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# Modulational instability of nematic phase

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**Abstract.** We numerically observe the effect of homogeneous magnetic field on the modulationally stable case of polar phase in $F = 2$ spinor Bose–Einstein condensates (BECs). Also we investigate the modulational instability of uniaxial and biaxial (BN) states of polar phase. Our observations show that the magnetic field triggers the modulational instability and demonstrate that irrespective of the magnetic field effect the uniaxial and biaxial nematic phases show modulational instability.

**Keywords.** Spinor Bose–Einstein condensates; modulational instability.

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## 1. Introduction

The study of multicomponent Bose–Einstein condensates (BEC) in optical trap have opened up a new field in atomic optics, namely, spinor BECs. These BECs with spin degrees of freedom resulting from optical trap due to the hyperfine spin, explored novel phenomena like creation of vortex states without core, skyrmions, modulational instability, etc. [1–5]. Of these the study on modulational instability (MI) phenomenon, exponential growth of homogeneous condensate, is especially an interesting one because it acts as a basic mechanism to understand some dynamic processes in spinor BECs.

The MI phenomenon is quite common in nonlinear systems. It is well demonstrated in the fields of nonlinear fibre-optics and BECs. Recent studies on MI in spinor BECs have revealed that it can be used to explore the properties of matter waves like spin domain fragmentation, pattern formation, chaotic dynamics, etc. [5,6]. The observations on $F = 1$ spinor BECs have showed that out of the two phases, ferromagnetic phase only shows MI [5]. But later it has been found that in the presence of a homogeneous magnetic field antiferromagnetic $F = 1$ BECs can undergo spatial modulational instability followed by the subsequent generation of spin domains [7]. Further studies on $F = 1$ ferromagnetic phase showed that metastable phases of an antiferromagnetic spin-1 condensate in a simple model with pure contact interactions can exhibit a roton-like minimum in the excitation spectrum [8]. In our recent paper we showed that for $F = 2$ spinor BECs, all phases show MI, but at the same time it has also been found that for antiferromagnetic phase, MI depends on the relative phase between the components and initial population [9].

As the instability of $F = 1$ antiferromagnetic (stable at zero magnetic field) phase is observed in the presence of homogeneous magnetic field, it is interesting to observe the effect of homogeneous magnetic field for the phases of $F = 2$ spinor BECs on the modulationally stable case 2 for ref. [6]. The detailed study for $F = 2$ BECs found that the already defined phases like ferromagnetic, polar and cyclic have the transformation properties of a ferromagnet, a nematic and a tetrahedron respectively, where nematic may be either uniaxial (UN) or biaxial (BN) [10]. Normally, nematic phases are degenerate at mean-field level. The quadratic Zeeman effect and zero-point fluctuation or correction to the mean-field level lift this degeneracy [11,12]. Recently, Kronjager *et al* have observed the spontaneous pattern formation associated with unstable modes of the Gross-Pitaevskii equation for the external magnetic field on antiferromagnetic phase for $F = 2$ spinor condensates [13]. So, it is interesting to study the modulation instability on polar phase for $F = 2$ spinor condensates. In this paper, we study the possibility of MI on nematic phases for zero fields and in the presence of the field.

## 2. Model

In a homogeneous magnetic field $B$, the energy of an atom depends on $B$ as $-p(f_z)_{\rm mn} + q(f_z^2)_{\rm mn}$, where $f_z$ is the $z$-component of the spin matrix with elements as $(f_z)_{\rm mn} = m\delta_{\rm mn}$. The parameters $p$ and $q$ are defined as: $p = -\frac{1}{2}\mu_{\rm B}B$ and $q = (\mu_{\rm B}B)^2/4E_{\rm hf}$, where $\mu_{\rm B}$ is the Bohr magneton and $E_{\rm hf}(= \hbar\omega_{\rm hf})$ is the hyperfine splitting between the states. The multicomponent Gross–Pitaevskii (GP) equations describing the dynamics of $F = 2$ spinor condensates is of the form [14]

$$i\hbar\frac{\partial\varphi_{\pm 2}}{\partial t} = (L \mp 2p + 4q)\varphi_{\pm 2} + c_2\left(\pm 2f^z\varphi_{\pm 2} + f^{\mp}\varphi_{\pm 1}\right) + \frac{c_4}{\sqrt{5}}\Theta\varphi^*_{\mp 2}, \tag{1a}$$

$$i\hbar\frac{\partial\varphi_0}{\partial t} = (L)\varphi_0 + c_2\frac{\sqrt{6}}{2}\left(f^+\varphi_{+1} + f^-\varphi_1\right) + \frac{c_4}{\sqrt{5}}\Theta\varphi^*_0, \tag{1b}$$

