PRAMANA — journal of physics

© Indian Academy of Sciences

Vol. 82, No. 3
March 2014
pp. 465–476

Solitons and periodic solutions to a couple of fractional nonlinear evolution equations

M MIRZAZADEH[1,*], M ESLAMI[2] and ANJAN BISWAS[3,4]

[1]Department of Engineering Sciences, Faculty of Technology and Engineering, East of Guilan, University of Guilan, Rudsar, Iran
[2]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
[3]Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA
[4]Faculty of Science, Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
*Corresponding author. E-mail: mirzazadehs2@guilan.ac.ir

DOI: 10.1007/s12043-013-0679-0; ***e*Publication:** 26 February 2014

Abstract. This paper studies a couple of fractional nonlinear evolution equations using first integral method. These evolution equations are foam drainage equation and Klein–Gordon equation (KGE), the latter of which is considered in (2 + 1) dimensions. For the fractional evolution, the Jumarie's modified Riemann–Liouville derivative is considered. Exact solutions to these equations are obtained.

Keywords. First integral method; solitons; foam drainage equation; Klein–Gordon equation.

PACS Nos 02.30.Jr; 05.45.Yv

1. Introduction

The theory of nonlinear evolution equations (NLEEs) has applications in several areas of applied mathematics, theoretical physics and engineering sciences [1–40]. Therefore, it is imperative to carry out a deeper investigation of these equations. So far, these NLEEs were studied with integral-order temporal derivative or integer-order evolution. It is about time to generalize these evolution terms to fractional order so that this will lead to fractional-order evolution equations. The advantage of fractional derivatives, over integer derivatives, is that it is related to the memory and heredity of several materials and processes. In addition, temporal fractional evolution is more realistic to describe many physical phenomena [39].

This leads to a more generalized setting so that the results of NLEEs with integer order can be considered as a special case. This paper addresses a couple of NLEEs that appear in engineering science and theoretical physics with fractional evolution. They are foam drainage equation and Klein–Gordon equation, the latter of which is considered in $(2+1)$ dimensions.

Foam drainage is the flow of liquid through foam when the effects of capillarity and gravity are taken into consideration. The physics of foam drainage and the issue of creaming in emulsions lead to this equation. This study was inspired by forced drainage experiments where foam was injected from the top with liquid supply. Subsequently, the temporal and spatial variations of the liquid supply is determined [40]. This is an important area of research in the detergent industry.

Klein–Gordon equation (KGE), on the other hand, is a relativistic field equation for scalar particles (spin-0). KGE is a relativistic generalization of the well-known Schrödinger's equation. While there are other relativistic wave equations, KGE has been the most frequently studied equation for describing particle dynamics in quantum field theory [4].

Jumarie [19] presented a modification of the Riemann–Liouville definition which appears to provide a framework for fractional calculus. In this paper, the fractional evolution terms for the evolution equations will be from Jumarie's point of view. Also, the methodology of extracting solutions for these couple of NLEEs will be the first integral approach. The main idea of this algorithm is to implement division theorem for two variables in the complex domain based on the ring theory of commutative algebra.

The paper is arranged as follows. In §2, we describe briefly the modified Riemann–Liouville derivative. Section 3 gives the algorithmic approach to the first integral method. In §4 and 5, we apply this method to the nonlinear fractional foam drainage equation and the generalized forms of time-fractional Klein-Gordon equation.

2. Jumarie's modified Riemann–Liouville derivative

The Jumarie's modified Riemann–Liouville derivative of order α is defined as [21]

$$D_t^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\xi)^{-\alpha-1}(f(\xi)-f(0))\mathrm{d}\xi, \tag{1}$$

for $\alpha < 0$,

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))\mathrm{d}\xi, \tag{2}$$

for $0 < \alpha < 1$,

$$D_t^\alpha f(t) = (f^{(n)}(t))^{(\alpha-n)}, \tag{3}$$

if $n \leq \alpha \leq n+1,\ n \geq 1$.

