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Numerical solution of the one-dimensional Burgers' equation: Implicit and fully implicit exponential finite difference methods

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Abstract. This paper describes two new techniques which give improved exponential finite difference solutions of Burgers' equation. These techniques are called implicit exponential finite difference method and fully implicit exponential finite difference method for solving Burgers' equation. As the Burgers' equation is nonlinear, the scheme leads to a system of nonlinear equations. At each time-step, Newton's method is used to solve this nonlinear system. The results are compared with exact values and it is clearly shown that results obtained using both the methods are precise and reliable.

Keywords. Burgers' equation; exponential finite difference method; implicit exponential finite difference method; fully implicit exponential finite difference method.

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1. Introduction

In this paper, we consider the one-dimensional non-linear Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - v\frac{\partial^2 u}{\partial x^2} = 0, \qquad a < x < b, \tag{1}$$

with the initial condition

$$u(x, 0) = g(x), \qquad a < x < b$$

and the boundary conditions

$$u(a, t) = h_1(t) \quad \text{and} \quad u(b, t) = h_2(t), \qquad t > 0,$$

where v is the positive coefficient of kinematic viscosity and g, h_1 and h_2 are the prescribed functions of the variables.

Burgers' equation is found to describe various kinds of phenomena such as mathematical model of turbulence and the approximate theory of flow through a shock wave travelling in a viscous fluid [1].

In literature, many numerical methods have been proposed and implemented for approximating solution of the Burgers' equation. Many authors have used numerical techniques based on finite difference [1–8], finite element [9–13] and boundary element [14] methods in attempting to solve the equation. Kadalbajoo *et al* [15] used a parameter-uniform implicit difference scheme for solving time-dependent Burgers' equation. The explicit exponential finite difference method was originally developed by Bhattacharya [16] for solving heat equation. Bhattacharya [17] and Handschuh and Keith [18] used exponential finite difference method for solving Burgers' equation. Bahadır solved the KdV equation by using the exponential finite difference technique [19].

In this paper, we design two new schemes for solving the Burgers' equation. Some examples are presented to show the ability of these methods to solve the equation. It is clearly seen that both numerical methods are reasonably in good agreement with the exact solution.

2. Methods of solution

We obtain numerical solutions of the Burgers' equation by implicit exponential finite difference method and fully implicit exponential finite difference method for three standard problems. The accuracy of the proposed methods are measured using the L_2 and L_∞ error norms defined by

$$L_2 = \|u - U\|_2 = \left(h \sum_{i=0}^{N} |u_i - U_i|^2 \right)^{1/2},$$
$$L_\infty = \|u - U\|_\infty = \max_{0 \le i \le N} |u_i - U_i|. \tag{2}$$

The solution domain is discretized into cells described by the node set (x_i, t_n) in which $x_i = ih$ $(i = 0, 1, 2, ..., N)$ and $t_n = nk$ $(n = 0, 1, 2, ...)$, $h = \Delta x$ is the spatial mesh size and $k = \Delta t$ is the time-step.

2.1 *Implicit exponential finite difference scheme*

The implicit exponential finite difference method (I-EFDM) for eq. (1) takes the following nonlinear form:

$$U_i^{n+1} = U_i^n \exp\left\{ \frac{v \Delta t}{(\Delta x)^2} \left[-\frac{\Delta x U_i^n}{2v} \frac{\left(U_{i+1}^{n+1} - U_{i-1}^{n+1}\right)}{U_i^n} + \frac{\left(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}\right)}{U_i^n} \right] \right\} \tag{3}$$

which is valid for values of i lying in the interval $1 \le i \le N - 1$.

