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The integrability of an extended fifth-order KdV equation with Riccati-type pseudopotential

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Abstract. The extended fifth-order KdV equation in fluids is investigated in this paper. Based on the concept of pseudopotential, a direct and unifying Riccati-type pseudopotential approach is employed to achieve Lax pair and singularity manifold equation of this equation. Moreover, this equation is classified into three categories: extended Caudrey–Dodd–Gibbon–Sawada–Kotera (CDGSK) equation, extended Lax equation and extended Kaup–Kuperschmidt (KK) equation. The corresponding singularity manifold equations and auto-Bäcklund transformations of these three equations are also obtained. Furthermore, the infinitely many conservation laws of the extended Lax equation are found using its Lax pair. All conserved densities and fluxes are given with explicit recursion formulas.

Keywords. Extended fifth-order KdV equation; Riccati-type pseudopotential; Lax pair; Schwarzian derivative.

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1. Introduction

Soliton theory is widely used in physics, biology, chemistry, etc. [1–5]. The study on the integrability of nonlinear evolution equations (NLEEs) play an important role in soliton theory, which can be regarded as a pretest and the first step of its exact solvability [6]. There are many significant properties, such as Bäcklund transformation (BT), Lax pair, infinitely many conservation laws, Hamiltonian structure and Painlevé test that can characterize the integrability of NLEEs. In a fundamental and outstanding paper [7], the Wahlquist–Estabrook prolongation structure method was first proposed and used to find BTs and inverse scattering equations of NLEEs in (1+1) dimensions, in which the concept of pseudopotential was introduced. Based on the properties of the Riccati ordinary differential equation [8], Nucci considered the Riccati-type pseudopotentials of NLEEs, and extended it to derive Lax pairs, auto-BTs and singularity manifold equations of the NLEEs both in (1+1) dimensions and (2+1) dimensions [9,10]. Moreover, this method was extended to investigate variable coefficient NLEEs, which included generalized KdV equation, generalized modified KdV equation and generalized Boussinesq equation [11,12].

It is well known that KdV equation models a variety of nonlinear phenomena, including ion-acoustic waves in plasmas and shallow water waves [13–23]. The canonical KdV equation is a linear dispersive equation of third order. However, the KdV equations appear in three, five, seven or more order forms [13]. In the present paper, we consider the extended fifth-order KdV (efKdV) equation in the form [24]

$$q_t + c_1 q^2 q_x + c_2 q_x q_{2x} + c_3 q q_{3x} + c_4 q_{5x} + c_5 q q_x + c_6 q_{3x} + c_7 q_x = 0, \quad (1)$$

where q is a differentiable function of x and t, and $c_i (i = 1, \ldots, 7)$ are arbitrary constants. The efKdV eq. (1) can reduce to a series of integrable models or can describe such physical phenomena as the amplitude of the shallow-water wave and/or surface wave in fluids. Obviously, eq. (1) contains quite a number of KdV-type equations, e.g., the KdV equation, the Sawada–Kotera (SK) equation [14], the Caudrey–Dodd–Gibbon (CDG) equation [15], the Lax equation [16], the Kaup–Kuperschmidt (KK) equation [17–19], the Ito equation [20], the combined KdV-CDG equation and KdV-Lax equation [21].

The aim of this paper is to investigate Lax pair, auto-BT, singularity manifold equation and infinitely many conservation laws of the efKdV equation (1) by using its Riccati-type pseudopotential. In §2, the Riccati-type pseudopotential is obtained with the help of an ansatz of the pseudopotential u for efKdV equation (1). Based on the pseudopotential system, both the Lax pair and singularity manifold equation of (1) are obtained by means of two different transformations. In §3, the efKdV equation (1) is classified into three types of equations by choosing different coefficients, which include extended CDGSK equation (20), extended Lax equation (35) and extended KK equation (29). Moreover, both the singularity manifold equations and auto-BTs of these equations are derived with the help of three kinds of different transformations and Möbius transformation, respectively. In §4, the infinitely many conservation laws of the extended Lax equation are found by using its Lax pair. All conserved densities and fluxes are given with explicit recursion formulas. Conclusions are presented in §5.