We list some important properties for the modified Riemann–Liouville derivative as follows:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \qquad r > 0, \tag{4}$$

$$D_t^\alpha (f(t)g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \tag{5}$$

$$D_t^\alpha f(g(t)) = f_g'(g(t)) D_t^\alpha g(t) = D_g^\alpha f(g(t))(g'(t))^\alpha. \tag{6}$$

3. First integral method

A fractional partial differential equation, say in two or three independent variables $x,\ y,\ t$, is given by

$$P\left(u,\ \frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial^\alpha u}{\partial t^\alpha},\ \frac{\partial^\alpha u}{\partial x^\alpha},\ \frac{\partial^\alpha u}{\partial y^\alpha},\ \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}},\ldots\right)=0, \tag{7}$$

where $u=u(x,y,t)$ is an unknown function, P is a polynomial in $u=u(x,y,t)$ and its various partial derivatives, in which the highest order derivative and nonlinear term are involved.

The main steps of the first integral method [21] are summarized as follows:

Step I: Suppose

$$u(x,y,t)=U(\xi),\qquad \xi=\frac{k}{\Gamma(1+\alpha)}x^\alpha+\frac{l}{\Gamma(1+\alpha)}y^\alpha-\frac{\lambda}{\Gamma(1+\alpha)}t^\alpha, \tag{8}$$

and then eq. (7) can be turned into the following nonlinear ordinary differential equation (ODE):

$$Q\left(U(\xi),\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi},\frac{\mathrm{d}^2U(\xi)}{\mathrm{d}\xi^2},\ldots\right)=0, \tag{9}$$

where $U(\xi)$ is an unknown function, Q is a polynomial in the variable $U(\xi)$ and its derivatives. If all terms contain derivatives, then eq. (9) is integrated where integration constants are considered zero.

Step II: We assume that eq. (9) has a solution in the form

$$U(\xi)=X(\xi), \tag{10}$$

and introduce a new independent variable

$$Y(\xi)=\frac{\mathrm{d}X(\xi)}{\mathrm{d}\xi},$$

which leads to a new system of

$$\frac{\mathrm{d}X(\xi)}{\mathrm{d}\xi}=Y(\xi),$$

$$\frac{\mathrm{d}Y(\xi)}{\mathrm{d}\xi}=G(X(\xi),Y(\xi)). \tag{11}$$

Step III: By using the Division Theorem for two variables in the complex domain C which is based on the Hilbert–Nullstellensatz Theorem [30], we can obtain one first integral to eq. (11) which can reduce eq. (9) to a first-order integrable ordinary differential equation. An exact solution to eq. (7) is then obtained by solving this equation directly.

Division Theorem: Suppose $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$, and $P(w,z)$ is irreducible in $C[w,\nu]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that

$$Q(w,z)=P(w,z)G(w,z).$$

4. Foam drainage equation

In this section, we shall apply the first integral method to obtain exact solutions of the nonlinear foam drainage equation with time and space-fractional derivatives [31]

$$D_t^{\alpha} u = \frac{u}{2} D_x^{\alpha} D_x^{\alpha} u - 2u^2 D_x^{\alpha} u + (D_x^{\alpha} u)^2, \qquad 0 < \alpha \le 1. \tag{12}$$

The foam drainage equation is a model of the flow of liquid through channels and nodes (intersection of four channels) between the bubbles, driven by gravity and capillarity [32].

As in [21], we make transformation

$$u(x,t) = U(\xi), \qquad \xi = \frac{l x^{\alpha}}{\Gamma(1+\alpha)} - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \tag{13}$$

and generate the reduced nonlinear ODE in the form

$$\frac{l^2}{2} U U'' + \lambda U' + l^2 (U')^2 - 2l U^2 U' = 0, \tag{14}$$

where the prime denotes the differential with respect to ξ.

Using (10) and (11), eq. (14) is equivalent to the two-dimensional autonomous system

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}\xi} &= Y, \\ \frac{\mathrm{d}Y}{\mathrm{d}\xi} &= \frac{4}{l} XY - \frac{2\lambda}{l^2} \frac{Y}{X} - 2\frac{Y^2}{X}. \end{aligned} \tag{15}$$

Now, we make the transformation

$$\mathrm{d}\eta = \frac{\mathrm{d}\xi}{X} \tag{16}$$

in eq. (15) to avoid the singular line $X = 0$ temporarily. Thus, system (15) becomes

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}\eta} &= XY, \\ \frac{\mathrm{d}Y}{\mathrm{d}\eta} &= \frac{4}{l} X^2 Y - \frac{2\lambda}{l^2} Y - 2Y^2. \end{aligned} \tag{17}$$

