta_1\alpha_2}{\zeta_1(\alpha_2-\alpha_3)}}\ln|R(\Gamma)|+\sqrt{\frac{\zeta_0+\zeta_1\alpha_1}{\zeta_1(\alpha_1-\alpha_3)}}\ln|S(\Gamma)|\right\},\quad \alpha_1>\alpha_2>\alpha_3, \tag{80}$$

where

$$R(\Gamma) = \frac{\alpha_2-\Gamma}{\zeta_0(\Gamma+\alpha_2-2\alpha_3)+2\sqrt{(\zeta_0+\zeta_1\Gamma)(\zeta_0+\zeta_1\alpha_2)(\Gamma-\alpha_3)(\alpha_2-\alpha_3)}+\zeta_1(2\Gamma\alpha_2-\alpha_3(\Gamma+\alpha_2))}, \tag{81}$$

and

$$S(\Gamma)=\frac{\zeta_0(\Gamma+\alpha_1-2\alpha_3)+2\sqrt{(\zeta_0+\zeta_1\Gamma)(\zeta_0+\zeta_1\alpha_1)(\Gamma-\alpha_3)(\alpha_1-\alpha_3)}+\zeta_1(2\Gamma\alpha_1-\alpha_3(\Gamma+\alpha_1))}{\Gamma-\alpha_2}, \tag{82}$$

$$\pm(\eta-\eta_0)=\frac{-2A_4}{\alpha_1-\alpha_3}\sqrt{\frac{\zeta_0+\zeta_1\alpha_3}{\zeta_1(\alpha_1-\alpha_2)}}E(\varphi_3,l_3),\quad \alpha_1>\alpha_2>\alpha_3, \tag{83}$$

where

$$A_4=\sqrt{\frac{\zeta_1}{\xi_5}},\quad \varphi_3=\arcsin\sqrt{\frac{(\Gamma-\alpha_3)(\alpha_2-\alpha_1)}{(\Gamma-\alpha_1)(\alpha_2-\alpha_3)}},$$
$$l_3^2=\frac{(\alpha_3-\alpha_2)(\zeta_0+\zeta_1\alpha_1)}{(\alpha_1-\alpha_2)(\zeta_0+\zeta_1\alpha_3)}. \tag{84}$$

$$\pm(\eta-\eta_0)=\frac{2A_4(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_2)\sqrt{\zeta_1(\alpha_2-\alpha_3)(\zeta_0+\zeta_1\alpha_4)}} \times\left(\frac{\zeta_0+\zeta_1\Gamma}{\alpha_1-\alpha_2}\pi(\varphi_3,n_2,l_3)-\frac{\zeta_0+\zeta_1\alpha_2}{\alpha_2-\alpha_4}F(\varphi_3,l_3)\right), \tag{85}$$

where

$$\varphi_3=\arcsin\sqrt{\frac{(\Gamma-\alpha_3)(\alpha_2-\alpha_1)}{(\Gamma-\alpha_1)(\alpha_2-\alpha_3)}},\quad l_2^2=\frac{(\alpha_3-\alpha_2)(\zeta_0+\zeta_1\alpha_1)}{(\alpha_1-\alpha_2)(\zeta_0+\zeta_1\alpha_3)}, \tag{86}$$
$$n_2=-\frac{(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)}{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}.$$

4. Discussion

A more general extended trial equation method for nonlinear partial differential equations is given as follows:

Step 1. The extended trial equation (5) can be reduced to the following more general form:

$$u=\frac{A(\Gamma)}{B(\Gamma)}=\frac{\sum_{i=0}^{\delta}\tau_i\Gamma^i}{\sum_{j=0}^{\mu}\omega_j\Gamma^j}, \tag{87}$$

where

$$(\Gamma')^2=\Lambda(\Gamma)=\frac{\Phi(\Gamma)}{\Psi(\Gamma)}=\frac{\xi_\theta\Gamma^\theta+\cdots+\xi_1\Gamma+\xi_0}{\zeta_\epsilon\Gamma^\epsilon+\cdots+\zeta_1\Gamma+\zeta_0}. \tag{88}$$

Here, τ_i $(i=0,\ldots,\delta)$, ω_j $(j=0,\ldots,\mu)$, ξ_ς $(\varsigma=0,\ldots,\theta)$ and ζ_σ $(\sigma=0,\ldots,\epsilon)$ are the constants to be determined.

Step 2. Taking trial equations (87) and (88), we derive the following equations:

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \frac{\left(A'(\Gamma)B(\Gamma) - A(\Gamma)B'(\Gamma)\right)^2}{B^4(\Gamma)}, \tag{89}$$

$$u'' = \frac{(A'(\Gamma)B(\Gamma) - A(\Gamma)B'(\Gamma))\left\{(\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma))B(\Gamma) - 4\Phi(\Gamma)\Psi(\Gamma)B'(\Gamma)\right\} + 2\Phi(\Gamma)\Psi(\Gamma)B(\Gamma)(A''(\Gamma)B(\Gamma) - A(\Gamma)B''(\Gamma))}{2B^3(\Gamma)\Psi^2(\Gamma)} \tag{90}$$

and other derivation terms such as u''', and so on.

Step 3. Substituting u', u'' and other derivation terms into eq. (4) yields the following equation:

$$\Omega(\Gamma) = \varrho_s \Gamma^s + \cdots + \varrho_1 \Gamma + \varrho_0 = 0. \tag{91}$$

According to the balance principle we can determine a relation of θ, ϵ, δ and μ.

Step 4. Letting the coeffcients of $\Omega(\Gamma)$ to be zero will yield an algebraic equation system $\varrho_i = 0$ $(i = 0, ..., s)$. Solving this equation system, we shall determine the values $\tau_0, ...\tau_\delta; \omega_0, \ldots, \omega_\mu; \xi_0 \ldots, \xi_\theta$ and $\zeta_0, \ldots, \zeta_\epsilon$.

Step 5. Substituting the results obtained in Step 4 into eq. (88) and integrating eq. (88), we can find the exact solutions of eq. (2).

5. Conclusions

To find some soliton and elliptic function solutions to the generalized Zakharov–Kuznetsov modified equal-width equation, the extended trial equation method is applied in this study. The productivity of this method is confidential and efficacious, and also a more general solution is obtained by this method. Otherwise, we have discussed a new version of the extended trial equation method. In future studies, the nonlinear partial differential equations can be solved with the help of this recommended method.

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