2. Riccati-type pseudopotential, Lax pair and singularity manifold equation

Riccati-type equation has wide applications in modern theory of solitons. For instance, the Riccati-type equation can be used to construct various exact solutions of NLEEs [25–27]. It is also found that if the equations satisfied by the pseudopotential are of Riccati-type, then one can easily obtain the Lax pairs and auto-BTs of the corresponding NLEEs [9,10]. Furthermore, if a pseudopotential has the form

$$u_x = ku^2 + F_1(q)u + F_0(q), \quad (2a)$$

$$u_t = G(u, q, q_x, \ldots, q_{4x}), \quad (2b)$$

where k is an arbitrary constant and G is a polynomial of second order in u, then the Lax pair with dependent variable $\psi = \psi(x, t)$ and the singularity manifold equation with dependent variable $\phi = \phi(x,t)$ can be obtained immediately by the following transformations:

$$u = -\frac{1}{k}(\ln \psi)_x \tag{3}$$

and

$$u = \frac{1}{2k}(\ln \phi_x)_x. \tag{4}$$

In the following, in terms of the natural ansatz for pseudopotential system (2) of efKdV equation (1), the Lax pair and singularity manifold equation will be presented.

To begin with, with the help of the integrability condition $u_{x,t} = u_{t,x}$ of pseudopotential system (2) and symbolic computation, we obtain a kind of solution as given below:

$$u_x = ku^2 + d_1 u + \frac{3c_3 - c_2}{10kc_4} q + d, \tag{5a}$$

$$\begin{aligned} u_t = \Big[& \frac{c_2 - 3c_3}{50c_4}(10c_4 q_{2x} + 10(c_4 d_1^2 - 4kc_4 d + c_6)q + (2c_2 - c_3)q^2)u \\ & - ((8kdc_4 - c_6)d_1^2 + c_4 d_1^4 + 16c_4 k^2 d^2 - 4kdc_6 + c_7)u \\ & + \frac{c_2 - 3c_3}{50c_4 k}(5c_4 q_{3x} + 5c_4 d_1 q_{2x} + ((2c_2 - c_3)q \\ & + 5c_4 d_1^2 + 5c_6 - 20kdc_4)q_x + \frac{(2c_2 - c_3)d_1}{2} q^2 \\ & - 5d_1(4kdc_4 - c_6 - d_1^2 c_4)q) \Big]_x, \end{aligned} \tag{5b}$$

under the constraint condition

$$\begin{aligned} c_1 &= -\frac{(2c_2 - c_3)(c_2 - 3c_3)}{10c_4}, \\ c_5 &= (2c_3 - c_2)d_1^2 + 4k(c_2 - 2c_3)d - \frac{3(c_2 - 3c_3)c_6}{5c_4}, \end{aligned} \tag{6}$$

with

$$u^2 = \frac{u_x - d_1 u - d}{k} + \frac{c_2 - 3c_3}{10k^2 c_4} q. \tag{7}$$

For simplicity, we take $k = 1$ and $d_1 = 0$, and the system (5) leads to the following Riccati-type pseudopotential u such that

$$u_x = u^2 + \frac{3c_3 - c_2}{10c_4} q + d, \tag{8a}$$

$$\begin{aligned} u_t = \Big[& \frac{c_2 - 3c_3}{50c_4}(10c_4 q_{2x} + 10(c_6 - 4c_4 d)q + (2c_2 - c_3)q^2)u \\ & - (16c_4 d^2 - 4c_6 d + c_7)u \\ & + \frac{c_2 - 3c_3}{50c_4}(5c_4 q_{3x} + ((2c_2 - c_3)q + 5c_6 - 20c_4 d)q_x) \Big]_x, \end{aligned} \tag{8b}$$

under the constraint condition

$$c_1 = -\frac{(2c_2 - c_3)(c_2 - 3c_3)}{10c_4}, \quad c_5 = 4(c_2 - 2c_3)d - \frac{3(c_2 - 3c_3)c_6}{5c_4}, \tag{9}$$

with the requirement that $u_{x,t} = u_{t,x}$ whenever (1) is satisfied.