2.2 *Fully implicit exponential finite difference scheme*

The fully implicit exponential finite difference method (FI-EFDM) for eq. (1) takes the following nonlinear form:

$$U_i^{n+1} = U_i^n \exp\left\{ \frac{\upsilon \Delta t}{(\Delta x)^2} \left[-\frac{\Delta x U_i^{n+1}}{2\upsilon} \frac{\left(U_{i+1}^{n+1} - U_{i-1}^{n+1}\right)}{U_i^n} + \frac{\left(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}\right)}{U_i^n} \right] \right\} \tag{4}$$

which is valid for values of i lying in the interval $1 \le i \le N-1$.

Here U_i^n denotes the exponential finite difference approximation to the exact solution $u(x,t)$. Equations (3) and (4) are systems of nonlinear difference equations. Let us consider these nonlinear systems of equations in the form

$$\mathbf{F}(\mathbf{V}) = \mathbf{0}, \tag{5}$$

where $\mathbf{F} = [f_1, f_2, ..., f_{N-1}]^T$ and $\mathbf{V} = [U_1^{n+1}, U_2^{n+1}, ..., U_{N-1}^{n+1}]^T$. Newton's method applied to eq. (5) results in the following iteration:

(1) Set $\mathbf{V}^{(0)}$, an initial guess.
(2) For $m = 0, 1, 2, ...$ until convergence do:
Solve $J(\mathbf{V}^{(m)})\delta^{(m)} = -\mathbf{F}(\mathbf{V}^{(m)})$;
Set $\mathbf{V}^{(m+1)} = \mathbf{V}^{(m)} + \delta^{(m)}$,

where $J(\mathbf{V}^{(m)})$ is the Jacobian matrix which is evaluated analytically. The solution at the previous time-step is taken as the initial estimate. The Newton's iteration at each time-step is stopped when $\left\|\mathbf{F}(\mathbf{V}^{(m)})\right\|_\infty \le 10^{-5}$. The convergence is generally obtained in two or three iterations.

3. Numerical results

Problem 1.

We first solve the Burgers' equation (1) and the initial condition

$$u(x,0) = \sin(\pi x), \quad 0 < x < 1$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0$$

and the exact solution given by

$$u(x,t) = \frac{2\pi\upsilon \sum_{n=1}^{\infty} a_n \exp\left(-n^2\pi^2\upsilon t\right) n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n \exp\left(-n^2\pi^2\upsilon t\right) \cos(n\pi x)} \tag{6}$$

Table 1. Comparison of the I-EFDM solutions with the exact solution at $t = 0.1$ for $v = 1$ and $k = 10^{-5}$ using various mesh sizes.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	$h = 0.01$	Exact
0.1	0.109737	0.109595	0.109560	0.109556	0.109538
0.2	0.210184	0.209905	0.209835	0.209826	0.209792
0.3	0.292464	0.292059	0.291958	0.291945	0.291896
0.4	0.348637	0.348127	0.348000	0.347984	0.347924
0.5	0.372384	0.371806	0.371662	0.371644	0.371577
0.6	0.359872	0.359279	0.359131	0.359113	0.359046
0.7	0.310656	0.310116	0.309981	0.309965	0.309905
0.8	0.228393	0.227979	0.227875	0.227817	0.227817
0.9	0.121000	0.120774	0.120718	0.120687	0.120687
L_2	0.000579	0.000164	0.000060	0.000048	
L_∞	0.000827	0.000234	0.000086	0.000068	

with

$$a_0 = \int_0^1 \exp\left\{-(2\pi v)^{-1}\,[1-\cos(\pi x)]\right\} \mathrm{d}x$$

$$a_n = 2\int_0^1 \exp\left\{-(2\pi v)^{-1}\,[1-\cos(\pi x)]\right\} \cos(n\pi x)\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots .$$

The results for Problem 1 are displayed in tables 1–3 and figure 1. The numerical solution obtained by implicit exponential finite difference method and the exact solution for different values of h are presented in table 1. Table 2 compares the numerical results obtained by fully implicit exponential finite difference method and the exact solutions