Now, we are applying the Division Theorem to seek the first integral to system (17). Suppose that $X = X(\eta)$, $Y = Y(\eta)$ are the nontrivial solutions to (17), and $Q(X,Y) = \sum_{i=0}^{N} a_i(X) Y^i$ is an irreducible polynomial in the complex domain C such that

$$Q(X(\eta), Y(\eta)) = \sum_{i=0}^{N} a_i(X(\eta)) Y^i(\eta) = 0, \tag{18}$$

where $a_i(X)$, $i = 0, 1, \ldots, N$ are polynomials of X and $a_N(X) \neq 0$. Equation (18) is called the first integral to system (17). Note that $\mathrm{d}Q/\mathrm{d}\eta$ is a polynomial in X and Y, and

$q\,[X(\eta), Y(\eta)] = 0$ implies $\mathrm{d}Q/\mathrm{d}\eta|_{(17)} = 0$. According to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain C such that

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}\eta} &= \frac{\mathrm{d}Q}{\mathrm{d}X}.\frac{\mathrm{d}X}{\mathrm{d}\eta} + \frac{\mathrm{d}Q}{\mathrm{d}Y}.\frac{\mathrm{d}Y}{\mathrm{d}\eta} \\ &= (g(X) + h(X)Y)\sum_{i=0}^{N} a_i(X)Y^i. \end{aligned} \tag{19}$$

In this example, we assume that $N = 1$ in eq. (18). Then, by equating the coefficients of Y^i, $i = 2, 1, 0$, on both sides of (19), we have

$$Xa_1'(X) = h(X)a_1(X) + 2a_1(X), \tag{20}$$

$$Xa_0'(X) = \left(g(X) - \frac{4}{l}X^2 + \frac{2\lambda}{l^2}\right)a_1(X) + h(X)a_0(X), \tag{21}$$

$$g(X)a_0(X) = 0. \tag{22}$$

As $a_1(X)$ and $h(X)$ are polynomials, from eq. (20), we deduce that $h(X) = -2$ and $a_1(X)$ must be a constant. For simplicity, we can take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 0$ and $\deg(a_0(X)) = 2$ only.

Suppose

$$\begin{aligned} g(X) &= A_0, \\ a_0(X) &= B_0 + B_1X + B_2X^2, \quad B_2 \neq 0, \end{aligned} \tag{23}$$

where $A_0,\ B_0,\ B_1,\ B_2$ are all constants to be determined.

Substituting $a_0(X),\ a_1(X)$ and $g(X)$ into (21) and (22) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it we get the solution set

$$A_0 = 0, \qquad B_0 = \frac{\lambda}{l^2}, \qquad B_1 = 0, \quad B_2 = -\frac{1}{l}, \tag{24}$$

where l and λ are arbitrary constants.

Using the conditions (24) in (18), we obtain

$$Y = \frac{\lambda}{l^2} + \frac{1}{l}X^2. \tag{25}$$

Combining eq. (25) with eq. (17) and changing to the original variables, we find exact solutions to eq. (12) as:

Case I. $\lambda l < 0$:

$$u_1(x,t) = -\sqrt{-\frac{\lambda}{l}}\tanh\left[\sqrt{-\frac{\lambda}{l^3}}\left(\frac{lx^\alpha}{\Gamma(1+\alpha)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0\right)\right] \tag{26}$$

and

$$u_2(x,t) = -\sqrt{-\frac{\lambda}{l}} \coth\left[\sqrt{-\frac{\lambda}{l^3}}\left(\frac{lx^\alpha}{\Gamma(1+\alpha)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0\right)\right]. \tag{27}$$

Case II. $\lambda l > 0$:

$$u_3(x,t) = \sqrt{\frac{\lambda}{l}} \tan\left[\sqrt{\frac{\lambda}{l^3}}\left(\frac{lx^\alpha}{\Gamma(1+\alpha)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0\right)\right] \tag{28}$$

and

$$u_4(x,t) = -\sqrt{\frac{\lambda}{l}} \cot\left[\sqrt{\frac{\lambda}{l^3}}\left(\frac{lx^\alpha}{\Gamma(1+\alpha)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0\right)\right]. \tag{29}$$

Solutions (26) and (27) are topological soliton and singular soliton solution respectively while (28) and (29) are singular periodic solutions.