$$i\hbar\frac{\partial\varphi_{\pm 1}}{\partial t} = (L \mp p + 4q)\varphi_{\pm 1} + c_2\left(f^{\pm}\varphi_{\pm 2} \pm f^z\varphi_{\pm 1} + \frac{\sqrt{6}}{2}f^{\mp}\varphi_0\right) - \frac{c_4}{\sqrt{5}}\Theta\varphi^*_{\mp 1}, \tag{1c}$$

where

$$L = -\frac{\hbar^2}{2m}\nabla^2 + V_{\rm T} + c_0\left(|\varphi_2|^2 + |\varphi_1|^2 + |\varphi_0|^2 + |\varphi_{-1}|^2 + |\varphi_{-2}|^2\right),$$

$$\Phi = (\varphi_2, \varphi_1, \varphi_0, \varphi_{-1}, \varphi_{-2})$$

are the wave functions of the condensate,

$$n = \sum_{\alpha=-2}^{2} \varphi_\alpha^* \varphi_\alpha$$

is the total number density, $V_{\mathrm{T}}(r)$ is the trapping potential,

$$\Theta = \frac{1}{\sqrt{5}}\left(2\varphi_2\varphi_{-2} - 2\varphi_1\varphi_{-1} + \varphi_0^2\right)$$

is the singlet-pair amplitude, annihilates a pair of bosons at a position, and

$$\begin{aligned} f^+ &= (f^-)^* = 2\left(\varphi_2^*\varphi_1 + \varphi_{-1}^*\varphi_{-2}\right) + \sqrt{6}\left(\varphi_1^*\varphi_0 + \varphi_0^*\varphi_{-1}\right) \\ f^z &= 2\left(|\varphi_2|^2 - |\varphi_{-2}|^2\right) + |\varphi_1|^2 - |\varphi_{-1}|^2. \end{aligned}$$

Here $c_0\frac{(4g_2+3g_4)}{7}$, $c_2 = \frac{(g_4-g_2)}{7}$ and $c_4 = \frac{(7a_0-10a_2+3a_4)}{7}$ are coupling constants, which govern the nonlinear atomic interactions between different spin components, and $g_i = 4\pi\hbar^2 a_{\mathrm{i}}/m (i = 0, 2, 4)$, $a_i$'s are scattering lengths of two colliding bosons.

The linear Zeeman term, $-p(f_z)_{\mathrm{mm}}$, stands for the relative phase shift corresponding to the rotation of spin vector about the $z$-axis. So its effect can be omitted by the transformations

$$\varphi_m \to \mathrm{e}^{-pmt/\hbar}\varphi_m \quad \text{and} \quad f^\pm \to \mathrm{e}^{\mp ipt/\hbar} f^\pm.$$

The dimensionless form of the multicomponent GP equations after the above transformation is given below.

$$i\frac{\partial\varphi_{\pm2}}{\partial t} = (L + 4\tilde{q})\varphi_{\pm2} + \tau_2\left(\pm 2 f^z\varphi_{\pm2} + f^\mp\varphi_{\pm1}\right) + \frac{\tau_4}{\sqrt{5}}\Theta\varphi_{\mp2}^*, \tag{2a}$$

$$i\frac{\partial\varphi_0}{\partial t} = (L)\varphi_0 + \tau_2\frac{\sqrt{6}}{2}\left(f^+\varphi_{+1} + f^-\varphi_{-1}\right) + \frac{\tau_4}{\sqrt{5}}\Theta\varphi_0^*, \tag{2b}$$

$$i\frac{\partial\varphi_{\pm1}}{\partial t} = (L + \tilde{q})\varphi_{\pm1} + \tau_2\left(f^\pm\varphi_{\pm2} \pm f^z\varphi_{\pm1} + \frac{\sqrt{6}}{2}f^\mp\varphi_0\right) - \frac{\tau_4}{\sqrt{5}}\Theta\varphi_{\mp1}^*, \tag{2c}$$

where

$$L = -\nabla^2 + V_{\mathrm{T}} + \tau_0\left(|\varphi_2|^2 + |\varphi_1|^2 + |\varphi_0|^2 + |\varphi_{-1}|^2 + |\varphi_{-2}|^2\right)$$

with wave functions, time and spatial coordinates are measured in the units of $(\hbar/2m\omega_z)^{-3/2}$, $\omega_z^{-1}$ and $(\hbar/2m\omega_z)^{1/2}$. Here coupling constants,

$$\tau_0 = \frac{8\pi(4a_2 + 3a_4)}{7b_0}, \quad \tau_2 = \frac{8\pi(a_4 - a_2)}{7b_0}, \quad \tau_4 = \frac{8\pi(7a_0 - 10a_2 + 3a_4)}{7b_0}$$

with

$$b_0 = \sqrt{\frac{\hbar}{2m\omega_z}} \quad \text{and} \quad \tilde{q} = \frac{q}{\hbar\omega_z}.$$