Next, in terms of the transformation

$$u = -(\ln \psi)_x, \tag{10}$$

the Riccati-type pseudopotential system (8) can be directly linearized to the following system:

$$\psi_{2x} = \frac{c_2 - 3c_3}{10c_4} q\psi - d\psi, \tag{11a}$$

$$\begin{aligned}\psi_t = {} & \frac{c_2 - 3c_3}{50c_4}[(10c_4 q_{2x} + (2c_2 - c_3)q^2 - 10(4c_4 d - c_6)q)\psi_x \\ & - (16c_4 d^2 - 4c_6 d + c_7)\psi_x + ((c_3 - 2c_2)qq_x \\ & + 5(4c_4 d - c_6)q_x - 5c_4 q_{3x})\psi],\end{aligned} \tag{11b}$$

under the constraint condition (9). It is easy to check that the integrability condition of (11) exactly gives the efKdV equation (1) under the constraint condition (9). Thus, the system (11) can be regarded as the Lax pair of eq. (1).

Moreover, if we use the linearizing transformation

$$u = \frac{v_1}{v_2}, \tag{12}$$

instead of transformation (10), the Lax pair of the AKNS form can also be obtained in the form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} 0 & d - [(c_2 - 3c_3)/10c_4]q \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{13a}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{13b}$$

where

$$\begin{aligned}A = {} & \frac{c_2 - 3c_3}{50c_4}[5c_4 q_{3x} + ((2c_2 - c_3)q + 5c_6 - 20c_4 d)q_x], \\ B = {} & \frac{c_2 - 3c_3}{50c_4}\Big[5q_{4x} + ((c_2 + 2c_3)q - 5(2c_4 d - c_6))q_{2x} \\ & + \frac{(6c_2 - 13c_3)d - c_6(c_2 - 3c_3)}{c_4}q^2 - \frac{2c_2 - c_3}{10c_4}q^3 + (2c_2 - c_3)q_x^2 \\ & + 5(8c_4 d^2 - 2c_6 d + c_7)q\Big] - (16c_4 d^2 - 4c_6 d + c_7)d, \\ C = {} & \frac{c_2 - 3c_3}{50c_4}[-10c_4 q_{2x} - (2c_2 - c_3)q^2 + 10(4c_4 d - c_6)q] \\ & + 4d(4c_4 d - c_6) + c_7.\end{aligned} \tag{14}$$

Furthermore, under the transformation

$$u = \frac{1}{2}(\ln \phi_x)_x, \tag{15}$$

the pseudopotential system (8) leads to the singularity manifold equation of efKdV equation (1) in the form

$$\frac{\phi_t}{\phi_x} = -c_4\{\phi, x\}_{2x} + \frac{c_4(2c_2 - c_3)}{2(c_2 - 3c_3)}\{\phi, x\}^2 - \left(\frac{10c_3c_4d}{c_2 - 3c_3} + c_6\right)\{\phi, x\} + 6c_6d - c_7 - \frac{10(2c_2 - 7c_3)c_4}{c_2 - 3c_3}d^2, \tag{16}$$

in which $\{\phi, x\}$ is the Schwarzian derivative

$$\{\phi, x\} = \left(\frac{\phi_{2x}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{2x}}{\phi_x}\right)^2. \tag{17}$$

3. Auto-Bäcklund transformation

Conformal invariance, namely the invariance of Schwartz form under the Möbious transformation, plays an important role in the soliton theory. Integrable models are related to the conformal invariance, such as new integrable models, Darboux transformation, Bäcklund transformation, infinitely many conservation laws etc., and infinitely many non-local symmetries can be obtained from the conformal invariance of an integrable model [28,29]. In the following, firstly, the efKdV equation (1) is classified into three kinds of extended fifth-order KdV equations, and their corresponding singularity manifold equations are also presented. Then, based on the concept of conformal invariance, their auto-BTs are also obtained.

In order to seek the new singularity manifold equation of eq. (1), we consider the transformation

$$u = r(\ln \phi_x)_x, \tag{18}$$

which leads to the following results:

$$r = \frac{1}{2}, \quad c_2 = c_2, \quad c_6 = c_6, \tag{19a}$$

$$r = -1, \quad c_2 = c_3, \quad c_6 = 5dc_4, \tag{19b}$$

$$r = -\frac{1}{4}, \quad c_2 = \frac{5}{2}c_3, \quad c_6 = 20dc_4, \tag{19c}$$

$$r = -\frac{1}{2}, \quad c_2 = 2c_3, \quad c_6 = c_6. \tag{19d}$$

Equation (19a) just corresponds to the transformation (15), which means we just need to discuss the remaining cases.