Table 2. Comparison of the FI-EFDM solutions with the exact solution at $t = 0.1$ for $v = 1$ and $k = 10^{-5}$ using various mesh sizes.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	$h = 0.01$	Exact
0.1	0.109738	0.109596	0.109561	0.109556	0.109538
0.2	0.210185	0.209906	0.209836	0.209827	0.209792
0.3	0.292465	0.292060	0.291959	0.291946	0.291896
0.4	0.348638	0.348128	0.348000	0.347985	0.347924
0.5	0.372385	0.371807	0.371662	0.371645	0.371577
0.6	0.359871	0.359278	0.359130	0.359113	0.359046
0.7	0.310655	0.310115	0.309981	0.309964	0.309905
0.8	0.228392	0.227978	0.227874	0.227862	0.227817
0.9	0.120999	0.120774	0.120717	0.120711	0.120687
L_2	0.000579	0.000164	0.000060	0.000048	
L_∞	0.000827	0.000234	0.000086	0.000068	

Table 3. Comparison of the numerical solutions with the exact solution at different times for $v = 1.0$, $v = 0.01$, $h = 0.0125$ and $k = 10^{-5}$.

		$v = 1.0$			$v = 0.01$		
x	t	I-EFDM	FI-EFDM	Exact	I-EFDM	FI-EFDM	Exact
0.25	0.10	0.253690	0.253691	0.253638	0.566353	0.566355	0.566328
	0.15	0.156651	0.156651	0.156601	0.512175	0.512179	0.512148
	0.20	0.096484	0.096484	0.096442	0.466611	0.466614	0.466583
	0.25	0.059251	0.059251	0.059218	0.428021	0.428024	0.427995
0.50	0.10	0.371662	0.371662	0.371577	0.947453	0.947454	0.947414
	0.15	0.226901	0.226901	0.226824	0.900157	0.900159	0.900098
	0.20	0.138536	0.138536	0.138473	0.848433	0.848436	0.848365
	0.25	0.084585	0.084585	0.084538	0.796831	0.796835	0.796762
0.75	0.10	0.272650	0.272649	0.272582	0.860116	0.860119	0.860134
	0.15	0.164429	0.164429	0.164369	0.922814	0.922817	0.922756
	0.20	0.099482	0.099482	0.099435	0.962051	0.962053	0.961891
	0.25	0.060382	0.060382	0.060347	0.974916	0.974916	0.974689

at $t = 0.1$ for $v = 1$ and $k = 10^{-5}$ using various mesh sizes. It is observed from tables 1 and 2 that the values of L_2 and L_∞ decrease with decrease of h. Comparison of both numerical solutions with exact solution at different times for $v = 1.0$, $v = 0.01$, $h = 0.0125$ and $k = 10^{-5}$ are given in table 3. The obtained solutions by I-EFDM and FI-EFDM are compared with other methods [3,5,11,20] in table 4. All comparisons show that the present methods offer better results than the others. In order to show how the numerical solutions of Problem 1 obtained with fully implicit exponential finite difference method, we give the graph in figure 1.

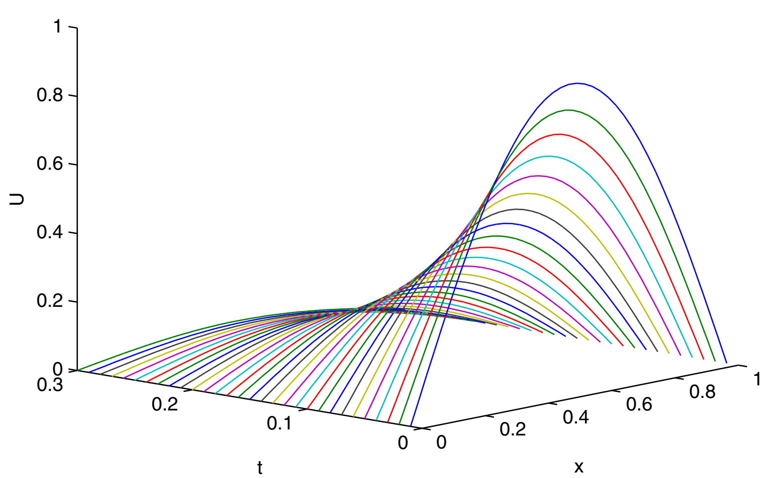

Figure 1. Solution with FI-EFDM at different times for $v = 1$, $h = 0.025$, $k = 10^{-4}$.

