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Comprehensive decay law for emission of charged particles and exotic cluster radioactivity

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Abstract. A general decay formula for the emission of charged particles from metastable nuclei is developed based on the basic phenomenon of resonances occurring in quantum scattering process under Coulomb-nuclear potential. It relates the half-lives of radioactive decays with the Q values of the outgoing elements with masses and charges of the nuclei involved in the decay. The relation is found to be a generalization of the Geiger–Nuttall law in α radioactivity and explains well all the known emissions of charged particles including clusters, alpha and proton.

Keywords. Decay of proton, alpha and cluster; analytic expression for decay half-life; general decay law.

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1. Introduction

Besides the well-known α decays, there are various other decays, namely, emission of proton from nuclei lying beyond the proton drip line and decay of nuclear cluster with charge number greater than 4, emitted from radioactive parent nuclei with Z as large as $Z = 117$ in the superheavy region. The experimental results of half-lives and decay energies have been accumulated recently in all these categories of decay events [1–9]. Further, besides data of ground-state transition of nuclei emitting particles with zero angular momentum (ℓ), experimental data of half-lives of outgoing particles including proton and α with l-dependent Q-values have been reported in recent times [8,10].

There is a famous age old formula called Geiger–Nuttall (GN) law, to explain the measured values of half-life of α-decay with a mathematical characteristic of linear variation logarithm of half-life, with inverse square root of Q-value in $l = 0$ state of transition or decay. We wish to derive an analytical expression for the half-life decay akin to GN law by considering the metastable parent nucleus as a quantum two-body system of ejected particle and the daughter nucleus exhibiting resonance scattering phenomena under the combined effect of nuclear, Coulomb and centrifugal forces. The formula coefficients in our expression for the half-life are derived naturally and the angular momentum dependence is found inbuilt in the formulation. The formula is suited to explain the measured results of decay half-lives of positively charged particles namely, alpha, cluster and proton from heavy and superheavy nuclei.

2. Formulation

In the approach we have proposed recently [11–13] for the calculation of Q-value energy and decay half-life $T_{1/2}$ on the α decay of radioactive heavy ions, the $\alpha+$ nucleus system is considered as a Coulomb-nuclear potential scattering problem and the accurately determined resonance energy (E) of the quasibound state is taken as the Q-value of the decaying system. The width or lifetime of the resonance state accounts for the decay half-life. The normalized regular solution $u(r)$ of the modified Schrödinger equation is matched at radius $r = R$ to the Coulomb Hankel outgoing spherical wave $f_{\mathrm{C}}(kr) = G_\ell(\eta, kr) + i F_\ell(\eta, kr)$ such that

$$u(r) = N_0[G_\ell(\eta, kR) + i F_\ell(\eta, kR)], \tag{1}$$

where R is the radial position outside the range of the nuclear field.

For a typical cluster–daughter system with cluster particle as the projectile and the daughter nucleus as the target, let μ represents the reduced mass of the system, wave number $k = \sqrt{(2\mu/\hbar^2)E}$ and η stands for the Coulomb parameter.

$$\eta = \mu \frac{Z_e Z_d e^2}{\hbar^2 k}.$$

With this, the mean life T (or width Γ) of the decay is expressed in terms of amplitude N_0 as

$$T = \frac{\hbar}{\Gamma} = \frac{\mu}{\hbar k} \frac{1}{|N_0|^2}. \tag{2}$$

As the wave function $u(r)$ decreases rapidly with radius outside the daughter nucleus, it can be normalized by $\int_0^R |u(r)|^2 \mathrm{d}r = 1$. Further, using the fact that for a value of radial distance sufficiently large, the value of $G_\ell(\eta, kR)$ is very large compared to $F_\ell(\eta, kR)$ by several orders of magnitude, T of eq. (2) is expressed as

$$T = \frac{\mu}{\hbar k} \frac{|G_\ell(\eta, kR)|^2}{P}, \tag{3}$$

where

$$P = \frac{|u(r)|^2}{\int_0^R |u(r)|^2 \mathrm{d}r}. \tag{4}$$

The result from the above expression gives values of mean life T or half-life $T_{1/2} = 0.693\, T$ for the decay of the charged particle carrying angular momentum ℓ with Q-value equal to the resonance energy E. This formula for the estimation of decay half-life is valid in emissions of all types of positively charged particles namely, proton, alpha and cluster.