## 3. Modulation instability analysis

We first observe the effect of the magnetic field on the polar phase for the stationary solutions obtained in our recent study [6], where case 2 of polar phase is stable against the small perturbation. But when we apply a magnetic field, it is observed that the stable state shows modulation instability (figures 1, 2). Similar result can be seen for $F = 1$ where stable antiferromagnetic phase shows MI in the presence of magnetic field [7]. For the numerical analysis, we use harmonic oscillator potential as our trapping potential, $V_{\mathrm{T}}(x, y, z) = \frac{m^2}{2}\left(\omega_x^2 + \omega_y^2 + \omega_z^2\right)$, where $\omega_x$, $\omega_y$ and $\omega_z$ are the trap frequency in $x$-direction, $y$-direction and $z$-direction with the 1D equivalent of eqs (2a)–(2c). As a result of large confinement in $x$- and $y$-directions (transverse) compared to the $z$-direction, the 1D equivalent of $V_{\mathrm{T}}(x, y, z)$ has the form $V_{\mathrm{T}}(z) = \frac{z^2}{4}$. For $^{87}$Rb ground state the hyperfine splitting $\omega_{\mathrm{hf}} \approx 2\pi \cdot 6.835$ GHz [15].

The experimental study in the presence of magnetic field shows that the $\Phi_{\mathrm{af}} = \frac{\sqrt{n}}{2}(1, 0, 0, 0, 1)^{\mathrm{T}}$ state is stable for polar state [9]. For the MI analysis on this state, we use the ansatz,

$$\Phi(r, t) = (\Phi_{\mathrm{af}} + \delta\varphi_j)\mathrm{e}^{-i\mu t},$$

where $\delta\varphi_j = (u_j + i v_j)\cos(\mathbf{kr})\mathrm{e}^{\omega t}$ is the perturbation. We linearize eqs (2a)–(2c) in $\delta\varphi_j$ for the homogeneous condensate. The manipulation of the resulting matrix gives eigenvalues. From the form of perturbation, it is clear that at least one real eigenvalue results in the exponential growth of the condensate. One of resulting eigenvalues is

