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On the determination of the mutual exclusion statistics parameter

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Abstract. Following the generalized definition of exclusion statistics to infinite-dimensional Hilbert space [Murthy and Shankar, *Phys. Rev. Lett.* **72**, 3629 (1994)] for a single-component anyonic system, we derive a simple relation between second mixed virial coefficient and the mutual exclusion statistics parameters using high-temperature expansion method for multicomponent anyonic system. The above result is derived without working in a specific model and is valid in any spatial dimensions.

Keywords. Anyons; virial coefficient; high-temperature expansion.

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1. Introduction

Like charge, mass, angular momentum etc., exclusion statistics obeyed by the elementary particles is an intrinsic property. We know that elementary particles are either fermions or bosons depending upon their exclusion statistics. If a given quantum state is occupied by a fermion, no second identical fermion can occupy it. On the other hand, for the case of bosons, there is no restriction on the occupancy of a given quantum state. However, recently it was established that in two dimensions there could be particles or excitations (as a result of strong correlations in a many-body system) whose exclusion statistics lies between fermions and bosons [1,2]. These particles are called anyons. In his seminal work [3], Haldane has characterized such anyons by proposing the idea of generalized exclusion statistics parameter g where

$$g = -\frac{d_{N+\Delta N} - d_N}{\Delta N}.$$

Here N is the number of particles and d_N is the dimension of the one-particle subspace in N-particle Hilbert space. It is obvious that $g = 0$ for bosons and $g = 1$ for fermions. For the state counting procedure, the above definition by Haldane implies the existence

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of discrete countable states. The concept of exclusion statistics is intimately connected with the exchange statistics. Similar to exclusion statistics parameter, exchange statistics is characterized by the exchange statistics parameter α. The exchange statistics parameter α denotes the phase of $e^{i\alpha\pi}$ picked up by many-body wave function when one particle is exchanged with other. Haldane has shown that g is related to α [3] for quasiparticles in fractional quantum Hall effect (FQHE) systems and spinons in quantum antiferromagnets. Later, Murthy and Shankar [4] have generalized the work of Haldane for the system in continuum where there is no cut-off in the energy scales and the single-particle Hilbert space dimension d_N can be infinite. They have shown that for a system with infinite-dimensional Hilbert space, g is solely determined by the second virial coefficient in high-temperature limit. Applying this result to the anyon gas, they have shown that $g = 2\alpha - \alpha^2$. It may be noted that, generally, exclusion statistics does not imply exchange statistics or vice versa but if it is there a connection can be drawn following the prescription in [4].

In this work we extend the work of [4] to a multicomponent anyonic system. This implies that there exist exclusion statistics between two species of anyons also. The generalized mutual exclusion statistics parameter g_{ij} is defined by the linear relation [3], $\Delta d_i = \sum_j g_{ij} \Delta N_j$, where Δd_i is the reduction of the available single-particle states for the ith species of anyon and ΔN_j is a set of allowed changes of the particle numbers of the jth species. By a straightforward generalization of the method shown in [4], we find that the sum of the mutual exclusion statistics parameters $(g_{ij} + g_{ji})$ is determined by the second mixed virial coefficient. The mixed second virial coefficient depends on the mixed two-particle partition function. This enables us to have an estimate of the sum of mutual exclusion statistics. For a simple case where the mixed two-particle partition function can be calculated, we show that $g_{ij} + g_{ji} = \lambda(2 - \lambda)$ where λ is the mutual exchange parameter of the two anyonic species. We conclude with a discussion.

2. High-temperature limit and virial coefficients

We consider a system of many anyonic species having mutual exclusion statistical interaction among them. Each species of anyon i is characterized by its own exclusion statistics parameter g_{ii} and a set of mutual exclusion statistic parameter g_{ij} characterizing the exclusion statistics between the species i and j. In general, one expect that $g_{ij} \neq g_{ji}$. Different species may refer to particles of the same kind but different quantum numbers [5,6], for example, momentum. The distribution function for such a multicomponent anyonic system [7] can be written as

$$D_N^{[n_i]} = \prod_i^m \frac{(d_{1i} - \sum_{j\neq i} n_j g_{ij} + (1 - g_{ii})(n_i - 1))!}{n_i!(d_{1i} - \sum_{j\neq i} n_j g_{ij} - 1 - g_{ii}(n_i - 1))!} . \quad (1)$$