Remark 1. The following Riccati equation

$$U'(\xi) = a_0 + a_1 U(\xi) + a_2 U^2(\xi), \tag{30}$$

admits the following exact solutions [33]:

Type 1: When $\Delta = a_1^2 - 4a_0a_2 > 0$, the solutions of eq. (30) are

$$U_1(\xi) = -\frac{\sqrt{\Delta}}{2a_2} \tanh\left[\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)\right] - \frac{a_1}{2a_2},$$

$$U_2(\xi) = -\frac{\sqrt{\Delta}}{2a_2} \coth\left[\frac{\sqrt{\Delta}}{2}(\xi+\xi_0)\right] - \frac{a_1}{2a_2}, \tag{31}$$

which respectively represent topological and singular soliton solutions.

Type 2: When $\Delta = a_1^2 - 4a_0a_2 < 0$, the solutions of eq. (30) are

$$U_3(\xi) = \frac{\sqrt{-\Delta}}{2a_2} \tan\left[\frac{\sqrt{-\Delta}}{2}(\xi+\xi_0)\right] - \frac{a_1}{2a_2},$$

$$U_4(\xi) = -\frac{\sqrt{-\Delta}}{2a_2} \cot\left[\frac{\sqrt{-\Delta}}{2}(\xi+\xi_0)\right] - \frac{a_1}{2a_2}, \tag{32}$$

and these are singular periodic solutions.

Type 3: When $\Delta = a_1^2 - 4a_0a_2 = 0$, the solution of eq. (30) is

$$U_5(\xi) = -\frac{1}{a_2(\xi+\xi_0)} - \frac{a_1}{2a_2}, \tag{33}$$

which is a rational solution and more general solutions are presented in [33].

5. Klein–Gordon equation in 2 + 1 dimensions

The Klein–Gordon equation that is known as the Schrödinger's relativistic wave equation arises in the study of quantum mechanics [34–36]. This equation is used as the equation of motion of a massive spinless particle in classical and quantum field theories. In this section, we investigate the generalized forms of time-fractional Klein–Gordon equations in $1+2$ dimensions. The generalized time-fractional Klein–Gordon equation (gFKGE) in $2+1$ dimensions [37] is modelled by the equation

$$\frac{\partial^{2\alpha}u^m}{\partial t^{2\alpha}} - k^2\left(\frac{\partial^2 u^m}{\partial x^2} + \frac{\partial^2 u^m}{\partial y^2}\right) + F(u) = 0, \qquad t > 0, \qquad 0 < \alpha \le 1, \quad (34)$$

where the dependent variable $u(x, y, t)$ represents the wave profile. Also, k is a constant and m is a positive integer with $m \ge 1$. In fact, if $m = 1$, eq. (34) reduces to the regular FKGE in $1+2$ dimensions [38].

In this paper, the following two forms of the function $F(u)$ will be considered:

$$F(u) = au^m - bu^n + cu^{2n-m}, \quad (35)$$

$$F(u) = au^m - bu^{m-n} + cu^{n+m}. \quad (36)$$

These two cases will be respectively labelled as Forms I and II. In these two forms a, b and c are real constants.

Form I

In this case, eqs (34) and (35) together give

$$\frac{\partial^{2\alpha}u^m}{\partial t^{2\alpha}} - k^2\left(\frac{\partial^2 u^m}{\partial x^2} + \frac{\partial^2 u^m}{\partial y^2}\right) + au^m - bu^n + cu^{2n-m} = 0, t > 0, 0 < \alpha \le 1. \quad (37)$$

Equation (37) will be converted to the ODE

$$(\lambda^2 - 2k^2)(U^m)'' + aU^m - bU^n + cU^{2n-m} = 0, \quad (38)$$

on using the transformation

$$u(x, y, t) = U(\xi), \quad \xi = x + y - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}. \quad (39)$$

Due to the difficulty in obtaining the first integral of eq. (38), we propose a transformation denoted by $U = V^{1/(n-m)}$. Then eq. (38) is converted to

$$\begin{aligned}(\lambda^2 - 2k^2)m(2m-n)(V')^2 &+ (\lambda^2 - 2k^2)m(n-m)VV'' \\ &+ a(n-m)^2V^2 - b(n-m)^2V^3 \\ &+ c(n-m)^2V^4 = 0. \quad (40)\end{aligned}$$