3.1 *Extended fifth-order CDGSK equation*

The case (19b) implies that eq. (1) can be read as

$$q_t + \frac{c_3^2}{5c_4}q^2q_x + c_3q_xq_{2x} + c_3qq_{3x} + c_4q_{5x} + 2dc_3qq_x + 5dc_4q_{3x} + c_7q_x = 0, \tag{20}$$

which is related to the case of CDG equation [15] and SK equation [14].

With the help of the pseudopotential system (8) and the transformation

$$u = -(\ln \phi_x)_x, \tag{21}$$

singularity manifold equation of eq. (20) can be directly obtained as

$$\frac{\phi_t}{\phi_x} = -c_4\{\phi, x\}_{2x} - 4c_4\{\phi, x\}^2 + 5c_4 d^2 - c_7, \tag{22}$$

with

$$q = -\frac{5c_4}{c_3} d - \frac{5c_4 \phi_{3x}}{c_3 \phi_x}. \tag{23}$$

As is well-known, the Schwarzian derivative (17) is invariant under the Möbius transformation

$$\phi^* = \frac{a_1 + b_1 \phi}{c_1 + d_1 \phi}, \tag{24}$$

which means singularity manifold equation (22) is also invariant under Möbius transformation (24). For simplicity, if we choose $b_1 = c_1 = 0, a_1 = d_1 = 1$, this implies that

$$\phi^* = \phi^{-1}, \tag{25}$$

is also a solution of eq. (22), and the corresponding

$$q^* = -\frac{5c_4}{c_3} d - \frac{30c_4}{c_3}\frac{\phi_x^2}{\phi^2} + \frac{30c_4}{c_3}\frac{\phi_{2x}}{\phi} - \frac{5c_4}{c_3}\frac{\phi_{3x}}{\phi_x}, \tag{26}$$

is another solution of (20).

By means of (23) and (26), the spatial part of an auto-BT for eq. (20) can be obtained

$$\begin{aligned} &90c_3c_4(Q^*_{2x} + Q_{2x})(Q^* - Q) + 900c_4^2 d(Q^* - Q) \\ &\quad + c_3^2(Q^* - Q)^3 + 900c_4^2(Q^*_{2x} - Q_{2x}) = 0, \end{aligned} \tag{27}$$

with

$$Q = \int q \mathrm{d}x \quad \text{and} \quad Q^* = \int q^* \mathrm{d}x. \tag{28}$$

3.2 *Extended fifth-order KK equation*

The case (19c) implies that eq. (1) can be read as

$$q_t + \frac{c_3^2}{5c_4} q^2 q_x + \frac{5c_3}{2} q_x q_{2x} + c_3 q q_{3x} + c_4 q_{5x} + 8dc_3 q q_x + 20dc_4 q_{3x} + c_7 q_x = 0, \tag{29}$$

which is related to the case of KK equation [17–19].

The singularity manifold equation of eq. (29) can be derived by virtue of pseudopotential system (8) and transformation

$$u = -\frac{1}{4}(\ln \phi_x)_x. \tag{30}$$

The result reads as

$$\frac{\phi_t}{\phi_x} = -c_4\{\phi, x\}_{2x} - \frac{c_4}{4}\{\phi, x\}^2 + 80c_4 d^2 - c_7, \tag{31}$$

with

$$q = -\frac{5c_4}{c_3}\{\phi, x\} - \frac{15c_4}{4c_3}\frac{\phi_{2x}^2}{\phi_x^2} - \frac{20c_4}{c_3}d. \tag{32}$$

It is straightforward to obtain another solution of eq. (29):

$$q^* = -\frac{15c_4}{c_3}\left(\frac{\phi_x^2}{\phi^2} - \frac{\phi_{2x}}{\phi} - \frac{\phi_{2x}^2}{4\phi_x^2} + \frac{\phi_{3x}}{3\phi_x} + \frac{4}{3}d\right), \tag{33}$$

and the auto-Bäcklund transformation

$$\begin{aligned} &900c_4^2(Q^* - Q)(Q_{2x}^* - Q_{2x}) + c_3^2(Q^* - Q)^4 \\ &\quad + 90c_4[40c_3 d + c_3(Q_x^* + Q_x)](Q^* - Q)^2 - 675c_4^2(Q_x^* - Q_x)^2 = 0. \end{aligned} \tag{34}$$