$$\omega_1^2 = \frac{1}{5}\left(-5k^4 + 10k^2\tau_0 n - 2k^2\tau_4 n\right),$$

where the chemical potential,

$$\mu = 4q + \tau_0 n + \frac{\tau_4}{\sqrt{5}}n$$

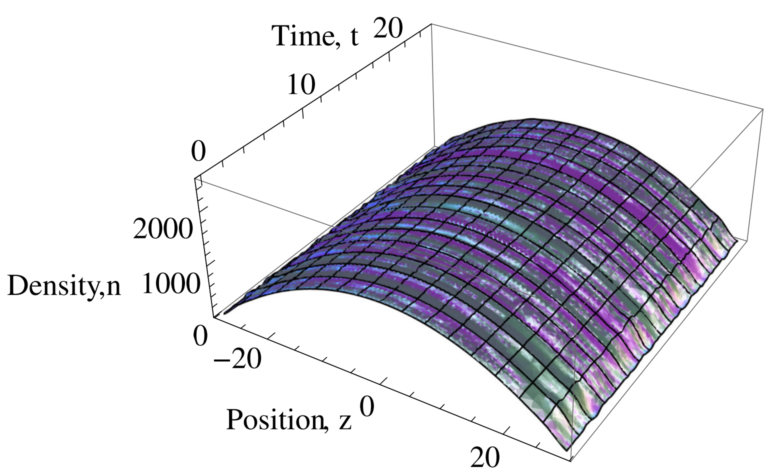


**Figure 1.** Time evolution of the 1D components for the magnetic field free case.

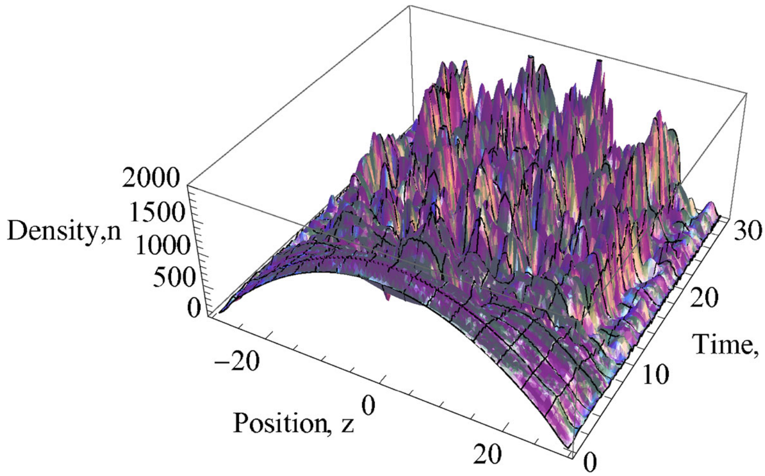


**Figure 2.** Time evolution of the 1D components for the field, $B = 0.175$, showing spatial development of MI.

is given by eqs (2a)–(2c) on the substitution of $\Phi_{af}$. The conditions for polar phase, $\tau_2 > 0$ and $\tau_4 < 0$, show that $\omega_1$ is real for small values of $\mathbf{k}$. It establishes that the dynamically stable ground state is modulationally unstable. Since the coefficient of quadratic Zeeman term is not presented in the eigenvalue $\omega_1$, we can also conclude that irrespective of the field, the ground state $\Phi_{\rm af} = \frac{\sqrt{n}}{2}(1, 0, 0, 0, 1)^{\rm T}$ of polar phase is modulationally unstable.

The ground state of the polar phase is represented by uniaxial nematic (UN) or biaxial nematic (BN) with order parameters $\Phi_{\rm UN} = \left(0, 0, \sqrt{n}, 0, 0\right)^{\rm T}$ for UN and $\Phi_{\rm BN} = \frac{\sqrt{n}}{2}(1, 0, 0, 0, 1)^{\rm T}$ for BN respectively [12]. The effect of magnetic field is mainly due to the quadratic Zeeman term and it can have either positive or negative coefficient. It has been found that positive coefficient favours UN phase and the negative coefficient favours square phase [11]. Also, it has been found that on increasing the magnetic field, the square state approaches the BN state. Since our state $\Phi_{\rm af}$ is similar to the BN state, it is clear that BN phase shows modulation instability. Similar analysis on UN state, $\Phi_{\rm UN} = \left(0, 0, \sqrt{n}, 0, 0\right)^{\rm T}$ with chemical potential, $\mu = 4q + \tau_0 n + \frac{\tau_4}{\sqrt{5}} n$ gives one of the eigenvalues, $\omega_1^2 = \frac{1}{5}\left(-5k^4 + 10k^2\tau_0 n + 2k^2\tau_4 n\right)$. As $\tau_0 > \tau_4$, the eigenvalue $\omega_1$ has real values. It demonstrates the modulation instability of the UN phase. Here $\omega_1$ indicates that, irrespective of the sign of the quadratic Zeeman term, UN phase is modulationally unstable.

## 4. Conclusion

As a first step, we numerically observed the effect of magnetic field on the modulationally stable case of polar phase in $F = 2$ spinor condensates. Our observations showed the periodic pattern formation. So we concluded that the magnetic field triggers the formation of periodic patterns in spinor systems. In order to get the picture for MI of uniaxial and biaxial states, we have analytically observed the eigenvalues resulting from the multicomponent GP equation after adding a small perturbation. Our results demonstrate that, irrespective of the quadratic Zeeman effect, the UN and BN phases show modulational instability.

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## References


[1] K-P Marzlin, W Zhang and B C Sanders, *Phys. Rev. A* **62**, 013602 (2000)
[2] C K Law, H Pu and N P Bigelow, *Phys. Rev. Lett.* **81**, 5257 (1998)
[3] T-L Ho, *Phys. Rev. Lett.* **81**, 742 (1998)
[4] T Ohmi and K Machida, *J. Phys. Soc. Jpn.* **67**, 1822 (1998)
[5] N P Robins, W Zhang, E A Ostrovskaya and Y S Kivshar, *Phys. Rev. A* **64**, 021601 (2001)
[6] T Mithun and K Porsezian, *Phys. Rev. A* **85**, 013616 (2012)
[7] M Matuszewski, T J Alexander and Y S Kivshar, *Phys. Rev. A* **78**, 023632 (2008)
[8] M Matuszewski, *Phys. Rev. Lett.* **105**, 020405 (2010)
[9] H Schmaljohann *et al*, *Phys. Rev. Lett.* **92**, 040402 (2004)
[10] A M Turner, R Barnett, E Demler and A Vishwanath, *Phys. Rev. Lett.* **98**, 190404 (2007)
[11] R Barnett, A Turner and E Demler, *Phys. Rev. Lett.* **97**, 180412 (2006)
[12] Y Kawaguchi and M Ueda, *Phys. Rep.* **520**, 253 (2012)
[13] J Kronjager, C Becker, P S Panahi, K Bongs and K Sengstock, *Phys. Rev. Lett.* **105**, 090402 (2010)
[14] H Saito and M Ueda, *Phys. Rev. A* **72**, 053628 (2005)
[15] J Kronjager, C Becker, P Navez, K Bongs and K Sengstock, *Phys. Rev. Lett.* **97**, 110404 (2006)