Table 4. Comparison of the results for Problem 1 at different times for $v = 0.1$, $h = 0.0125$ and $k = 10^{-4}$.

x	t	RHC [3]	RPA [5]	[11]	NM [20]	I-EFDM	FI-EFDM	Exact
0.25	0.4	0.317062	0.308776	0.31215	0.30415	0.308936	0.308962	0.308894
	0.6	0.248472	0.240654	0.24360	0.23629	0.240775	0.240795	0.240739
	0.8	0.202953	0.195579	0.19815	0.19150	0.195709	0.195725	0.195676
	1.0	0.169527	0.162513	0.16473	0.15861	0.162599	0.162612	0.162565
0.50	0.4	0.583408	0.569527	0.57293	0.56711	0.569727	0.569762	0.569632
	0.6	0.461714	0.447117	0.40588	0.44360	0.447307	0.447337	0.447206
	0.8	0.373800	0.359161	0.36286	0.35486	0.359343	0.359368	0.359236
	1.0	0.306184	0.291843	0.29532	0.28710	0.292026	0.292046	0.291916
0.75	0.4	0.638847	0.625341	0.63038	0.61874	0.625659	0.625676	0.625438
	0.6	0.506429	0.487089	0.49268	0.47855	0.487495	0.487513	0.487215
	0.8	0.393565	0.373827	0.37912	0.36467	0.374187	0.374203	0.373922
	1.0	0.305862	0.029726	0.03038	0.27860	0.287700	0.287714	0.287474

RHC – Restrictive Hopf–Cole method.
RPA – Restrictive Pade approximation.

Problem 2.

The initial condition for the current problem is

$$u(x, 0) = 4x(1 - x)\,, \qquad 0 < x < 1$$

Table 5. Comparison of the numerical solutions with the exact solution at different times for $v = 1.0$, $v = 0.01$, $h = 0.0125$ and $k = 10^{-5}$.

		$v = 1.0$			$v = 0.01$		
x	t	I-EFDM	FI-EFDM	Exact	I-EFDM	FI-EFDM	Exact
0.25	0.10	0.261534	0.261535	0.261480	0.607370	0.607373	0.607363
	0.15	0.161529	0.161529	0.161478	0.549431	0.549435	0.549421
	0.20	0.099513	0.099513	0.099470	0.499841	0.499845	0.499828
	0.25	0.061121	0.061121	0.061088	0.457427	0.457431	0.457413
0.50	0.10	0.383509	0.383510	0.383422	0.956022	0.956023	0.956007
	0.15	0.234135	0.234135	0.234055	0.914453	0.914454	0.914426
	0.20	0.142953	0.142953	0.142888	0.867170	0.867172	0.867136
	0.25	0.087282	0.087282	0.087233	0.818373	0.818376	0.818337
0.75	0.10	0.281643	0.281643	0.281573	0.886728	0.886730	0.886767
	0.15	0.169800	0.169800	0.169738	0.938493	0.938495	0.938437
	0.20	0.102704	0.102704	0.102655	0.969863	0.969864	0.969741
	0.25	0.062326	0.062326	0.062290	0.979625	0.979625	0.979469

Table 6. Comparison of the results for Problem 2 at different times for $v = 0.1$ and $h = 0.0125$.