3.3 *Extended fifth-order Lax equation*

The case (19d) implies that eq. (1) can be read as

$$q_t + \frac{3c_3^2}{10c_4}q^2 q_x + 2c_3 q_x q_{2x} + c_3 q q_{3x} + c_4 q_{5x} + \frac{3c_3c_6}{5c_4}q q_x + c_6 q_{3x} + c_7 q_x = 0. \tag{35}$$

In terms of the transformation

$$u = -\frac{1}{2}(\ln \phi_x)_x, \tag{36}$$

and system (8) with condition (19d), the singularity manifold equation of eq. (35) can be obtained as

$$\begin{aligned} \frac{\phi_t}{\phi_x} &= -c_4\{\phi, x\}_{2x} - \frac{3c_4}{2}\{\phi, x\}^2 + (10c_4 d - c_6)\{\phi, x\} \\ &\quad + 6c_6 d - 30c_4 d^2 - c_7, \end{aligned} \tag{37}$$

with

$$q = -\frac{5c_4}{c_3}\{\phi, x\} - \frac{5c_4}{c_3}\frac{\phi_{2x}^2}{\phi_x^2} - \frac{10c_4}{c_3}d. \tag{38}$$

It is straightforward to obtain another solution of eq. (35)

$$\begin{aligned} q^* &= -\frac{5c_4}{c_3}\{\phi^*, x\} - \frac{5c_4}{c_3}\frac{\phi_{2x}^{*2}}{\phi_x^{*2}} - \frac{10c_4}{c_3}d \\ &= -\frac{5c_4}{c_3}\left(\frac{4\phi_x^2}{\phi^2} - \frac{4\phi_{2x}}{\phi} - \frac{\phi_{2x}^2}{2\phi_x^2} + \frac{\phi_{3x}}{\phi_x} + 2d\right), \end{aligned} \tag{39}$$

and the auto-Bäcklund transformation

$$\begin{aligned} &400c_4^2(Q_x^* - Q_x)^2 - 800c_4^2(Q^* - Q)(Q_{2x}^* - Q_{2x}) - c_3^2(Q^* - Q)^4 \\ &\quad -80c_4[20c_4 d + c_3(Q_x^* + Q_x)](Q^* - Q)^2 = 0. \end{aligned} \tag{40}$$

4. Infinitely many conservation laws

One of the many remarkable properties that characterizes the soliton equations is the existence of an infinite sequence of conservation laws. In this section, we shall construct the infinitely many conservation laws of the extended Lax equation (35) by using its Lax pair.

Lax pair of eq. (35) reads as

$$\psi_{2x} = -\frac{c_3}{10c_4} q\psi - d\psi, \tag{41a}$$

$$\begin{aligned}\psi_t =& -\frac{1}{50c_4}[10c_3c_4q_{2x} + 3c_3^2q^2 + (10c_3c_6 - 40dc_3c_4)q \\ &- 200dc_4c_6 + 800d^2c_4^2 + 50c_4c_7]\psi_x \\ &+ \frac{c_3}{50c_4}[5c_4q_{3x} + (3c_3q - 20dc_4 + 5c_6)q_x]\psi.\end{aligned} \tag{41b}$$

Now, we introduce a transformation

$$\psi_x = \eta\psi, \tag{42}$$

on account of which, the Lax pair (41) can be transformed into a Riccati-type equation

$$\eta_x + \eta^2 + \frac{c_3}{10c_4} q + d = 0 \tag{43}$$

and a divergence-type equation

$$\begin{aligned}\eta_t &+ [6c_4\eta^5 - (2c_6 - 20dc_4)\eta^3 - 10c_4\eta^2\eta_{2x} - (10c_4\eta_x^2 - c_7 + 6dc_6 \\ &-30d^2c_4)\eta - (10dc_4 - c_6)\eta_{2x} + c_4\eta_{4x}]_x = 0.\end{aligned} \tag{44}$$