Using (10) and (11), eq. (40) is equivalent to the two-dimensional autonomous system

$$\frac{\mathrm{d}X}{\mathrm{d}\xi} = Y,$$
$$\frac{\mathrm{d}Y}{\mathrm{d}\xi} = \frac{(\lambda^2-2k^2)m(n-2m)Y^2 - a(n-m)^2X^2 + b(n-m)^2X^3 - c(n-m)^2X^4}{(\lambda^2-2k^2)m(n-m)X}. \tag{41}$$

Making the following transformation

$$\mathrm{d}\eta = \frac{\mathrm{d}\xi}{(\lambda^2-2k^2)m(n-m)X}, \tag{42}$$

system (41) becomes

$$\frac{\mathrm{d}X}{\mathrm{d}\eta} = (\lambda^2-2k^2)m(n-m)XY,$$
$$\frac{\mathrm{d}Y}{\mathrm{d}\eta} = (\lambda^2-2k^2)m(n-2m)Y^2 - a(n-m)^2X^2 + b(n-m)^2X^3 - c(n-m)^2X^4. \tag{43}$$

Suppose that $N = 1$ in (18). From now on, we shall omit some details because the procedure is the same. By comparing with the coefficients of $Y^i, i = 2, 1, 0$, on both sides of (19), we have

$$(\lambda^2-2k^2)m(n-m)Xa_1'(X) = h(X)a_1(X) - (\lambda^2-2k^2)m(n-2m)a_1(X), \tag{44}$$

$$(\lambda^2-2k^2)m(n-m)Xa_0'(X) = g(X)a_1(X) + h(X)a_0(X), \tag{45}$$

$$a_1(X)(b(n-m)^2X^3 - a(n-m)^2X^2 - c(n-m)^2X^4) = g(X)a_0(X). \tag{46}$$

As $a_1(X)$ and $h(X)$ are polynomials, from eq. (44), we deduce that $h(X) = (\lambda^2-2k^2)m(n-2m)$ and $a_1(X)$ must be a constant. For simplicity, we can take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 2$ and $\deg(a_0(X)) = 2$ only.

Suppose

$$g(X) = A_0 + A_1X + A_2X^2,\ \ A_2 \neq 0,$$
$$a_0(X) = B_0 + B_1X + B_2X^2,\ \ B_2 \neq 0, \tag{47}$$

where $A_0,\ A_1,\ A_2,\ B_0,\ B_1,\ B_2$ are constants to be determined.

Substituting eq. (47) into eq. (45), we obtain

$$A_0 = (\lambda^2-2k^2)m(2m-n)B_0,$$
$$A_1 = (\lambda^2-2k^2)m^2B_1,$$
$$A_2 = (\lambda^2-2k^2)mnB_2. \tag{48}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ into (46) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it we obtain

$$\begin{aligned} c &= \frac{nmb^2}{a(n+m)^2}, \\ B_0 &= 0, \\ B_1 &= \frac{m-n}{m}\sqrt{\frac{a}{2k^2-\lambda^2}}, \\ B_2 &= \frac{(n-m)b}{a(n+m)}\sqrt{\frac{a}{2k^2-\lambda^2}} \end{aligned} \tag{49}$$

and

$$\begin{aligned} c &= \frac{nmb^2}{a(n+m)^2}, \\ B_0 &= 0, \\ B_1 &= -\frac{m-n}{m}\sqrt{\frac{a}{2k^2-\lambda^2}}, \\ B_2 &= -\frac{(n-m)b}{a(n+m)}\sqrt{\frac{a}{2k^2-\lambda^2}}, \end{aligned} \tag{50}$$

where a, b, k and λ are arbitrary constants.

Using conditions (49) and (50) in (18), we obtain

$$Y = \mp\frac{m-n}{m}\sqrt{\frac{a}{2k^2-\lambda^2}}\left(X - \frac{bm}{a(n+m)}X^2\right). \tag{51}$$

Combining eq. (51) with eq. (43) and changing to the original variables, we find exact solutions to eq. (37) as

$$\begin{aligned} u(x,y,t) = \Bigg\{\pm\frac{a(n+m)}{2mb}\Bigg[1 - \tanh\Bigg(\pm\frac{m-n}{2m}\sqrt{\frac{a}{2k^2-\lambda^2}} \\ \times \left(x + y - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0\right)\Bigg)\Bigg]\Bigg\}^{1/(n-m)} \end{aligned} \tag{52}$$

and

$$\begin{aligned} u(x,y,t) = \Bigg\{\pm\frac{a(n+m)}{2mb}\Bigg[1 - \coth\Bigg(\pm\frac{m-n}{2m}\sqrt{\frac{a}{2k^2-\lambda^2}} \\ \times \left(x + y - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0\right)\Bigg)\Bigg]\Bigg\}^{1/(n-m)} \end{aligned} \tag{53}$$

for $a(2k^2 - \lambda^2) > 0$. Solutions (52) and (53) respectively represent topological and singular solitons.