In the above, m denotes the number of anyonic species in the system, n_i is the number of particles belonging to the species i and d_{1i} is the single-particle dimension of the species i. The total number of particles in the system and total energy satisfy the following relation:

$$E = \sum_i^m \epsilon_i n_i, \quad N = \sum_i^m n_i. \quad (2)$$

The distribution function given in eq. (1) is applicable to the non-interacting, ideal gas of anyons whose only interaction is statistical. This implies that the above formula cannot be generalized straightforwardly to the case of interacting anyons, for example, in the presence of Coulomb interaction. Newtonian particles carrying flux tubes in two spatial dimensions and anyons in a magnetic field are known to satisfy eq. (2) in the lowest Landau level (LLL) [6,8,9]. The low-energy regime of the strongly correlated electron system interacting with Coulomb interactions can be shown to obey mutual exclusion statistics in one and two dimensions [10].

Following eq. (1), various thermodynamical properties are calculated in [7]. The remarkable success of the application of eq. (1) was in fractional quantum Hall effect [7]. In what follows, we take the single-particle dimension d_{1i} of every species to be identical and equal to d without loss of generality. The generalization of eq. (1) to infinite-dimensional Hilbert space is achieved by taking $d \to \infty$ [4]. It is easy to find that from the distribution function in eq. (1), we can write in the limit of $d \to \infty$,

$$-\sum_i n_i S_i + \sum_i n_i(n_i - 1)\left(\frac{1}{2} - g_{ii}\right) = \lim_{d\to\infty} d\left(\frac{D_N^{[n_i]}\Pi n_i!}{d^N} - 1\right), \tag{3}$$

where $S_i = \sum_{i\neq j} n_j g_{ij}$. In arriving at eq. (3) from eq. (1), we first expanded the right-hand side of eq. (1) in powers of d, then we multiplied both sides by the factor $\Pi n_i!$ and divided by d^N followed by a rearrangement of terms of both sides. To use eq. (3) in a meaningful way and to connect to thermodynamical quantity we need to consider the high-temperature limit where $KT = 1/\beta$, is much higher than any other energy scales of the system. In this limit we can establish a connection between the partition function and the dimension of the Hilbert space, as explained in the following. The definition of the partition function is given by

$$Z_N = \sum_i \mathrm{e}^{-\beta\epsilon_i}. \tag{4}$$

In the above, the sum over i runs through all the available states. The degeneracy ($\epsilon_i = \epsilon_j$ for some i and j) is taken care of automatically. It is clear from the above equation that if $(\beta)^{-1}$ is larger than all the available ϵ_i, the sum in the RHS can be replaced by the number of states, i.e. the Hilbert space dimension. This motivates us to rewrite eq. (3) in the following form:

$$-\sum_i n_i S_i + \sum_i n_i(n_i - 1)\left(\frac{1}{2} - g_{ii}\right) = \lim_{\beta\to 0} C Z_1\left(\frac{\Pi n_i! Z_{[n_i]}}{Z_1^N} - 1\right), \tag{5}$$

where Z_1 is the single-particle partition function ($Z_1 = \mathrm{e}^{-\beta w}/(1 - \mathrm{e}^{\beta w})^2$) and $Z_{[n_i]}$ is the N-particle partition function for a given distribution $[n_i]$ among m different species. The constant C is an overall constant of proportionality. While deriving eq. (5) from eq. (3) we have replaced the dimension D by the partition function Z(times a constant) which is valid in the limit $\beta \to 0$. It can be shown that $C = 4$ for a system confined in harmonic potential in two dimensions. To proceed with the right-hand expression of eq. (5), we notice that one can write without loss of generality, the following expansion of N particle partition function Z_N:

$$\frac{Z_N}{Z_1^N} = \frac{1}{N!} + f_2^N(\beta w)^2 + f_3^N(\beta w)^4 + \cdots. \tag{6}$$

Here we consider the expansions in powers of $(\beta w)^2$ as we consider the particles to be confined in a harmonic oscillator potential in two space dimensions. The meaning of the above expression is quite simple. For N non-interacting particles, all the f_{2i}^N are identical to zero. For the interacting particle, the information of interaction is encoded in various f_{2i}^N. Generalizing eq. (6) for multispecies anyonic system, we write the following high-temperature expansion for the factor $Z_{[n_i]}$:

$$\frac{Z_{[n_i]}}{Z_1^N} = \frac{1}{(\Pi_i^m n_i!)} + f_2^{[n_i]}(\beta w)^2 + f_3^{[n_i]}(\beta w)^4, \tag{7}$$

where coefficients $f_k^{[n_i]}$ are to be determined. Now, if we sum the above expressions for a different set of distribution $[n_i]$ we get,

$$\frac{Z_N}{(Z_1^N)} = \frac{m^N}{N!} + F_2^N(\beta w)^2 + F_3^N(\beta w)^4 + \cdots , \tag{8}$$

where

$$F_2^N = \sum_{[n_i]} f_2^{[n_i]} \quad \text{and} \quad Z_N = \sum_{[n_i]} Z_N^{[n_i]}.$$