We, in this paper, wish to simplify the expression (3) and put it in the well-known linear form of GN law for the variation of $\mathrm{Ln}T_{1/2}$ as a function of Q-value of the particle emitted with some amount of angular momentum ℓ.

In Coulomb-nuclear problem, there are specific values of η and $\rho = kR$ for which the Coulomb Hankel function G_ℓ can be expressed in a simple mathematical form giving quite accurate results. When $2\eta > \rho$ with $\ell \geq 0$ [14],

$$G_\ell = \frac{2(2\eta)^\ell}{(2\ell+1)!C_\ell(\eta)}\ (2\eta\rho)^{1/2}\ \kappa_{2\ell+1}[2(2\eta\rho)^{1/2}], \tag{5}$$

$$C_\ell^2(\eta) = \frac{(1+\eta^2)(4+\eta^2)\cdots(\ell^2+\eta^2)2^{2\ell}}{(2\ell+1)((2\ell)!)^2}\ \frac{2\pi\eta}{(2\ell+1)}\ (\mathrm{e}^{2\pi\eta}-1)^{-1}, \tag{6}$$

$$\begin{aligned}\kappa_\gamma(z) = \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\Bigg[1 + \frac{4\gamma^2-1}{8z} + \frac{(4\gamma^2-1)(4\gamma^2-9)}{2!(8z)^2} \\ + \frac{(4\gamma^2-1)(4\gamma^2-9)(4\gamma^2-25)}{3!(8z)^2} + \cdots\Bigg],\end{aligned} \tag{7}$$

where $\gamma = 2\ell + 1$ and $z = 2(2\eta\rho)^{1/2}$. On further simplification, we get

$$\begin{aligned}G_\ell^2 &= \frac{4}{\pi}(\mathrm{e}^{2\pi\eta}-1)\ \frac{\rho\,\eta^{2\ell}\,\kappa^2}{(1+\eta^2)(4+\eta^2)\cdots(\ell^2+\eta^2)} \\ &= \left(\frac{\rho}{2\eta}\right)^{1/2}(\mathrm{e}^{2\pi\eta}-1)\exp(-4(2\eta\rho)^{1/2}) \\ &\quad\times\frac{\eta^{2\ell}}{(1+\eta^2)(4+\eta^2)\cdots(\ell^2+\eta^2)}\ M_\ell,\end{aligned} \tag{8}$$

where

$$\begin{aligned}\sqrt{M}_\ell = 1 + \frac{4(2\ell+1)^2-1}{16(2\eta\rho)^{1/2}} + \frac{[4(2\ell+1)^2-1](4(2\ell+1)^2-9]}{2[16(2\eta\rho)]^2} \\ + \frac{[4(2\ell+1)^2-1][4(2\ell+1)^2-9][4(2\ell+1)^2-25]}{6[16(2\eta\rho)]^3}.\end{aligned} \tag{9}$$

Substituting G_ℓ^2 given by (8) in (3), we get

$$\begin{aligned}T_{1/2} \simeq \frac{0.693}{\hbar kP}\left(\frac{\rho}{2\eta}\right)^{1/2}(\mathrm{e}^{2\pi\eta}-1)\exp(-4(2\eta\rho)^{1/2}) \\ \times\frac{\eta^{2\ell}}{(1+\eta^2)(4+\eta^2)\cdots(\ell^2+\eta^2)}\ M_\ell.\end{aligned} \tag{10}$$