By inserting the expansion

$$\eta = \varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n(q, q_x, \ldots)\varepsilon^{-n}, \tag{45}$$

into (43) and equating the coefficients for power of ε, we explicitly obtain the recursion relation for the conserved densities $\mathscr{I}_n$'s as follows:

$$\mathscr{I}_1 = -\frac{c_3}{20c_4} q, \tag{46a}$$

$$\mathscr{I}_2 = -\frac{1}{2}\mathscr{I}_{1,x} = \frac{c_3}{40c_4} q_x, \tag{46b}$$

$$\ldots, \tag{46c}$$

$$\mathscr{I}_{n+1} = -\frac{1}{2}\left(\mathscr{I}_{n,x} + \sum_{k=1}^{n} \mathscr{I}_k\mathscr{I}_{n-k}\right), \quad n = 2, 3, \ldots \tag{46d}$$

with $d = -\varepsilon^2$.

In addition, substituting expansion (45) into the divergence-type equation (44) leads to the infinitely many conservation laws

$$\mathscr{I}_{n,t} + \mathscr{F}_{n,x} = 0, \quad n = 1, 2, \ldots, \tag{47}$$

of eq. (35), where the fluxes $\mathscr{F}_n$'s are given by

$$\begin{aligned}
\mathscr{F}_1 &= \frac{c_3}{200c_4^2}[10c_4^2 q_{4x} + 10c_4(c_6 + c_3 q)q_{2x} \\
&\quad + 5c_3c_4q_x^2 + q(c_3^2q^2 + 10c_4c_7 + 3c_3c_6q)], && \text{(48a)}\\
\mathscr{F}_2 &= \frac{-c_3}{400c_4^2}[10c_4^2 q_{5x} + 20c_3c_4q_xq_{2x} + (3c_3^2q^2 + 10c_4c_7 + 6c_3c_6q)q_x \\
&\quad + 10c_4(c_6 + c_3q)q_{3x}], && \text{(48b)}\\
&\ldots, && \text{(48c)}\\
\mathscr{F}_n &= 40c_4 \sum_{i+j+k=n+2} \mathscr{I}_i\mathscr{I}_j\mathscr{I}_k + 30c_4 \sum_{i+j+k+l=n+1} \mathscr{I}_i\mathscr{I}_j\mathscr{I}_k\mathscr{I}_l \\
&\quad - 6c_6 \sum_{i+j=n+1} \mathscr{I}_i\mathscr{I}_j + 6c_4 \sum_{i+j+k+l+m=n} \mathscr{I}_i\mathscr{I}_j\mathscr{I}_k\mathscr{I}_l\mathscr{I}_m \\
&\quad - 2c_6 \sum_{i+j+k=n} \mathscr{I}_i\mathscr{I}_j\mathscr{I}_k + c_6\mathscr{I}_{n,2x} + c_4\mathscr{I}_{n,4x} \\
&\quad - 10c_4 \sum_{i+j+k=n} \mathscr{I}_i\mathscr{I}_j\mathscr{I}_{k,2x} - 10c_4 \sum_{i+j+k=n} \mathscr{I}_{i,x}\mathscr{I}_{j,x}\mathscr{I}_k \\
&\quad + c_7\mathscr{I}_n + 10c_4\mathscr{I}_{n+2,2x}. && \text{(48d)}
\end{aligned}$$

We present the recursion formulas (46) and (48) for generating an infinite sequence of conservation laws for eq. (35). The first equation of conservation law (47) is exactly the extended Lax equation (35).

5. Conclusions

In this paper, we have obtained the Riccati-type pseudopotential, Lax pair and singularity manifold equations for the efKdV equation (1). By choosing different coefficients, the corresponding results and the auto-BTs are also obtained on three conditions which includes extended CDGSK equation, extended Lax equation and extended KK equation. Moreover, we construct the infinitely many conservation laws of the extended Lax equation by using its Lax pair. In conclusion, the efKdV equation (1) is completely integrable under condition (6) in the sense that it admits the Lax pair, singularity manifold equation, auto-BT and infinitely many conservation laws. These results imply that we can investigate the other integrability of this equation, such as finding nonlocal symmetries from its conformal invariance.

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