		$k = 10^{-5}$		$k = 10^{-4}$				
x	t	RHC [3]	RPA [5]	[11]	NM [20]	I-EFDM	FI-EFDM	Exact
0.25	0.4	0.306529	0.317399	0.32091	0.31247	0.317567	0.317595	0.317523
	0.6	0.236051	0.246058	0.24910	0.24148	0.246175	0.246196	0.246138
	0.8	0.190181	0.199437	0.20211	0.19524	0.199589	0.199606	0.199555
	1.0	0.156646	0.165529	0.16782	0.16153	0.165633	0.165647	0.165599
0.50	0.4	0.565994	0.584429	0.58788	0.58176	0.584627	0.584664	0.584537
	0.6	0.438926	0.457888	0.46174	0.45414	0.458077	0.458110	0.457976
	0.8	0.348328	0.367320	0.37111	0.36283	0.367507	0.367533	0.367398
	1.0	0.280038	0.298271	0.30183	0.29336	0.298455	0.298476	0.298343
0.75	0.4	0.626990	0.645527	0.65054	0.63858	0.645850	0.645866	0.645616
	0.6	0.477908	0.502564	0.50825	0.49362	0.502969	0.502987	0.502676
	0.8	0.360630	0.385232	0.39068	0.37570	0.385613	0.385630	0.385336
	1.0	0.272623	0.295779	0.30057	0.28663	0.296092	0.296106	0.295857

and the boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t > 0$$

with the exact solution also given by eq. (6) but with the following coefficients:

$$a_0 = \int_0^1 \exp[-x^2(3v)^{-1}(3 - 2x)]\mathrm{d}x$$

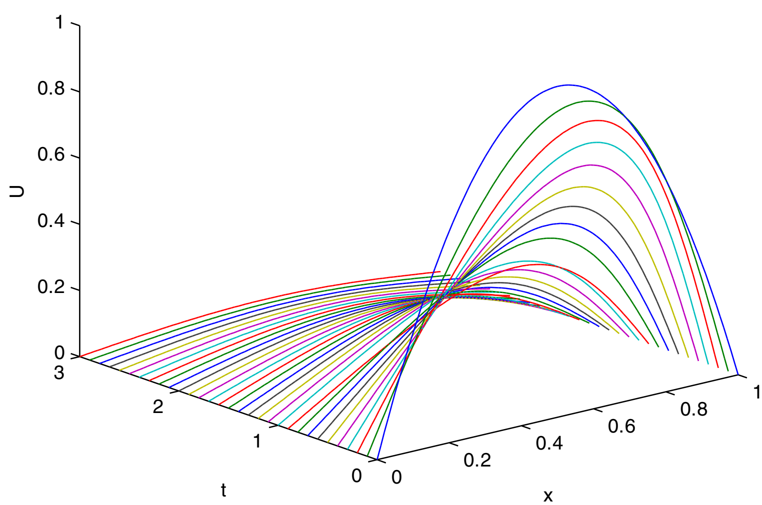

Figure 2. Solution with FI-EFDM at different times for $v = 0.1$, $h = 0.01$ and $k = 10^{-4}$.

Table 7. Comparison of the numerical solutions with the exact solution at different times for $a = 0$, $b = 8$, $v = 0.5$, $h = 0.025$ and $k = 10^{-4}$.

	$t = 1.5$			$t = 3.0$			$t = 4.5$		
x	I-EFDM	FI-EFDM	Exact	I-EFDM	FI-EFDM	Exact	I-EFDM	FI-EFDM	Exact
0.5	0.153285	0.153286	0.153273	0.064268	0.064268	0.064262	0.037993	0.037993	0.037989
1.0	0.265789	0.265791	0.265771	0.118814	0.118815	0.118804	0.071874	0.071875	0.071869
1.5	0.304137	0.304138	0.304125	0.155098	0.155099	0.155087	0.097937	0.097938	0.097931
2.0	0.261417	0.261417	0.261421	0.167631	0.167632	0.167623	0.113393	0.113394	0.113387
2.5	0.172156	0.172157	0.172169	0.156298	0.156299	0.156296	0.116989	0.116989	0.116984
3.0	0.088063	0.088064	0.088070	0.127378	0.127379	0.127382	0.109492	0.109493	0.109491
3.5	0.035822	0.035822	0.035820	0.091319	0.091320	0.091325	0.093685	0.093685	0.093685
4.0	0.011863	0.011863	0.011859	0.057971	0.057972	0.057975	0.073603	0.073604	0.073605
4.5	0.003249	0.003249	0.003246	0.032844	0.032844	0.032844	0.053298	0.053298	0.053300
5.0	0.000742	0.000742	0.000741	0.016737	0.016737	0.016735	0.035714	0.035714	0.035717
L_2	0.000021	0.000022		0.000022	0.000023		0.000408	0.000408	
L_∞	0.000018	0.000019		0.000038	0.000038		0.000743	0.000743	