As $2\pi\eta$ is a large quantity, $(e^{2\pi\eta}-1)\approx e^{2\pi\eta}$. Hence,

$$T_{1/2} \simeq \frac{0.693}{\hbar k P}\left(\frac{\rho}{2\eta}\right)^{1/2}\exp(2\pi\eta - 4(2\eta\rho)^{1/2}) \times \frac{\eta^{2\ell}}{(1+\eta^2)(4+\eta^2)\cdots(\ell^2+\eta^2)} M_\ell. \tag{11}$$

Taking logarithm of both sides, we can present the result (11) in GN form as

$$\text{Ln } T_{1/2} = a\chi' + b\rho' + c + d, \tag{12}$$

where

$$\chi' = Z_e Z_d\sqrt{\frac{A}{Q}},$$
$$\rho' = \sqrt{A Z_e Z_d\left(A_e^{1/3} + A_d^{1/3}\right)},$$

and

$$A = \frac{A_e A_d}{A_e + A_d}.$$

The constants a, b, c and the ℓ-dependent parameter d are expressed as

$$a = \pi e^2\sqrt{2m}/\hbar,$$
$$b = -4\sqrt{2me^2 r_0}/\hbar,$$
$$c = \text{Ln}\left[\frac{0.693}{3P}\sqrt{\frac{m}{2e^2}A\left(A_e^{1/3}+A_d^{1/3}\right)r_0/Z_c Z_d}\ \ 10^{-23}\right],$$
$$d = -\left[\frac{2}{2\eta^2+1}+\frac{8}{2\eta^2+4}+\cdots+\frac{2\ell^2}{2\eta^2+\ell^2}\right] + \text{Ln } M_\ell,$$

where the nucleon mass $m = 931.5$ MeV, square of electronic charge $e^2 = 1.4398$ MeV fm, $\hbar = 197.329$ MeV fm and radial distance parameter

$$r_0 = \frac{R}{A_e^{1/3}+A_d^{1/3}}$$

expressed in fm unit.

For $\ell = 0$ with $2\eta > \rho$, a more accurate formula for the Coulomb Hankel function G_0 is given by [14]

$$G_0 = \left(\frac{t}{1-t}\right)^{1/4}\exp(-\delta), \tag{13}$$

where

$$t = \rho/2\eta \text{ and } \delta = 2\eta\left[\{t(1-t)\}^{1/2} + \sin^{-1} t^{1/2} - \tfrac{\pi}{2}\right].$$

Using the above expression for G_0, the half-life for $\ell = 0$ is expressed in the form

$$\text{Ln } T_{1/2} = -2\delta + \text{Ln}\left[\frac{0.693}{\hbar k P}\left(\frac{t}{1-t}\right)^{1/2}\right]. \tag{14}$$

When $2\eta > \rho$, we have $t < 1$ and $y(=t^{1/2}) < 1$. We consider some non-vanishing terms in the expansion of $\sin^{-1} y \approx \frac{\pi}{2} - (1-y)^{1/2}(a_0 + a_1 y + a_2 y^2 + a_3 y^3)$, where $a_0 = 1.5707288, a_1 = -0.2121144, a_2 = 0.074240, a_3 = -0.018729$ [14].

Considering terms up to t^2 in the binomial expansion of $(1 - t^{1/2})^{1/2}$ and taking $\left(\frac{t}{1-t}\right)^{1/2} \approx t^{1/2}$ for $t \ll 1$, eq. (14) reduces to

$$\text{Ln } T_{1/2} = a'\chi' + b'\rho' + c, \tag{15}$$

where the coefficients a' and b' are expressed as $a' = 2a_0 e^2\sqrt{2m}/\hbar$ and $b' = -b_f\sqrt{2me^2 r_0}/\hbar$ with

$$\begin{aligned} b_f &= 2 + a_0 - 2a_1 + \left(\frac{a_0}{4} + a_1 - 2a_2\right)t^{1/2} \\ &\quad + \left(\frac{a_0}{8} + \frac{a_1}{4} + a_2 - 2a_3 - 1\right)t \\ &\quad + \left(\frac{5}{64}a_0 + \frac{a_1}{8} + \frac{a_2}{4} + a_3\right)t^{3/2} \\ &\quad + \left(\frac{5}{64}a_1 + \frac{a_2}{8} + \frac{a_3}{4} - \frac{1}{4}\right)t^2 \\ &\quad + \left(\frac{5}{64}a_2 + \frac{a_3}{8}\right)t^{5/2} \\ &\quad + \left(\frac{5}{64}a_3 - \frac{1}{8}\right)t^3. \end{aligned} \tag{16}$$