Form II

Here, eqs (34) and (36) together imply

$$\frac{\partial^{2\alpha} u^m}{\partial t^{2\alpha}} - k^2 \left(\frac{\partial^2 u^m}{\partial x^2} + \frac{\partial^2 u^m}{\partial y^2} \right) + au^m - bu^{m-n} + cu^{n+m} = 0, t > 0, 0 < \alpha \leq 1. \tag{54}$$

We consider the travelling wave solutions

$$u(x, y, t) = U(\xi), \tag{55}$$

for

$$\xi = x + y - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}. \tag{56}$$

Similarly, as before, eq. (54) leads to the solutions for the following two cases:

Case I. $a(\lambda^2 - 2k^2) < 0$

$$u(x, y, t) = \left\{ \pm 2\sqrt{\frac{bm}{a(2m-n)}} \tanh\left[\frac{n}{2m} \sqrt{\frac{a}{2(2k^2 - \lambda^2)}} \times \left(x + y - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0 \right) \right] \right\}^{2/n} \tag{57}$$

and

$$u(x, y, t) = \left\{ \pm 2\sqrt{\frac{bm}{a(2m-n)}} \coth\left[\frac{n}{2m} \sqrt{\frac{a}{2(2k^2 - \lambda^2)}} \times \left(x + y - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0 \right) \right] \right\}^{2/n} \tag{58}$$

which respectively represent topological soliton solution and singular soliton solution to the equation.

Case II. $a(\lambda^2 - 2k^2) > 0$

$$u(x, y, t) = \left\{ \pm 2\sqrt{-\frac{bm}{a(2m-n)}} \tan\left[\frac{n}{2m} \sqrt{-\frac{a}{2(2k^2 - \lambda^2)}} \times \left(x + y - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0 \right) \right] \right\}^{2/n} \tag{59}$$

and

$$u(x, y, t) = \left\{\pm 2\sqrt{-\frac{bm}{a(2m-n)}}\cot\left[\frac{n}{2m}\sqrt{-\frac{a}{2(2k^2-\lambda^2)}}\times\left(x+y-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}+\xi_0\right)\right]\right\}^{2/n} \tag{60}$$

which are singular periodic solutions to the equations.

6. Conclusions

This paper studied a couple of NLEEs with fractional evolution. They are the foam drainage equation and Klein–Gordon equation in (2 + 1) dimensions. The integration tool used was the first integral method. Topological solitons, singular solitons as well as singular periodic solutions are obtained. It needs to be noted that the singular periodic solutions and soliton solutions are not simultaneous solutions of these NLEEs. In fact, solitons and singular periodic solutions are valid in supplementary constraint conditions or domain restrictions.

The results obtained in this paper are clear indications of the generalized version of the well-known results that stem out of NLEEs with integer derivatives. It was also observed that the final form of the solutions are non-trivial generalization of the solutions with integer evolution. Thus, upon setting $\alpha = 1$ in (26)–(29) as well as in (52), (53) and (57)–(60), these solutions collapse to the solutions that are retrieved from integer evolution. It must be however noted that the reverse is not possible. Given the soliton solution of the NLEE with integer evolution, it is not quite clear how the soliton solution structure of the NLEE with fractional evolution can be written.

The effect of α shows that for KGE, the centre position of topological and singular solitons as well as for singular periodic solutions are shifted. However, for foam drainage equation the soliton solutions as well as singular periodic solutions carry an additional generalization. This happens to the free parameter of these waves, namely the first term inside the argument of (26)–(29).

The future of this research is on a strong footing. Later, studies of this paper will be extended to the perturbed version of these NLEEs. Both deterministic and stochastic perturbation terms will be looked into. The solutions of this extended research will be reported later elsewhere.