$$a_n = 2\int_0^1 \exp[-x^2\,(3v)^{-1}\,(3-2x)]\cos(n\pi x)\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots.$$

In table 5, we compare the numerical results of Problem 2 obtained from both new methods (I-EFDM and FI-EFDM) with the exact solutions for both $v = 1.0$ and 0.01. In table 6, we compare the numerical results of our methods (I-EFDM and FI-EFDM) with the methods proposed in [3,5,11,20]. The comparisons showed that the present methods offer better results than the others. For $v = 0.1$, the computed solution of Problem 2 by FI-EFDM are displayed in figure 2.

Problem 3.

The initial condition for the current problem is

$$u(x, 1) = \frac{x}{1+\exp\left[1/4v\left(x^2-\frac{1}{4}\right)\right]}, \qquad a < x < b$$

and the boundary conditions

$$u(a, t) = u(b, t) = 0, \qquad t > 0$$

with the analytical solution

$$u(x, t) = \frac{x/t}{1+[t/\exp(1/8v)]^{1/2}\exp(x^2/4vt)}.$$

The numerical results of Problem 3 are displayed in table 7 for $v = 0.5$, $a = 0$, $b = 8$ with $h = 0.025$ and $k = 10^{-4}$. It is observed from the table that the values of L_2 and L_∞ are small enough. Figure 3 illustrates the fully implicit exponential finite difference solutions of Problem 3 at different values of t for $a = 0$, $b = 1.2$, $v = 0.005$, $h = 0.01$ and $k = 10^{-4}$. The figures for solutions from both methods are not drawn since they are very close to each other for the three problems. It is clearly seen from all the tables that the obtained numerical results with both methods present in this paper are in good agreement with the exact solution.

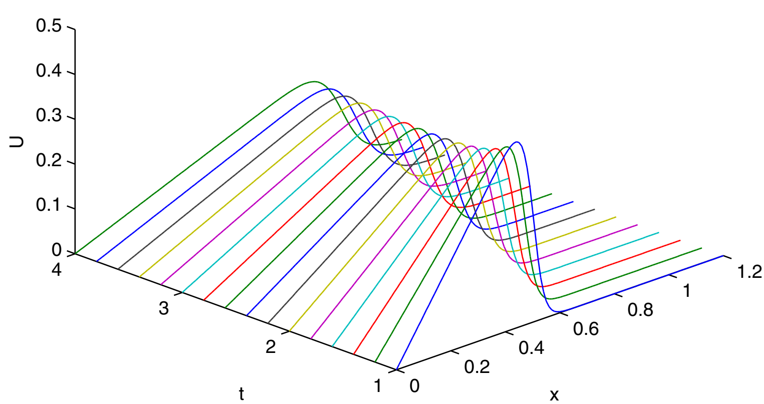

Figure 3. Solution with FI-EFDM at different times for $a = 0$, $b = 1.2$, $v = 0.005$, $h = 0.01$ and $k = 10^{-4}$.

4. Conclusion

In this paper, we defined two exponential finite difference methods for solving Burgers' equation. Numerical solutions for three different test problems were given. The results showed that both implicit exponential finite difference method and fully implicit exponential finite difference method offer high accuracy in the numerical solution of the one-dimensional Burgers' equation.

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