It can be pointed out here that the coefficient a' is the same as 'a' used in (12) because $2a_0 \approx \pi$ and b' differs from 'b' of (12) through the factor b_f, which replaces the multiplying factor 4 in the expression $b = -4\sqrt{2me^2 r_0}/\hbar$ given in (12). Taking (15) for better result for $\ell = 0$ along with the ℓ-dependent expression (12), the final formula for Ln $T_{1/2}$ in the form of decimal logarithm in unit of second $\log_{10} T_{1/2}(s) = \text{Ln } T_{1/2}/\text{Ln } 10 = 0.4342944819\,\text{Ln } T_{1/2}$ is given by

$$\log_{10} T_{1/2}(s) = a'\chi' + b'\rho' + c + d, \tag{17}$$

where

$$
\begin{aligned}
a' &= 2a_0 e^2 \sqrt{2m}/\hbar \text{ Ln } 10, \\
b' &= -b_f \sqrt{2me^2 r_0}/\hbar \text{ Ln } 10, \\
c &= \text{Ln}\left[\frac{0.693}{3P}\sqrt{\frac{m}{2e^2} A\left(A_e^{1/3} + A_d^{1/3}\right) r_0/Z_c Z_d}\; 10^{-23}\right]/\text{Ln } 10, \\
d &= -\left[\frac{2}{2\eta^2+1} + \frac{8}{2\eta^2+4} + \cdots + \frac{2\ell^2}{2\eta^2+\ell^2}\right]/\text{Ln } 10 + \text{Ln } M_\ell/\text{Ln } 10.
\end{aligned} \tag{18}
$$

The constant 'd' is ℓ-dependent and it helps one to estimate the values of decay life-time of particles pushed out with angular momentum $\ell > 0$. The values of these coefficients in (17) are determined using their respective analytical expressions (18).

We use $R = 9.5$ fm and $P = 10^{-3}$ for all types of emissions of charged particles. These values are obtained from the general behaviour of the wave function at a given resonance energy, occurring in several decaying systems. The radial distance $R = 9.5$ fm is a distance over which the value of amplitude of the resonant wave function reduces to a small value of $1/e$. Having fixed the values of R and P this way on the basis of physical property of resonant wave function, one can take eq. (17) as a universal decay law valid for emissions of all kinds of positively charged particles from radioactive nuclei, where the system of emitted particle and the daughter nucleus with characteristic Q-value fulfills the condition $2\eta > \rho$.

3. Numerical results and discussion

We analyse the events of decay of positive particles like cluster, alpha and proton from a large number of radioactive nuclei including light and superheavy ones. The value of the quantity $\log_{10} T_{1/2}(s) - b'\rho'(s)$ representing half-life $T_{1/2}$ in seconds as a function of χ' is compared with the corresponding results of $T_{1/2}$ derived from the experimental values in respective systems of decay. The value of χ' depends on the system specified by mass number A and proton number Z and the Q-value with specific ℓ. The value of χ' is found to be very small for proton+daughter system and large for cluster+daughter nucleus giving a value in between them for the α+daughter system.