Combining eqs (5), (7), and (6) we get

$$-\sum_{i\neq j} g_{ij} + \sum_i \left(\frac{1}{2} - g_{ii}\right) = C Z_1(\beta w)^2 F_2^N \frac{(N-2)!}{m^{(N-2)}}. \tag{9}$$

From now on we would call these species as a and b. Then the above equation will be,

$$-(g_{ab} + g_{ba}) + \left(\frac{1}{2} - g_{aa}\right) + \left(\frac{1}{2} - g_{bb}\right) = 4Z_1(\beta w)^2 F_2^N \frac{(N-2)!}{2^{N-2}}. \tag{10}$$

The above equation reduces to

$$\frac{1}{2} - g = 4Z_1(\beta w)^2 f_2^N \frac{(N-2)!}{2^{N-2}},$$

for a single-component system as it should be [4]. This f_2^N would relate g to the second virial coefficient. Now we shall show that F_2^N is actually related to F_2^2. The grand canonical partition function of this two-component system is given by

$$Z = \sum_{n_a=0}^{\infty} \sum_{n_b=0}^{\infty} \mathrm{e}^{-\beta(\mu_a n_a + \mu_b n_b)} Z_{n_a,n_b} = \sum_{n_a=0}^{\infty} \sum_{n_b=0}^{\infty} z_a^{n_a} z_b^{n_b} Z_{n_a,n_b}, \tag{11}$$

where $z_a = \mathrm{e}^{-\beta\mu_a}$, $z_b = \mathrm{e}^{-\beta\mu_b}$, μ_a and μ_b are the chemical potentials of the a-type and b-type anyons respectively, z_a and z_b are the respective fugacity parameters. The equation of state of the system in fugacity expansion may be written as [11]

$$\beta P = \frac{1}{V}\ln(Z) = \frac{Z_1}{V} \sum_{l_a+l_b=1}^{\infty} b_{l_a,l_b} z_a^{l_a} z_b^{l_b}. \tag{12}$$

The general expression for b_{l_a,l_b} is given by

$$b_{l_a,l_b} = (Z_1^{l-1}) \sum_{p_i} (-1)^{(\sum_{i_a} p_i - 1)} \left(\sum_{i_a} p_i - 1 \right)! \Pi_i (Z_{i_a,i_b}/Z_1^{i_a+i_b})^{p_i}/p_i!. \quad (13)$$

The summation over p_i is constrained by

$$\sum_{i_a=0}^{l_a} i_a p_i = l_a \quad \text{and} \quad \sum_{i_b=0}^{l_b} i_b p_i = l_b$$

with an additional constraint $l_a + l_b \geq 1$. Now substituting the expression for $Z_{i_a,i_b}/Z_1^{i_a+i_b}$ in the above equation we get the following expressions for b_{l_a,l_b}:

$$b_{l_a,l_b} = \frac{1}{(\beta w)^{2l-2}} \left[\sum_{n_a+n_b=0}^{l_a+l_b-2} (-1)^{n_a+n_b} \frac{f_2^{l_a-n_a,l_b-n_b}}{n_a! n_b!} (\beta w)^2 + \cdots \right]. \quad (14)$$

If we demand all b_{l_a,l_b} to remain finite, the terms up to the power $(\beta w)^{2l-2}$ should go to zero, in particular the coefficient of $(\beta w)^2$ should be zero. This leads to the following relations:

$$f_2^{n_a,n_b} = \frac{n_a n_b}{n_a! n_b!} f_2^{1_a,1_b} + \frac{n_a(n_a-1)}{n_a! n_b!} f_2^{2_a,0} + \frac{n_b(n_b-1)}{n_a! n_b!} f_2^{0,2_b}. \quad (15)$$

The above equations when summed up yields,

$$F_2^N = \frac{2^{N-2}}{(N-2)!} (f_2^{0,2_b} + f_2^{2_a,0} + f_2^{1_a,1_b}) = \frac{2^{N-2}}{(N-2)!} F_2^2. \quad (16)$$