References

[1] K S Miller and B Ross, *An introduction to the fractional calculus and fractional differential equations* (Wiley, New York, 1993)
[2] A A Kilbas, H M Srivastava and J J Trujillo, *Theory and applications of fractional differential equations* (Elsevier, San Diego, 2006)
[3] I Podlubny, *Fractional differential equations* (Academic Press, San Diego, 1999)

[4] A Biswas, C Zony and E Zerrad, *Appl. Math. Comput.* **203**(**1**), 153 (2008)
[5] A Biswas, *Int. J. Theor. Phys.* **48**, 256 (2009)
[6] A Biswas, *Nonlinear Dyn.* **58**, 345 (2009)
[7] A Biswas, *Phys. Lett. A* **372**, 4601 (2008)
[8] A Biswas, *Appl. Math. Lett.* **22**, 208 (2009)
[9] W X Ma, *Phys. Lett. A* **180**, 221 (1993)
[10] W Malfliet, *Am. J. Phys.* **60(7)**, 650 (1992)
[11] W X Ma, T W Huang and Y Zhang, *Phys. Scr.* **82**, 065003 (2010)
[12] W X Ma and Z N Zhu, *Appl. Math. Comput.* **218**, 11871 (2012)
[13] N K Vitanov and Z I Dimitrova, *Commun. Nonlinear Sci. Numer. Simulat.* **15(10)**, 2836 (2010)
[14] N K Vitanov, Z I Dimitrova and H Kantz, *Appl. Math. Comput.* **216(9)**, 2587 (2010)
[15] R Hirota, *Phys. Rev. Lett.* **27**, 1192 (1971)
[16] W X Ma, Y Zhang, Y N Tang and J Y Tu, *Appl. Math. Comput.* **218**, 7174 (2012)
[17] W X Ma, *Stud. Nonlinear Sci.* **2**, 140 (2011)
[18] W X Ma and J-H Lee, *Chaos, Solitons and Fractals* **42**, 1356 (2009)
[19] G Jumarie, *Comput. Math. Appl.* **51**, 1367 (2006)
[20] Z S Feng, *J. Phys. A: Math. Gen.* **35**, 343 (2002)
[21] B Lu, *J. Math. Anal. Appl.* **395(2)**, 684 (2012)
[22] A Bekir and O Unsal, *Pramana – J. Phys.* **79**, 3 (2012)
[23] F Tascan, A Bekir and M Koparan, *Commun. Non. Sci. Numer. Simulat.* **14**, 1810 (2009)
[24] I Aslan, *Appl. Math. Comput.* **217**, 8134 (2011)
[25] I Aslan, *Math. Meth. Appl. Sci.* **35**, 716 (2012)
[26] I Aslan, *Pramana – J. Phys.* **76**, 533 (2011)
[27] I Aslan, *AU.P.B. Sci. Bull., Ser. A* **75**, 13 (2013)
[28] N Taghizadeh, M Mirzazadeh and F Farahrooz, *J. Math. Anal. Appl.* **374**, 549 (2011)
[29] M Mirzazadeh and M Eslami, *Nonlin. Anal. Model Control* **17(4)**, 481 (2012)
[30] T R Ding and C Z Li, *Ordinary differential equations* (Peking University Press, Peking, 1996)
[31] Y Zhang and Q Feng, *Appl. Math. Inf. Sci.* **7(4)**, 1575 (2013)
[32] D Weaire, S Hutzler, S Cox, N Kern, M D Alonso and W Drenckhan, *J. Phys.: Condens. Matter* **15**, S65 (2003)
[33] W X Ma and B Fuchssteiner, *Int. J. Nonlinear Mech.* **31**, 329 (1996)
[34] K C Basak, P C Ray and R K Bera, *Commun. Nonlinear Sci. Numer. Simulat.* **14(3)**, 718 (2009)
[35] A Biswas, C Zony and E Zerrad, *Appl. Math. Comput.* **203(1)**, 153 (2008)
[36] G Chen, *Phys. Lett. A* **339(3–5)**, 300 (2005)
[37] A Biswas, A Yildirim, T Hayat, O M Aldossary and R Sassaman, *Proceedings of the Romanian Academy, Series A* **13(1)**, 32 (2012)
[38] R Sassaman and A Biswas, *Nonlinear Dyn.* **61**, 23 (2010)
[39] P Chen and Y Li. *Existence of mild solutions of fractional evolution equations with mixed monotone local conditions, ZAMP*, DOI: 10.1007/s00033-013-0351-z
[40] G Verbist, D Weaire and A M Kraynik. *J. Phys.: Condens. Matter* **8(21)**, 3715 (1996)