Considering the decay in (ℓ=0) ground state to ground state, in the case of several nuclei undergoing emission of α-particles, the calculated results (solid curve) of the quantity $\log_{10} T_{1/2} - b'\rho'$ for different values of χ' are compared with the corresponding experimental values (solid dots) obtained from [6,7] in figure 1. It is seen that, the solid curve representing our calculated results looks like a perfect straight line. The measured data (solid dots) of half-lives expressed in the form of the quantity $\log_{10} T_{1/2} - b'\rho'$ are found to arrange themselves in a rectilinear way as functions of Q-values specified by χ'. This variations of the experimental results are explained satisfactorily by the respective

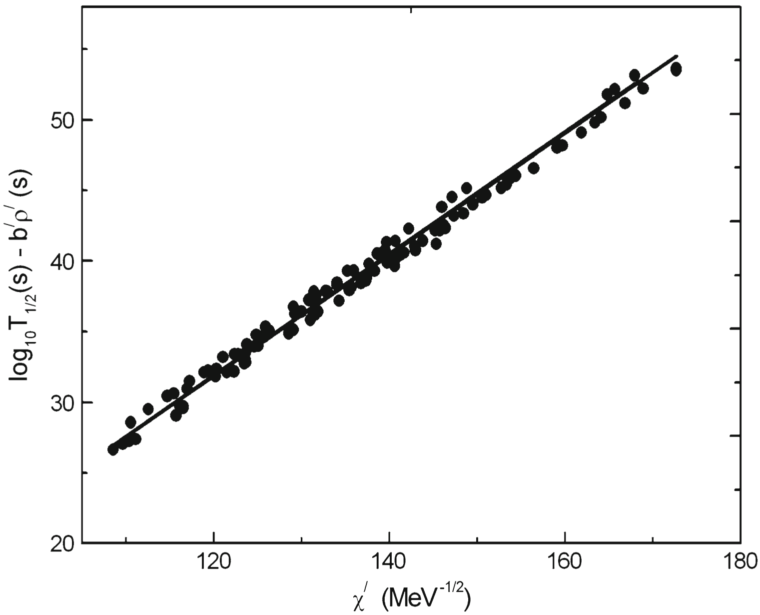

Figure 1. Plot of decay law (eq. (17)) for α decays in $\ell = 0$ state from 115 number of nuclei with $Z = 58$–117. The straight line is given as $a'\chi' + c + d$ of eq. (17).

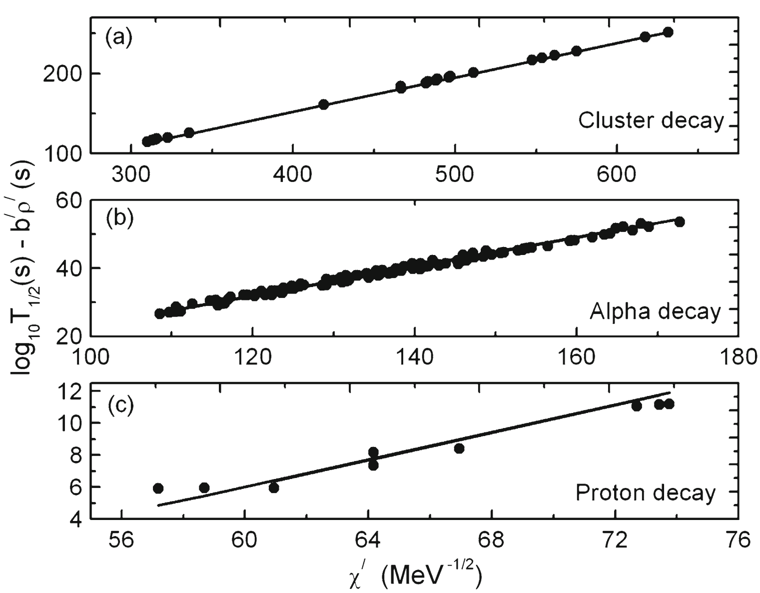

Figure 2. Same as figure 1 for decays of alpha (**b**) along with those of cluster (**a**) and proton (**c**) from nuclei with $Z = 52$–117.