Combining eqs (7) and (10) yields

$$-(g_{ab} + g_{ba}) + \left(\frac{1}{2} - g_{aa} \right) + \left(\frac{1}{2} - g_{bb} \right)$$
$$= 4Z_1 \left(\frac{Z_{1_a,1_b}}{Z_1^2} - 1 \right) + 4Z_1 \left(\frac{Z_{2_a,0}}{Z_1^2} - \frac{1}{2} \right) + 4Z_1 \left(\frac{Z_{0,2_b}}{Z_1^2} - \frac{1}{2} \right). \quad (17)$$

The above equation yields a relation among all the exclusion statistical parameters in the system. However, we already know the dependency of exclusion statistical parameter for the single component as,

$$\left(\frac{1}{2} - g_{aa} \right) = 4Z_1 \left(\frac{Z_{2a}}{Z_1^2} - \frac{1}{2} \right), \quad \left(\frac{1}{2} - g_{bb} \right) = 4Z_1 \left(\frac{Z_{2b}}{Z_1^2} - \frac{1}{2} \right). \quad (18)$$

Using the above relation we get the following relation for the mutual exclusion statistical parameter:

$$(g_{ab} + g_{ba}) = -4Z_1 \left(\frac{Z_{1_a,1_b}}{Z_1^2} - 1 \right). \quad (19)$$

The RHS is nothing but four times the mixed second virial coefficients [12]. The above equation relates the mutual exclusion statistics parameter g_{ab} with the two-particle partition function $Z_{1_a,1_b}$. If the a- and b-type anyons are statistically independent, then $Z_{ab} = Z_1^2$ implying both g_{ab} and g_{ba} are zero. In general, whenever the exclusion statistics is present among two different species of anyons, we expect that $g_{ab} \neq g_{ba}$. In that case, to get a unique expressions for individual evaluation for the parameter g_{ab} and g_{ba}, we need to go to next order to calculate the third virial coefficients which is given in terms of three-particle partition function and solve for g_{ab} and g_{ba}. The scope of such calculation is beyond the present work. A special case of $g_{ab} \neq g_{ba}$, is the exclusion statistics obeyed by the quasihole and quasiparticle in fractional quantum Hall system. In that case, one finds $g_{ab} = -g_{ba} = 1/3$ [3,7,13]. However, one may also have $g_{ab} = g_{ab}$, as shown in [14] in multispecies Calogero–Sutherland model. It is important to note that it is the combination $(g_{ab}+g_{ba})$ which appears together in eq. (19). Now we present an example where the left-hand side of eq. (19) can be calculated exactly. We consider a problem of two anyons where an anyon is modelled by attaching an electric charge e and a magnetic flux ϕ [15]. The species a (b) is attached with the electric charge and magnetic flux pair e_a, ϕ_a (and e_b, ϕ_b). One can show that if $e_a\phi_b = e_b\phi_a$, the resulting Hamiltonian in the presence of harmonic oscillator confinement is easily solvable, because, then the Hamiltonian can be decoupled into two parts, one representing the motion of the centre of mass and the other representing the motion of relative coordinate, each of which is exactly solvable. The expression for $Z_{1_a,1_b}$ is easily obtained and is given by

$$Z_{1_a,1_b} = Z_1 \frac{\cosh(1-\lambda)\beta w}{2\sinh^2\beta w}, \tag{20}$$

where $Z_1 = 1/(4\sinh^2(\beta w/2))$ is the single-particle partition function, $\beta = 1/(KT)$, w is the frequency of the confining harmonic potential and $\lambda = (e_1\phi_2 + e_2\phi_1)/2 = e_1\phi_2$. In the limit $\beta \to 0$, it is straightforward to show that eq. (19) yields

$$(g_{ab} + g_{ba}) = \lambda(2-\lambda). \tag{21}$$

Considering the above equation it is sufficient to consider the value of λ modulo 2. We also notice that the above equation correctly yields identical limits at $\lambda = 0$ and $\lambda = 2$ as it should.

3. Discussion

In the present work we considered a multicomponent anyonic system where mutual exclusion statistics is present among various species. We have followed generalizations of the definition of exclusion statistics to infinite-dimensional Hilbert space as first conceived in [4]. By employing high-temperature expansion method, we have shown that mixed second virial coefficients are determined by the mutual exclusion statistics parameters. Various general and particular cases where the mutual exclusion statistics parameter can be calculated are discussed. An example of a two-component anyonic system is presented (where flux and the charge attached with the anyons satisfy $e_a\phi_b = e_b\phi_a$) where explicit evaluation of the mutual exclusion statistics parameter is possible. We expect that this work would significantly help to determine the mutual exclusion statistical parameter and to understand the nature of exclusion statistical interaction by just considering the two-particle partition function involved.

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