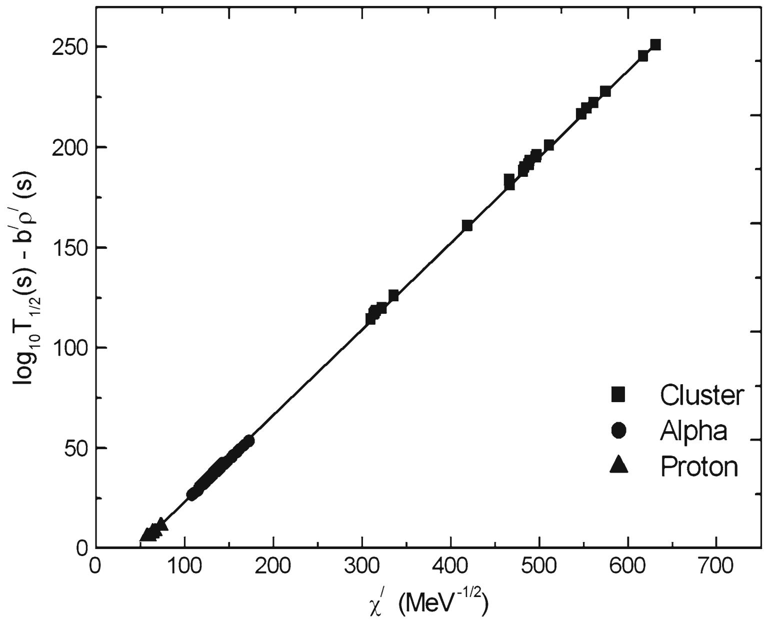

Figure 3. Same as figure 1 for decays of alpha along with those of cluster and proton presented together from nuclei with $Z = 52$–117.

calculated results (solid curve), given by eq. (17) with an inbuilt characteristic of linear nature of variation consistent with the GN law for α decays. For the estimation of accuracy in this fitting of 115 events, the absolute root mean square (RMS) deviation is found to be 1.01 which is well below a factor 2 to give a good explanation of the data. In figure 2, we present similar comparison of our calculated results with corresponding measured data for the decays of cluster in (**a**) and proton in (**c**). The experimental Q values and results of half-lives are obtained from [9] for the cluster and from [8] for the proton. We obtain equally good explanation of the measured data with linear characteristic of variation in these two cases with RMS deviation of 1.01 and 1.09 for the cluster and proton, respectively. In figure 3, the variations of the results of the quantity $\log_{10} T_{1/2} - b'\rho'$ with χ' in the case of alpha, cluster and proton decays are presented together. It is seen that, all of them fall in a single straight line with a common value for slope and intercept. This shows how the decays of charged particles of any Z values from heavy nuclei are governed by a single rule of law, namely, the linear GN law established in the case of α decays. This makes the nature of the present formulation broad and universal, showing no discrimination amongst the events of emission of charged particles of any Z value emerging from metastable parent nuclei with charge number ranging from small $Z = 52$ to very large $Z = 117$, in the superheavy region where the system of emitted particle and the daughter nucleus with characteristic Q-value obeys the condition $2\eta > \rho$. The decay half-life formula contains an angular momentum-dependent parameter derived naturally. Using this, we can explain the results of half-lives of particles emitted with non-zero angular momenta in α and proton decays [15].

4. Summary and conclusion

A relation between decay half-life and Q values is derived for the emission of positively charged particles from parent heavy nuclei. The logarithm of the half-life varies as a straight line with the inverse square root of Q value in ground-state to ground-state decay. This linear characteristic of variation is akin to the well-known (Geiger–Nuttall) GN law established for α decays. This formulation is applied to the analysis of the experimental data of half-lives for the decay of particles namely, cluster, α and proton. The respective data of half-lives for *s*-wave are explained by our calculated results in all the three categories of decay which obey the condition $2\eta > \rho$ equally well maintaining the linear nature of variation. In conclusion, we can say that the present formulation is comprehensive and universal for the decay of charged particles from radioactive heavy nuclei.

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